HURLSTONE AGRICULTURAL HIGH SCHOOL

MATHEMATICS

EXTENSION 1

2004

YEAR 12

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

EXTENSION 1

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GENERAL INSTRUCTIONS

- Reading Time – 5 minutes.
- Working Time – 2 hours.
- Attempt **all** questions.
- Questions are of equal value.
- **All** necessary working should be shown in every question.
- This paper contains seven (7) questions.
- Marks may not be awarded for careless or badly arranged work.
- Board approved calculators and MathAids may be used.
- **Each question is to be started in a new answer booklet**.
- This examination paper must **NOT** be removed from the examination room
- Place student number on each booklet

STUDENT NAME: ____________________________

TEACHER: ____________________________

Question 1 (commence a new answer booklet)(12 marks)

Marks

a. Solve for x, $\dfrac{2x+8}{x} < x$ **2**

b. $P(2,0)$ and $R(5,6)$ are two points on the number plane. Find the coordinates of the point M which divides PR externally in the ratio 4:1 **3**

c. $\displaystyle\int \frac{e^{2x}}{e^{2x}+1}\,dx$ **2**

d. Factorise $2^{n+1}+2^n$, and hence write **2**

$\dfrac{2^{1001}+2^{1000}}{3}$ as a power of 2.

e. Solve $\tan 2\theta + 4\tan\theta = 0$ for θ in the domain $0 \le x \le 2\pi$ **3**

Question 2 (commence a new answer booklet) (12 marks)

a) i) Using the sum to n terms of an Arithmetic series, $S_n = \frac{n}{2}(a+l)$,

Show that $(1+2+3+.......+n)^2 = \frac{1}{4}n^2(n+1)^2$. **1**

ii) Using the principle of Mathematical induction and the result in part (i) prove that **4**

$1^3 + 2^3 + 3^3 + ... + n^3 = (1+2+3+.......+n)^2$, for $n \geq 1$.

b) i) Sketch the curve $y = (x-1)^2 - 3$ **1**

ii) Find the largest positive domain such that the graph defines a function $f(x)$ which has an inverse function. **1**

iii) Find this inverse function. **1**

c) Show that the function $g(x) = \cos^{-1}(-x) + \cos^{-1}(x)$ is constant and find the value of this constant. **1**

d) Show $\frac{d}{dx}\left[\left\{\tan^{-1}\left(\frac{x}{3}\right)\right\}^2\right] = \frac{6\tan^{-1}\frac{x}{3}}{x^2+9}$

and hence find the exact value of $\frac{1}{\pi}\int_0^{\sqrt{3}} \frac{\tan^{-1}\left(\frac{x}{3}\right)}{x^2+9}\,dx$ **3**

Question 3 (commence a new answer booklet) (12 marks)

(a) The acceleration of a particle moving in a straight line is given by

$$\ddot{x} = e^{-2x}$$

where x is the displacement in metres and t is the time in seconds.

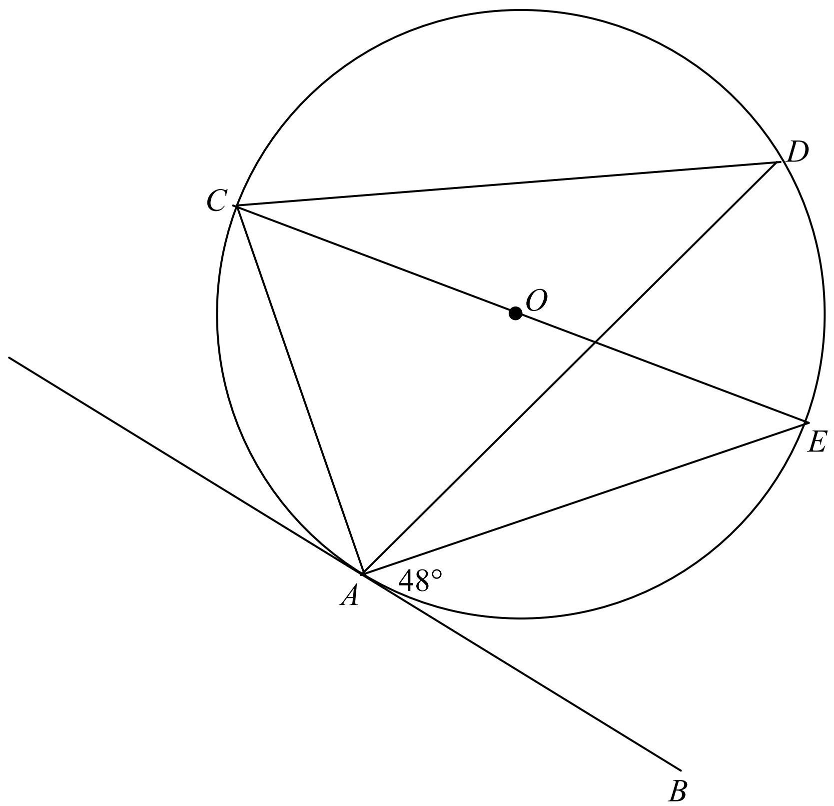
Initially, the particle is at rest at the origin.

(i) Show that $v^2 = 1 - e^{-2x}$ **2**

(ii) Explain why the velocity is never negative. **1**

(iii) What is the maximum speed the particle can reach? **1**

(b) **3**



AB is a tangent and *CE* is a diameter to the circle with centre *O*.
Angle *BAE* equals 48° and *D* lies on the circumference as shown in the diagram.

Find the size of angle *ADC*. Justify your answer.

Question 3 continued.

(c) $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are two points on the parabola $x^2 = 4ay$.

(i) If PQ passes through the point $R(2a, 3a)$, show that $pq = p + q - 3$. **1**

(You are given the equation of PQ as $y = \left(\dfrac{p+q}{2}\right)x - apq$)

(ii) If M is the mid-point of PQ, show that the coordinates of M are $\left[a(pq+3), \dfrac{a}{2}((pq+3)^2 - 2pq)\right]$. **2**

(iii) Hence, find the locus of M. **2**

Question 4 (commence a new answer booklet) (12 marks)

(a) Use the substitution $u = x^2 + 2$ to evaluate $\int_0^1 \frac{x}{(x^2+2)^2}\,dx$ **3**

(b) Use Newton's method to find a second approximation to the positive root of **3**

$$x - 2\sin x = 0 \;.$$

Take $x = 1\cdot 7$ as the first approximation.

(c) The population P of a town increases at a rate proportional to the number by which the town's population exceeds 5000. This can be expressed by the following differential equation:

$$\frac{dP}{dt} = k(P - 5000)$$

where t is the time in years and k is some constant.

(i) Show that $P = 5000 + Ae^{kt}$ is a solution of the equation. **1**

(ii) Initially the population was 8000, but after 3 years it had increased to 8500. **3**
Find the values of A and k.

(iii) After how many more years will the population reach 15000? **2**

Question 5 (commence a new answer booklet) (12 marks)

(a) The velocity v m/s of a particle moving in a straight line is given by

$$v^2 = 8 + 4x - 4x^2$$

where x is the displacement from O in metres.

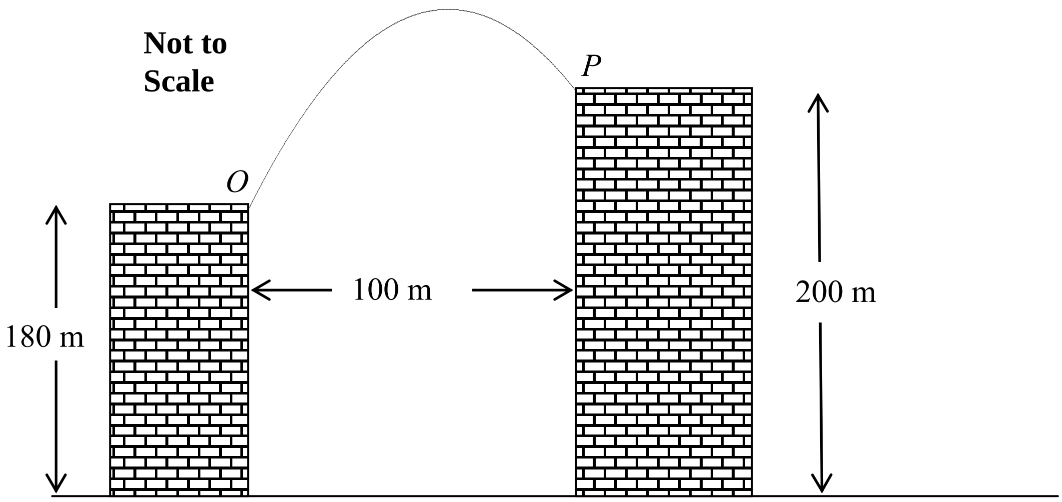
(i) Show that the motion is Simple Harmonic. **2**

(ii) Between what points is the particle oscillating? **1**

(iii) What is the period and amplitude? **2**

(iv) Find the maximum speed of the particle. **1**

(b) Two building are situated 100 metres apart on ground level. Their heights are 180 and 200 metres. An object is projected from the top of the shorter building at an angle of 45°, striking the top of the taller at P as illustrated in the diagram. Take g = 10 m/s^2.



Find:

(i) the initial velocity of projection. **5**

(ii) the time taken for the object to reach the point P. **1**

Question 6 (commence a new answer booklet) (12 marks)

(a) The equation $x^3 - mx + 2 = 0$ has two equal roots.

(i) Write down expressions for the sum of the roots and for the product of the roots. **1**

(ii) Hence find the value of m. **2**

(b) Show that the equation $2x^3 - 5x^2 + 3x - 2 = 0$ has only one real root. **2**

(c) Given $y = e^{\tan x}$

(i) Prove $\dfrac{d^2y}{dx^2} = e^{\tan x}.\sec^2 x(1+\tan x)^2$ **2**

(ii) Prove that there are no stationary points on the curve. **1**

(iii) Prove $\dfrac{d^2y}{dx^2} = 0$ when $x = \frac{3\pi}{4}, \frac{7\pi}{4}$ if $0 \le x \le 2\pi$. Show that neither of these points are points of inflexion. **2**

(iv) Sketch $y = e^{\tan x}$ $\quad 0 \le x \le 2\pi$ **2**

Question 7 (commence a new answer booklet) (12 marks)

a) A spherical bubble is expanding so that its volume is increasing at a constant rate of $10mm^3$ per second. What is the rate of increase of the radius when the surface area is $500mm^2$? **2**

(Given $V = \frac{4}{3}\pi r^3$ *and* $S = 4\pi r^2$)

b) i) Given that $2\cos^2 x = 1 + \cos 2x$,

prove that $8cos^4 x = 3 + 4\cos 2x + \cos 4x$ **1**

ii) Sketch on the same diagram, the curves **1**

$y = \cos x, y = \cos^2 x, for\ 0 \le x \le \frac{\pi}{2}.$

iii) Find the area enclosed between **2**

$y = \cos x, y = \cos^2 x, for\ 0 \le x \le \frac{\pi}{2}.$

iv) Using the result in part (i) or otherwise, find the volume generated when the area in part (iii) is rotated about the x axis. **2**

c. Given $f(x) = \sqrt{3}\sin x + \cos x$

i. Express $f(x)$ in the form $R\sin(x + \alpha)$. **2**

ii. Sketch $f(x)$ for $0 \le x \le 2\pi$ **1**

iii. Hence or otherwise, determine the number of solution of x for the equation $\sqrt{3}\sin x + \cos x - 1 = 0$ in the domain $0 \le x \le 2\pi$. Provide reasoning for your answer. **1**

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax\ \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$