INTERNATIONAL GRAMMAR SCHOOL
Concordia per Diversitatem

Name:...........................

INTERNATIONAL GRAMMAR SCHOOL

MATHEMATICS

YEAR 12

TRIAL EXAMINATION

JULY, 2000

3 UNIT

Time allowed — 2 hours
(Plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- ALL questions are of equal value.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Board-approved calculators may be used.
- Start each question on a *new page*. Number each question clearly.
- Label each page with your name.
- A table of Standard Integrals is attached.

Marks

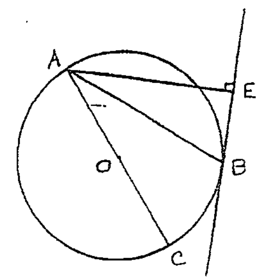
Q1. (a) Let A(-5,12) and B(4,9) be two points in the number plane. Find the coordinates of P which divides the interval AB externally in the ratio 5 : 2. **2**

(b) Find the size of the acute angle between the lines $y = 2x + 3$ and $y = 4x + 1$. (Answer to the nearest minute). **2**

(c) Express $f(x) = x^3 + 3x^2 - 10x - 24$ as a product of three linear factors. **3**

(d) Evaluate $\displaystyle\int_0^3 \frac{dx}{\sqrt{9-x^2}}$ **3**

(e) Two points A and B are placed on a circle and AC is a diameter. AE is perpendicular to the tangent at B. **2**



(i) Draw the diagram on your paper.

(ii) Prove AB bisects $\angle CAE$.

Q2. **Start a new booklet**

(a) Solve for x : $x \geq \frac{4}{x}$ **3**

(b) For $y = -3\sin^{-1}\frac{x}{2}$

(i) State the domain and range.
(ii) Sketch the curve. **3**

(c) Using the substitution $u = 9 - x^2$, evaluate $\displaystyle\int_0^3 x\sqrt{9-x^2}\,dx$ **3**

(d) The area bounded by the curve $y = \sin x$ between $x = 0$ and $x = \frac{\pi}{2}$ is rotated about the x-axis. Find the volume of the solid of revolution. **3**

Marks

Q3. **Start a new booklet**

(a) Express $3\cos x + 4\sin x$ in the form $A\cos(x-\alpha)$ where $A > 0$. Hence, or otherwise, solve $3\cos x + 4\sin x = -3$ for $0 \le x \le 360°$. 4

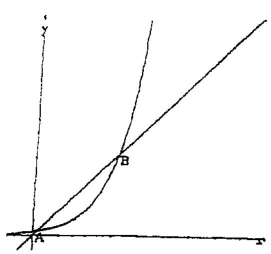
(b) In a co-educational class there are 4 girls and 7 boys. Their classroom has 5 rows of 5 desks neatly arranged. Each student occupies a desk with a chair. Find the number of seating arrangements possible if,

(i) students can sit anywhere,

(ii) all the girls want to occupy the first row.

(iii) Two particular girls and three particular boys fill the back row seated alternately. 4

(c)



The diagram shows the graphs of $y = e^{x-2}$ and $y = x$ with points of intersection at A and B.

(i) How many roots has the equation $e^{x-2} - x = 0$?

(ii) Taking $x = 3.3$ as the first approximation, use one application of Newton's Method to find a better approximation to the x-coordinate of B. 4

Marks

Q4. **Start a new booklet**

(a) Find x and y if $\frac{4^x}{16} = 8^{x+y}$ and $2^{2x+y} = 128$. 3

(b) If $x = 2 - \cos t$ and $y = 2t + 2\sin t$, 4

(i) find $\frac{dx}{dt}$ and $\frac{dy}{dt}$

(ii) Hence or otherwise, find $\frac{dy}{dx}$ in terms of $\frac{t}{2}$.

(c) A particle is oscillating in simple harmonic motion such that its displacement x metres from the origin is given by the equation $\frac{d^2x}{dt^2} = -9x$ where t is time in seconds. 5

(i) Show that $x = a\cos(3t+\alpha)$ is a solution of motion for this particle (a and α are constants).

(ii) When $t = 0$, $v = 3$ m/s and $x = 5$ m. Show that the amplitude of the oscillation is $\sqrt{26}$ metres.

(iii) What is the maximum speed of the particle?

Marks

5. **Start a new booklet**

(a) α, β, γ are the roots of the equation $2x^3 + 3x^2 - 4 = 0$ 3

Find

(i) $\alpha + \beta + \gamma$

(ii) $\alpha \beta \gamma$

(iii) $\alpha^2 + \beta^2 + \gamma^2$

(b) For the function $y = x^2 - 2x + 1$, find the largest possible domain such that this function has an inverse. Find the equation of this inverse and state its range. 3

(c) For the parabola $x^2 = 12y$, find 6

(i) the equation of the tangent at the point $P\ (6p, 3p^2)$ on the parabola.

(ii) the coordinates of the point T where the tangent meets the x axis.

(iii) Show that N, the midpoint of PT, has coordinates $(\frac{9p}{2}, \frac{3p^2}{2})$.

(iv) Find the equation of the locus of N.

Start a new booklet

(a) Find $\lim\limits_{x \to 0} \frac{\sin 3x}{5x}$ 2

(b) The daily growth of a colony of insects is 10% of the excess of the population over 1.2×10^6. 4

ie $\frac{dN}{dt} = 0.1\ (N - 1.2 \times 10^6)$.

Initially, the population is 2.7×10^6,

(i) Determine the population after $3\frac{1}{2}$ days.

(ii) If a scientist checks the population each day, which is the first day on which she should notice that the original population has tripled?

Marks

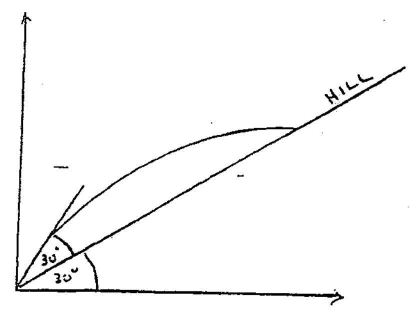
Q6. (continued).....

(c) A ball is thrown with a velocity of $30\sqrt{3}$ m/s at an angle of 60° to the horizontal. 6

(i) Assuming negligible air resistance and letting $g = 10\ ms^{-2}$, derive the equations of motion.

(ii) Find the time of flight and the range.

(iii) If the ball had been thrown with velocity $30\sqrt{3}$ m/s at an angle of 30° to a hill which is itself inclined at 30° to the horizontal (see diagram), determine the time of flight.



Q 7. **Start a new booklet**

(a) Prove by mathematical induction that for all values of n 5

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \ldots\ldots \frac{n}{(n+1)!} = \frac{(n+1)! - 1}{(n+1)!}$$

where n is a positive integer.

(b) (i) Show that $y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ has no stationary points. 7

(ii) Prove that the lines $y = \pm 1$ are asymptotes.

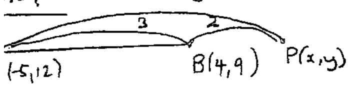
(iii) Sketch the curve.

(iv) If k is a positive constant, find the area in the first quadrant enclosed by the above curve and the three lines $y = 1$, $x = 0$ and $x = k$.

(v) Prove that for all values of k, this area is always less than $\log_e 2$.

it Trial July 2000

on 1



$\frac{3x+2(-5)}{5}$, $9 = \frac{3y+2(12)}{5}$

$3x-10$ $\quad 45 = 3y+24$

$3x$ $\quad 21 = 3y$

10 $\quad y = 7$

$P \equiv (10,7)$

$=2, \; m_2 = 4$

$n\,\alpha = \left|\frac{m_1 - m_2}{1+m_1m_2}\right|$

$= \left|\frac{2-4}{1+8}\right|$

$n\,\alpha = \frac{2}{9}$

$\alpha = 12^\circ 32'$

$(x) = x^3 + 3x^2 - 10x - 24$

$(3) = 27 + 27 - 30 - 24 = 0$

$3)$ is a factor

$$\begin{array}{r} x^2+6x+8 \\ x-3\,\overline{)\,x^3+3x^2-10x-24} \\ x^3-3x^2 \\ \hline 6x^2-10x \\ 6x^2-18x \\ \hline 8x-24 \\ 8x-24 \\ \hline \end{array}$$

$\therefore f(x) = (x-3)(x^2+6x+8)$

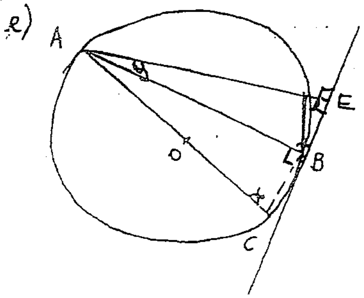
$f(x) = (x-3)(x+2)(x+4)$

d) $\int_0^3 \frac{dx}{\sqrt{9-x^2}} = \left[\sin^{-1}\frac{x}{3}\right]_0^3$

$= \sin^{-1}1 - \sin^{-1}0$

$= \frac{\pi}{2}$

e)



Construction: join BC

$\angle ABC = 90^\circ$ (angle in semicircle)

$\angle AEB = 90^\circ$ (adj suppl $\angle$)

let $\angle EAB = y$, $\angle ABE = x$

then $x + y = 90^\circ$ (angle sum of Δ)

$\angle ACB = x$ (angle in alt segment)

$\therefore \angle CAB = 180 - 90 - x$

$= 90 - x$ (angle sum of Δ)

$= y$

$\therefore \angle CAB = \angle BAE = y$

AB bisects $\angle CAE$

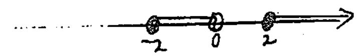
Question 2

a) $x \geq \frac{4}{x}$

$x \neq 0$;

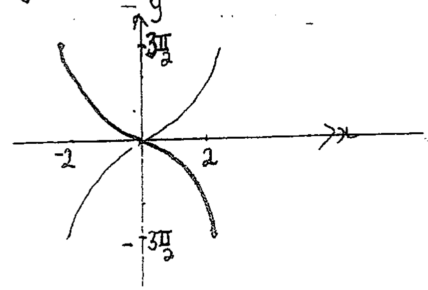
solve $x^2 = 4$

$x = \pm 2$ Try solutions



$\therefore -2 \leq x < 0$ or $x \geq 2$

b) $y = -3\sin^{-1}\frac{x}{2}$



Domain: $-2 \leq x \leq 2$

Range: $-\frac{3\pi}{2} \leq y \leq \frac{3\pi}{2}$

c) $u = 9 - x^2$ limits

$\frac{du}{dx} = -2x$ when $x=0$, $u=9$

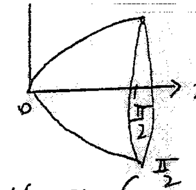
$\frac{du}{-2} = x\,dx$ when $x=3$, $u=0$

$\int_0^3 x\sqrt{9-x^2}\,.dx = -\frac{1}{2}\int_9^0 \sqrt{u}\;du$

$= \frac{1}{2}\int_0^9 u^{1/2}\,.du$

$= \frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_0^9$

d)



$\cos 2x = 1 - 2\sin$

$V = \pi\int_0^{\frac{\pi}{2}} \sin^2 x\,.dx$

$= \frac{\pi}{2}\int_0^{\frac{\pi}{2}} 1 - \cos 2x\;.dx$

$= \frac{\pi}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{2}}$

$= \frac{\pi}{2}\left(\left(\frac{\pi}{2} - 0\right) - (0-0)\right)$

$= \frac{\pi^2}{4}$ units^3

on 3

$s(x-\alpha) = 3\cos x + 4\sin x$

$s\alpha + A\sin x\sin\alpha = 3\cos x + 4\sin x$

$\cos\alpha = 3 \; \ldots \; ①$

$\sin\alpha = 4 \; \ldots \; ②$

$\tan\alpha = 4/3$

$\alpha = 53°$

$A^2(\cos^2\alpha + \sin^2\alpha) = 3^2 + 4^2$

$A^2 = 25$

$A = 5$

$\cos x + 4\sin x = 5\cos(x - 53°)$

$5\cos(x-53°) = -3$

$\cos(x-53°) = -3/5$

$x - 53° = 127°, 233°$

$x = 180°, 286°$

irls, 7 boys.

x x x x

x x x x

x x x x

x x x x

$\times {}^5P_4$

$6 \times {}^3P_3 \times {}^2P_2$

c) (i) 2 roots

(ii) $f(x) = e^{x-2} - x$

$f'(x) = e^{x-2} - 1$

$a_2 = 3.3 - \dfrac{f(3.3)}{f'(3.3)}$

$= 3.3 - \dfrac{0.369}{2.669}$

$= 3.16$

Question 4

a) $\dfrac{4^x}{16} = 8^{x+y}$ and $2^{2x+y} = 128$

change all to powers of 2

$2^{2x-4} = 2^{3x+3y}$

$2x - 4 = 3x + 3y$ (equating indices)

$-x - 3y = 4 \; \ldots \; ①$

$2^{2x+y} = 2^7$

$2x + y = 7 \; ②$ (equating indices)

solving $① \times 2$ & $②$ $\Rightarrow$

$-2x - 6y = 8$

$+ \quad 2x + y = 7$

$-5y = 15$

$y = -3 \quad \therefore x = 5$

b) $x = 2 - \cos t \qquad y = 2t + 2\sin t$

$\dfrac{dx}{dt} = +\sin t \qquad \dfrac{dy}{dt} = 2 + 2\cos t$

$\dfrac{dy}{dx} = \dfrac{dy}{dt} \times \dfrac{dt}{dx}$

$= \dfrac{2 + 2\cos t}{\sin t}$

where $\cos t = \dfrac{1 - t^2}{1 + t^2}$

$\sin t = \dfrac{2t}{1 + t^2}$

$\dfrac{dy}{dx} = \dfrac{2 + 2\left(\frac{1-t^2}{1+t^2}\right)}{\frac{2t}{1+t^2}}$

$\dfrac{dy}{dx} = \dfrac{2(1+t^2) + 2(1 - \quad)}{2t}$

$= \dfrac{2 + 2t^2 + 2 - 2t^2}{t}$

$= \dfrac{4}{t}$

c) $x = a\cos(3t + \alpha)$

$\dot{x} = -3a\sin(3t + \alpha)$

$\ddot{x} = -9a\cos(3t + \alpha)$

$\ddot{x} = -9(a\cos(3t + \alpha))$

$\ddot{x} = -9x$

$\therefore$ satisfies the ...

when $t = 0$ and $x = 5$

using ①

$5 = a\cos\alpha$

using ②

$3 = -3a\sin\alpha$

$-1 = a\sin\alpha$

$①^2 + ②^2 \Rightarrow 25 + 1 = a^2$

$a = \sqrt{26}$

$② \div ① \Rightarrow \tan\alpha = -1/5$

$\alpha = -0.1$

(iii) max speed at centre

from ① $0 = \sqrt{26}\cos(3t - 0.\ldots)$

$3t - 0.18 = \frac{\pi}{2}$

$\therefore v = -3\sqrt{26}\sin(3(0.56) - 0.1\ldots)$

$v^2 = 9(26 - 0)$

$v = \pm 3\sqrt{26} \quad \therefore$ max speed is $3\sqrt{26}$ m/

...ion 5

$\alpha+\beta+\gamma = -3/2$

$\alpha\beta\gamma = -(-4/2) = 2$

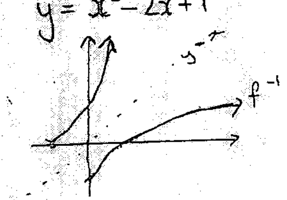
$(\alpha\beta+\beta\gamma+\alpha\gamma) = 0$

$\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\alpha\gamma)$

$= (-3/2)^2$

$= 9/4$

$y = x^2-2x+1$



...gest domain of f: $x \geq -1$

let $y = x^2-2x+1$

...ange $x = y^2-2y+1$

...ke y the subject

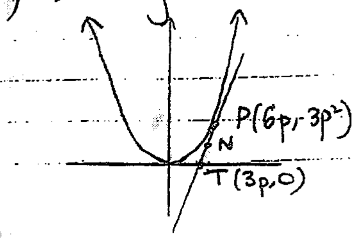
$x = (y-1)^2$

$\sqrt{x} = y-1$

$y = \sqrt{x}+1$

...nge: $y \geq -1$

c) $x^2 = 12y$



(i) $\frac{x^2}{12} = y$

$\frac{dy}{dx} = \frac{2x}{12} = \frac{x}{6}$

at $x=6p$, $\frac{dy}{dx} = \frac{6p}{6} = p$

$\therefore$ eqⁿ of tangent is

$(y-3p^2) = p(x-6p)$

$y-3p^2 = px-6p^2$

$y = px-3p^2$

(ii) tangent meets x-axis when $y=0$

$px = 3p^2$

$x = 3p$

$\therefore\ T(3p, 0)$

(iii) N is mid pt of PT

$N \equiv \left(\frac{6p+3p}{2}, \frac{3p^2}{2}\right)$

$N \equiv \left(\frac{9p}{2}, \frac{3p^2}{2}\right)$

(iv) let $X = \frac{9p}{2}$, $Y = \frac{3p^2}{2}$

$X^2 = \frac{81p^2}{4}$

$X^2 = \frac{27}{2}\left(\frac{3p^2}{2}\right) = \frac{27}{2}Y$

Question 6

a) $\lim_{x\to 0}\left(\frac{\sin 3x}{3x} \times \frac{3}{5}\right)$

$= \frac{3}{5}(1)$

$= \frac{3}{5}$

b) $\frac{dN}{dt} = 0.1(N - 1.2\times10^6)$

Solution is

$N = A \mp Be^{kt}$

where $A = 1.2\times10^6$

$k = 0.1$

when $t=0$, $N = 2.7\times10^6$

$2.7\times10^6 = 1.2\times10^6 + Be^0$

$B = 1.5\times10^6$

$N = 1.2\times10^6 + 1.5\times10^6 e^{0.1t}$

(i) when $t = 3\frac{1}{2}$

$N = 1.2\times10^6 + 1.5\times10^6 e^{0.35}$

$N = 3328\ 601$

(ii) when $N = 2.7\times10^6 \times 3$

$N = 8.1\times10^6$

$8.1\times10^6 = 1.2\times10^6 + 1.5\times10^6 e^{0.1t}$

$\frac{6.9\times10^6}{1.5\times10^6} = e^{0.1t}$

$\ln 4.6 = t$ $\quad t = 15.26$

c)(i) $\ddot{x} = 0$ $\quad \ddot{y} = -10$

$\dot{x} = V\cos\alpha$ $\quad \dot{y} = -10t + V\sin\alpha$

$\dot{x} = 30\sqrt{3}\cos 60°$ $\quad \dot{y} = -10t + 30\sqrt{3}\,s...$

$\dot{x} = 15\sqrt{3}$ $\quad \dot{y} = -10t + 45$

$x = 15\sqrt{3}t + 0$ $\quad y = -5t^2 + 4...$

(ii) For time of flight

when $y=0$, $5t^2 - 45t = 0$

$5t(t-9) = 0$

$t=0$, $t=9$

$\therefore$ 9 seconds

For range;

$x = 15\sqrt{3}\times 9 = 135\sqrt{3}$ me...

(iii) equation of hill, $y = \frac{1}{\sqrt{3}}x$

$-5t^2+45t = \frac{1}{\sqrt{3}}(15\sqrt{3})t$

$-5t^2+30t = 0$

$-5t(t-6) = 0$

$t=0$ or $t=6$

$\therefore$ time of flight is 6 secon...

tion 7

r $n=1$

s $= \frac{1}{2!} = \frac{1}{2}$

$= \frac{(1+1)!-1}{2!} = \frac{2-1}{2} = \frac{1}{2}$

me for $n=1$

sume true for $n=k$

$\frac{2}{3!} + \dots \frac{k}{(k+1)!} = \frac{(k+1)!-1}{(k+1)!}$

for $n=k+1$

$\underbrace{\frac{1}{2!} + \frac{2}{3!} + \dots \frac{k}{(k+1)!}} + \frac{k+1}{(k+2)!}$

$\frac{(k+1)!-1}{(k+1)!} + \frac{k+1}{(k+2)!}$

$= \frac{(k+2)[(k+1)!-1]}{(k+2)!} + \frac{k+1}{(k+2)!}$

$= \frac{(k+2)! - (k+2) + (k+1)}{(k+2)!}$

$= \frac{(k+2)!-1}{(k+2)!}$

$\frac{(k+2)!-1}{(k+2)!}$

true for $n=k+1$

true for $n=1$

$\therefore$ true for all n, positive integral

b) (i) $y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

$\frac{dy}{dx} = \frac{(e^x+e^{-x})(e^x+e^{-x}) - (e^x-e^{-x})(\dots}{(e^x+e^{-x})^2}$

$= \frac{(e^{2x}+2+e^{-2x}) - (e^{2x}-2+e\dots}{(e^x+e^{-x})^2}$

$\frac{dy}{dx} = \frac{4}{(e^x+e^{-x})^2}$

but $\frac{dy}{dx} \neq 0$ $\therefore$ no stat pts

as $4 \neq 0$ and $e^x + e^{-x} \neq 0$

(ii) $y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

$\lim_{x\to\infty} \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)$

$\lim_{x\to\infty} \left(\frac{\frac{e^x}{e^x} - \frac{e^{-x}}{e^x}}{\frac{e^x}{e^x} + \frac{e^{-x}}{e^{-x}}}\right)$

$\therefore \left(\frac{1-0}{1+0}\right) = 1$

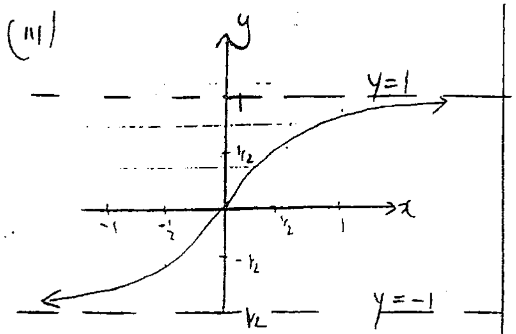
$\therefore y=1$ is an asymptote

Similarly $\lim_{x\to-\infty} \left(\frac{\frac{e^x}{e^{-x}} - \frac{e^{-x}}{e^{-x}}}{\frac{e^x}{e^{-x}} + \frac{e^x}{e^{-x}}}\right)$

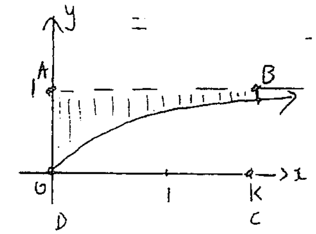
$= \frac{-1}{1} = -1$

$\therefore y=-1$ is an asymptote

(iii)



(iv)



Area of rectangle ABCD $= k \times 1$
$= k$

$\therefore$ Area under curve $= \int_0^k \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$

$= \ln(e^x+e^{-x})\Big]_0^k$

$= \ln(e^k+e^{-k}) - \ln 2$

$\therefore$ Shaded area $= k - (\ln(e^k+e^{-k}) - \ln 2)$

(v) Area $= k - \ln(e^k+e^{-k}) + \ln 2$

$= \log_e e^k - \ln(e^k+e^{-k}) + \ln 2$

$= \log_e \frac{e^k}{e^k+e^{-k}} + \ln 2$

Now, $\log_e\left[\frac{e^k}{e^k+e^{-k}}\right] < 0$

as $\frac{e^k}{e^k+e^{-k}} < 1$

$\therefore$ max value is $\log_e 2$