INTERNATIONAL GRAMMAR SCHOOL

Concordia per Diversitatem

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION 200

MATHEMATICS EXTENSION 1

YEAR 12

TIME ALLOWED: 2 HOURS

(Plus 5 minutes reading time)

DIRECTIONS

- Attempt **ALL** questions.
- Show all working clearly and neatly.
- Marks will be deducted for untidy and careless work.
- Board approved calculator may be used for this exam.
- All questions are of equal value.
- A table of integrals is provided.

YEAR 12 – TRIAL 2001 – EXTENSION 1

QUESTION 1 **MARKS**

a) Evaluate $\lim_{x \to 0} \frac{\sin 3x}{x}$ 1

b) The polynomial $P(x) = px^3 + 5x^2 - 3p$ has a factor $(x - 2)$. Find the value of p. 2

c) Differentiate $x \tan^{-1} x$ 2

d) Find the size of the acute angle between the tangents of $y = \ln(2x + 1)$ at the point where $x = 0$ and $x = \frac{1}{2}$ 2

e) Evaluate $\int_0^{\frac{1}{6}} \frac{9\,dx}{\sqrt{1 - 9x^2}}$ 3

f) Solve the inequation $\frac{1}{x} > \frac{1}{x + 2}$ 2

QUESTION 2 **MARKS**

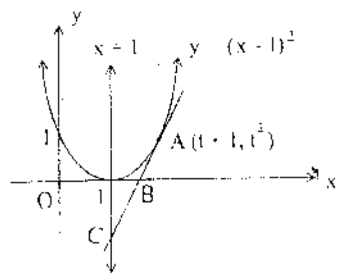
a) Solve the inequality $\frac{2x - 3}{x} > 1$ 2

b) Solve the equation $\sin 2\theta = 2\cos^2\theta$ for $0 \le \theta \le 2\pi$. 3

c) Use the substitution $u = 3\sin x$ to evaluate 3

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{1 + 3\sin x}}\,dx$$

d) The point $A(t + 1, t^2)$, $t \ne 0$, is a variable point on the parabola $y = (x - 1)^2$. The tangent at A meets the x axis at B and the line $x = 1$ at C.



i) Find the equation of the tangent at A. 2

ii) Show that B is the midpoint of AC. 2

QUESTION 3 **MARKS**

a) Consider the function $y = 2\cos^{-1}\frac{x}{3}$

i) State its domain. 1

ii) Sketch the graph of the function. 2

iii) Find the gradient of the tangent to the curve at the point where it crosses the y axis. 1

b) The velocity of a particle moving along the x axis is given by

$V^2 = -7 + 8x - x^2$

i) Find the acceleration of the particle. 2

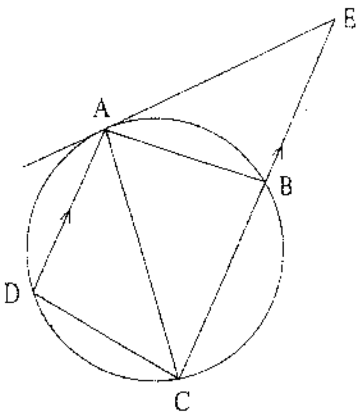
ii) Explain why the motion of the particle is simple harmonic and find its amplitude. 2

iii) Find the maximum speed. 1

c) Find the coefficient of x^2 in the expansion of 3

$$\left(\frac{x^4}{2} + \frac{2}{x^3}\right)^8$$

QUESTION 4 **MARKS**



NOT TO SCALE

a) In the diagram ABCD is a cyclic quadrilateral with AD parallel to BC. The tangent at A meet CB produced at E.

i) Show that ΔABE is similar to ΔADC. 3

ii) Hence, show that $AE \times DC = AC \times BE$. 1

b) The polynomial $P(x) = Ax^3 + Bx^2 + 2Ax - C$ has real roots $\sqrt{p}, \frac{1}{\sqrt{p}}, \alpha$

i) Explain why $\alpha = -\frac{C}{A}$ 1

ii) Show that $A^2 + C^2 = BC$ 2

c) Let $f(x) = e^{-x}$ and $g(x) = \log_e x$

i) Draw the graphs of $f(x)$ and $g(x)$ on the same set of axes for $x > 0$ 2

ii) Use your graph to show that the equation $e^{-x} - \log_e x = 0$ has only one root near $x = 1.4$ 1

iii) Use one application of Newton's method to find a better approximation of the root of the equation $e^{-x} - \log_e x = 0$ 2

QUESTION 5 — MARKS

a) Consider the function $f(x) = \dfrac{e^x}{e^x - 1}$

i) State the domain of $f(x)$. — 1

ii) Show that $f'(x) < 0$ for all x in the domain. — 2

iii) State the equations of the vertical and horizontal asymptotes. — 2

iv) Sketch the graph of $y = f(x)$. — 2

v) Explain why $f(x)$ has an inverse function. — 1

vi) Find the inverse function $y = f^{-1}(x)$. — 1

b) Newton's law of cooling states that the rate of change of the temperature T of a body at any time t is proportional to the difference in the temperature of the body and the temperature M of the surrounding medium.

i.e. $\dfrac{dT}{dt} = k(T - M)$, where k is a constant.

(i) Show that $T = M + Ae^{kt}$, where A is a constant, satisfies this equation. — 1

(ii) A freezer is maintained at a constant temperature of $-8\,^\circ C$. When water at $25\,^\circ C$ is placed in the freezer, the temperature of the water falls to $15\,^\circ C$ in 10 minutes. Find the temperature of the water after 10 more minutes, correct to the nearest degree. — 3

QUESTION 6 — MARKS

a) Use mathematical induction to prove that, for all integers n with $n \geq 1$ — 3

$$3.2! + 7.3! + 13.4! + \dots + (n^2 + n + 1)(n + 1)! = n(n + 2)!$$

b) The acceleration a of particle P moving along the x axis is given by

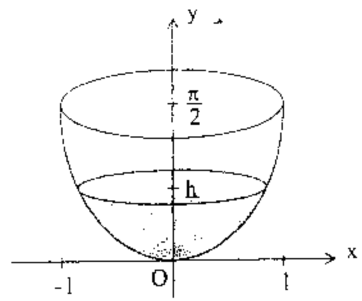
$a = -e^{-x}(1 + e^{-x})$ where x is the displacement of the particle from the origin in metres.

Initially, the particle is at the origin and its velocity is 2m/s.

i) Show that the velocity V m/s of the particle can be expressed by $V = 1 + e^{-x}$ — 3

ii) Find the time taken by the particle to reach a velocity of $1\frac{1}{2}$ m/s. — 2

c) A vessel is formed by rotating a part of the curve $y = \sin^{-1} x$, $0 \leq x \leq 1$ about the y axis. The vessel is being filled with water at constant rate of $2\text{cm}^3\text{s}^{-1}$.



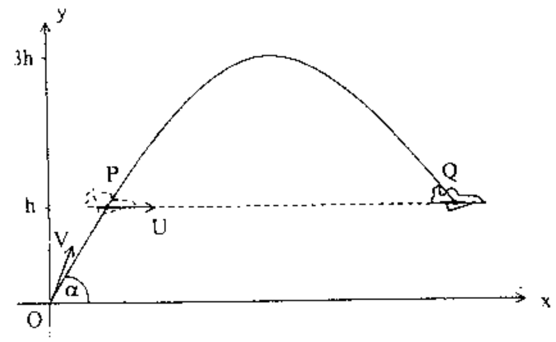
i) Show that the volume of the water in cubic centimeters when the depth is h cm can be expressed by — 2

$$V = \frac{\pi}{4}(2h - \sin 2h)$$

ii) Calculate the rate at which the water is rising when the depth is $\frac{\pi}{4}$ cm. — 2

QUESTION 7 MARKS

a)



An enemy fighter plane is flying horizontally at height h metres with a speed U ms^{-1}.

When it is at point P a ground rocket is fired towards it from the origin O with speed V ms^{-1} and angle of elevation α.

The rocket misses the plane, passing too late through point P. However, it goes on to reach a maximum height of 3h metres and then on its descent strikes the plane at Q.

With the axes shown in the diagram above, you may assume that the position of the rocket is given by

$x = Vt\cos\alpha$ and $y = -\frac{1}{2}gt^2 + Vt\sin\alpha$ where t is the time in seconds after firing and g is the acceleration due to gravity.

i) Show that initially the vertical component of the rocket's speed is $V\sin\alpha = \sqrt{6gh}$ 2

ii) If the rocket had not struck the fighter plane at Q, it would have returned to the x axis at a distance d from O. 2
Show that the horizontal component of the speed of the rocket is

$V\cos\alpha = \frac{gd}{2\sqrt{6gh}}$

iii) Show that the equation of the path of the rocket is 2

$y = \frac{12hx}{d}\left(1 - \frac{x}{d}\right)$

QUESTION 7 (continued) MARKS

iv) If the horizontal component of the rocket's speed is $100(3 + \sqrt{6})m/s$, find the time taken by the projectile to strike the plane at Q in terms of d. 2

v) Find U ms^{-1}, the speed of the fighter plane. 1

Either

b) i) Simplify

$(n-2)!\,{}^{2n+1}C_1 + (n-2)!\,{}^{2n+1}C_2 + \ldots + (n-2)!\,{}^{2n+1}C_n$ 2

ii) Find the smallest positive integer n such that :

$(n-2)!\,{}^{2n+1}C_1 + (n-2)!\,{}^{2n+1}C_2 + \ldots + (n-2)!\,{}^{2n+1}C_n > 1000\,000$ 1

OR

b)

A particle moves with simple harmonic motion and has a speed of 5 centimetres per second when passing through the centre O of its path. The period is π seconds. Find the speed of the particle when it is 1.5 centimetres from O. 3

STANDARD INTEGRALS

$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$

$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$

$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$

$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$

$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$

$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$

$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$

$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$

$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$

$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$

$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$

NOTE: $\ln x = \log_e x, \quad x > 0$

HSC TRIAL EXAMINATION PAPER 2001 SOLUTIONS + MAPPING GRID
MATHEMATICS – EXTENSION I

Question 1

(a) $\lim_{x \to 0} \frac{\sin 3x}{x} = \lim_{x \to 0} \frac{3 \sin 3x}{3x} = 3$

(1 mark)

(b) Since $x-2$ is a factor $\therefore P(2)=0$

$\therefore P(2) = 8p + 20 - 3p = 0 \quad \therefore p = -4$

(2 marks)

(c) $y = x \tan^{-1} x$

Using product rule:

Let $u = x \quad v = \tan^{-1} x$

$\therefore u' = 1 \quad v' = \frac{1}{1+x^2}$

$\therefore \frac{dy}{dx} = \tan^{-1} x + \frac{x}{1+x^2}$ (2 marks)

(d) $y = \ln(2x+1) \quad \therefore \frac{dy}{dx} = \frac{2}{2x+1}$

At $x = 0$, $\frac{dy}{dx} = 2$

At $x = \frac{1}{2}$, $\frac{dy}{dx} = 1$

$\therefore \tan \alpha = \left| \frac{2-1}{1+2 \times 1} \right| = \frac{1}{3}$

$\therefore \alpha = 18°26'$ (to nearest minute).

(2 marks)

(e) $\int_0^{1/6} \frac{9\,dx}{\sqrt{1-9x^2}} = \frac{1}{3} \int_0^{1/6} \frac{9\,dx}{\sqrt{\frac{1}{9} - x^2}}$

$= \int_0^{1/6} \frac{3\,dx}{\sqrt{\frac{1}{9} - x^2}} = 3\left[\sin^{-1} 3x\right]_0^{1/6}$

$= 3\left[\frac{\pi}{6} - 0\right] = \frac{\pi}{2}$

(3 marks)

(f) $\frac{1}{x} > \frac{1}{x+2}$

$\therefore \frac{1}{x} - \frac{1}{x+2} > 0 \quad \therefore \frac{2}{x(x+2)} > 0$

x		-2		0	
$x(x+2)$	$+$	‖	$-$	‖	$+$

$\therefore$ Solution is $x < -2$ or $x > 0$.

Question 2

(a) Arrangements $= \frac{8!2!}{3!2!4!} = 280$

(we divide by $3!, 2!$ & $4!$ because the red, blue & green are identical). (2 marks)

(b) $\sin 2\theta = 2\cos^2\theta, \quad 0 \leq \theta \leq 2\pi$

$2 \sin\theta \cos\theta = 2\cos^2\theta$

$\cos\theta(\sin\theta - \cos\theta) = 0$

$\therefore \cos\theta = 0, \quad \sin\theta = \cos\theta$

$\therefore \cos\theta = \cos\frac{\pi}{2}$

$\therefore \theta = \frac{\pi}{2} + 2k\pi \quad$ or $\quad \theta = -\frac{\pi}{2} + 2k\pi$

for $k = 0$, $\theta = \frac{\pi}{2}$ $\quad$ for $k = 1$, $\theta = \frac{3\pi}{2}$

$\sin\theta = \cos\theta$

$\therefore \tan\theta = \tan\frac{\pi}{4}$

$\therefore \theta = \frac{\pi}{4} + k\pi$

for $k = 0$, $\theta = \pi/4$

for $k = 1$, $\theta = 5\pi/4$

$\therefore$ Solutions are $\theta = \pi/4, \pi/2, 5\pi/4, \frac{3\pi}{2}$ in the domain $0 \leq \theta \leq 2\pi$ (3 marks).

(c) $\int_0^{\pi/2} \frac{\cos x\,dx}{\sqrt{1 + 3\sin x}}$

Let $u = 3\sin x$

$\therefore \cos x\,dx = du/3$

for $x = 0$, $u = 0$

$x = \frac{\pi}{2}$, $u = 3$

$= \frac{1}{3} \int_0^3 \frac{du}{\sqrt{1+u}}$

$= \frac{1}{3} \int_0^3 (1+u)^{-1/2}\,du = \frac{1}{3}\left(2\sqrt{1+u}\right]_0^3$

$= \frac{2}{3}\left[2 - 1\right] = \frac{2}{3}$

(3 marks)

(d)(i) $\frac{dy}{dx} = 2(x-1)$ (gradient function)

at $x = t+1$, $m_{tan} = 2t$

Using gradient point formula

$y - t^2 = 2t(x-t-1)$

$\therefore y - t^2 = 2tx - 2t^2 - 2t$

$\therefore y = 2tx - t^2 - 2t$ ① (2 marks)

(ii) Let $x=1$ in ① to find C

$y = 2t - t^2 - 2t = -t^2$ $\therefore C(1, -t^2)$

Let $y=0$ in ① to find B

$\therefore 2tx = t^2 + 2t$ $\therefore x = \frac{t+2}{2}$

$\therefore$ B is $\left(\frac{t+2}{2}, 0\right)$

$\therefore M_{AC} = \left(\frac{t+1+1}{2}, \frac{-t^2+t^2}{2}\right) = \left(\frac{t+2}{2}, 0\right)$

$\therefore$ B is mid-point of AC (2 marks)

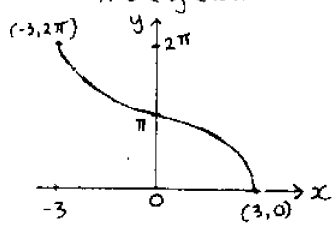
QUESTION 3

(a)(i) $y = 2\cos^{-1}\frac{x}{3}$

Domain: $-1 \le \frac{x}{3} \le 1$ $\therefore -3 \le x \le 3$ (1 mark)

(ii) Range: $0 \le \cos^{-1}\frac{x}{3} \le \pi$

$\therefore 0 \le y \le 2\pi$



(2 marks)

(iii) $y = 2\cos^{-1} x/3$

$\therefore dy/dx = \frac{2/3}{\sqrt{1-\frac{x^2}{9}}}$

At $x=0$, $m_{tan} = 2/3$ (1 mark)

(b)(i) $v^2 = -7 + 8x - x^2$

$\therefore \frac{1}{2}v^2 = -\frac{7}{2} + 4x - \frac{x^2}{2}$

$\therefore \frac{d(\frac{1}{2}v^2)}{dx} = 4 - x$

$\therefore$ Acceleration: $a = 4 - x$ (2 marks)

(ii) $a = -(x-4)$

$\therefore$ Acceleration is proportional to displacement but negative (i.e. directed towards the centre)

$\therefore$ Motion is simple harmonic, centred at $x=4$.

To find amplitude, let $v=0$.

$\therefore x^2 - 8x + 7 = 0$ $\therefore x=7$ or $x=1$

$\therefore$ Particle is oscillating between $x=1$ & $x=7$.

$\therefore$ Amplitude $= 3$ (2 marks)

(iii) Maximum speed occurs when $a=0$ (i.e. when $x=4$)

$\therefore v^2 = -7+32-16 = 9$ $\therefore v = 3$ m/s (1 mark)

(c) Considering term T_{r+1}

$\therefore T_{r+1} = {}^8C_r\left(\frac{x^4}{2}\right)^{8-r}\left(\frac{2}{x^2}\right)^r$

$= {}^8C_r \cdot \frac{x^{32-4r}}{2^{8-r}} \cdot \frac{2^r}{x^{2r}}$

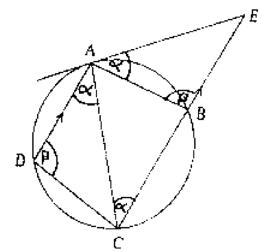
$= {}^8C_r \cdot 2^{2r-8} \cdot x^{32-6r}$

To get coefficient of x^2 we let $32 - 6r = 2$. $\therefore r = 5$

$\therefore$ Coefficient of $x^2 = {}^8C_5 \cdot 2^2 = 224$ (3 marks)

QUESTION 4

(a)



Data: ABCD is a cyclic quadrilateral AD ∥ BC

Aim: Prove that: (i) ΔABE ||| ΔADC

(ii) AE × DC = AC × BE

Construction: Figure

Proof: (i) Let ∠EAB = α

∴ ∠ACB = α (angle in alternate segment).

∴ ∠DAC = α (alternate angles, AD ∥ BC)

Let ∠ABE = β

∴ ∠CDA = β (exterior angle of cyclic quadrilateral equals opposite interior angle).

∠ACD = ∠AEB (remaining angles)

∴ ΔACD ||| ΔAEB (equiangular). (3 marks)

(ii) Since Δ's ADC & ABE are similar, their corresponding sides are in the same ratio.

Ratio of sides: $\frac{AE}{AC} = \frac{BE}{DC}$

∴ AE × DC = BE × AC (1 mark)

(b)(i) Product of roots: $\sqrt{p} \times \frac{1}{\sqrt{p}} \times \alpha = -\frac{C}{A}$

$\therefore \alpha = -\frac{C}{A}$ ① (1 mark)

(ii) Sum of roots: $\frac{1}{\sqrt{p}} + \sqrt{p} + \alpha = -\frac{B}{A}$

$\therefore \sqrt{p} + \frac{1}{\sqrt{p}} = \frac{C-B}{A}$ ②

Sum of roots 2 at a time:

$(\sqrt{p} \times \frac{1}{\sqrt{p}}) + \alpha\sqrt{p} + \frac{\alpha}{\sqrt{p}} = 2$

$\therefore 1 + \alpha(\sqrt{p} + \frac{1}{\sqrt{p}}) = 2$

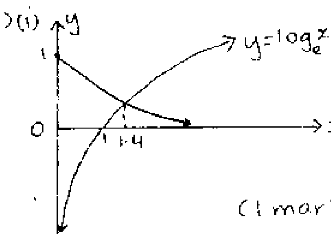
$\therefore \alpha(\sqrt{p} + \frac{1}{\sqrt{p}}) = 1$ ③

Sub ② & ① in ③

$\therefore -\frac{C}{A} \cdot \frac{C-B}{A} = 1$

$\therefore -C^2 + BC = A^2$ $\therefore A^2 + C^2 = BC$ (2 marks)

(c)(i)



(1 mark)

(ii) From the graph, we can see that the curves $y=e^{-x}$ & $y=\log_e x$ intersect near $x=1{\cdot}4$.

$\therefore$ The equation $e^{-x}=\log_e x$ (i.e. $e^{-x}-\log_e x=0$) has a root close to $x=1{\cdot}4$. (1 mark)

(iii) Let $h(x)=e^{-x}-\log_e x$

$\therefore h'(x)=-e^{-x}-\frac{1}{x}$

$\therefore h(1{\cdot}4)=e^{-1\cdot4}-\log_e 1{\cdot}4 \doteqdot -0{\cdot}089875272$

$h'(1{\cdot}4)=-e^{-1\cdot4}-\frac{1}{1\cdot4} \doteqdot -0{\cdot}960882678$

$\therefore x_2=1{\cdot}4-\frac{h(1\cdot4)}{h'(1\cdot4)} \doteqdot 1{\cdot}306465925$

$\therefore h(1{\cdot}306465925)=3{\cdot}4495834\times10^{-3}$

$\therefore x=1{\cdot}306465925$ is a better approximation of the root. (2 marks)

QUESTION 5

(a)(i) All real numbers except $x=0$ (1 mark)

(ii) $f'(x)=\frac{e^x(e^x-1)-e^{2x}}{(e^x-1)^2}$

$=\frac{-e^x}{(e^x-1)^2}$ (gradient function).

Since $e^x>0$ for all x & denominator is a perfect square greater than 0

$\therefore f'(x)<0$ for all x (2 marks)

(iii) When $x\to+\infty$, $y\simeq\frac{e^x}{e^x}\to1$

$\therefore y=1$ is a horizontal asymptote

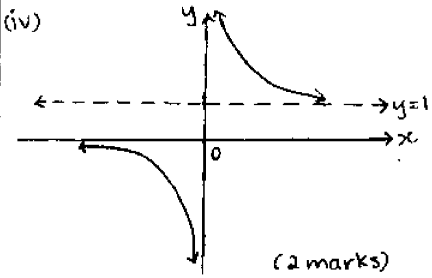
When $x\to-\infty$, $y\simeq\frac{0}{-1}\to0$ (n.b. $e^{-\infty}\to0$)

$\therefore y=0$ is a horizontal asymptote

When $x\to0$, $y\simeq\frac{1}{0}\to\pm\infty$

$\therefore x=0$ is a vertical asymptote (2 marks)

(iv)



(2 marks)

(v) Since $f(x)$ is a one-one function (i.e. for every x, there is only one y-value & vice versa).

$\therefore$ It has an inverse function (1 mark)

(vi) By interchanging x & y:

$x=\frac{e^y}{e^y-1} \longrightarrow \therefore e^y-x=e^y$

$\therefore e^y=\frac{x}{x-1}$

$\therefore \log_e e^y=\log_e\frac{x}{x-1}$

$\therefore y=\log_e\frac{x}{x-1}$ (1 mark)

(b)(i) P(ace) = 3/10   P(no ace) = 7/10

$\therefore$ P(one ace) $={}^6C_1(3/10)(7/10)^5$

$=0{\cdot}302526$ (1 mark)

(ii) P(at least 2 aces) = 1 − P(no ace) − P(ace)

$=1-{}^6C_0(3/10)^0(\frac{7}{10})^6-{}^6C_1(\frac{3}{10})(\frac{7}{10})^5$

$=1-0{\cdot}117649-0{\cdot}302526$

$=0{\cdot}579825$ (1 mark)

(iii) He has to serve ace, no ace, no ace, no ace, no ace, ace in this order.

$P=(\frac{3}{10})^2(\frac{7}{10})^4=0{\cdot}021609$ (1 mark)

QUESTION 6

(a) Step 1: For $n=1$, $3.2!=1(1+2)!$

$\therefore 6=6$ Hence statement is true for $n=1$.

Step 2: Assume that the statement is true for $n=k$

$3.2!+7.3!+\ldots+(k^2+k+1)(k+1)!$

$=k(k+2)!$ ①

Our aim is to prove it true for $n=k+1$

i.e. $3.2!+7.3!+\ldots+[(k+1)^2+k+2](k+2)!$

$=(k+1)(k+3)!$

Starting from ① and adding $[(k+1)^2+k+2](k+2)!$ to both sides:

$3.2!+7.3!+\ldots+[(k+1)^2+k+2](k+2)!$

$=k(k+2)!+[(k+1)^2+k+2](k+2)!$

$\therefore$ LHS $=[k^2+4k+3](k+2)!$ (factorizing)

$=(k+1)(k+3)(k+2)!$

$=(k+1)(k+3)!$ (Since $(k+3)!=(k+2)!\times(k+3)$).

Hence if the statement is true for $n=k$, it is also true for $n=k+1$.

Step 3: If the statement is true for $n=1$ & so it is true for $n=2$ & so on. Hence it is true for all $n\geq1$. (3 marks)

(b)(i) $\frac{d(\frac{1}{2}v^2)}{dx}=-e^{-x}-e^{-2x}$

$\therefore \frac{1}{2}v^2=\int(-e^{-x}-e^{-2x})dx$

$\therefore \frac{1}{2}v^2=e^{-x}+\frac{1}{2}e^{-2x}+c$

when $x=0$, $v=2$

$\therefore 2=1+\frac{1}{2}+c$ $\therefore c=\frac{1}{2}$

$\therefore \frac{1}{2}v^2=\frac{1}{2}e^{-2x}+e^{-x}+\frac{1}{2}$

$\therefore v^2=e^{-2x}+2e^{-x}+1$

$\therefore v^2=(e^{-x}+1)^2$ $\therefore v=\pm(e^{-x}+1)$

Since when $x=0$, $v=2$ (positive)

$\therefore$ Positive solution only is accepted.

$\therefore v=e^{-x}+1$ (3 marks)

(ii) $\frac{dx}{dt}=e^{-x}+1=\frac{1+e^x}{e^x}$

$\therefore \int dt=\int\frac{e^x dx}{1+e^x}$ $\therefore t=\ln(e^x+1)+d$

When $t=0$, $x=0$

$\therefore 0=\ln2+d$ $\therefore d=-\ln2$

$\therefore t=\ln(e^x+1)-\ln2=\ln(\frac{e^x+1}{2})$ ①

$v=1\frac{1}{2}$, $e^{-x}+1=1\frac{1}{2}$ $\therefore \frac{1}{e^x}=\frac{1}{2}$

$\therefore e^x = 2$ Sub in ①

$\therefore t = \ln\frac{3}{2}$ seconds

$\therefore$ It will take the particle $\ln\frac{3}{2}$ seconds to drop its velocity to 1·5 m/s. (2 marks)

(c)(i) $V = \pi\int x^2 dy$ $\quad y = \sin^{-1}x$

$\therefore \sin y = x \quad \therefore x^2 = \sin^2 y$

as $\cos 2y = 1 - 2\sin^2 y \quad \therefore \sin^2 y = \frac{1}{2}(1-\cos 2y)$

$\therefore x^2 = \frac{1}{2}(1-\cos 2y)$

$\therefore V = \frac{\pi}{2}\int_0^h (1-\cos 2y)\,dy$

$= \frac{\pi}{2}\left[y - \frac{1}{2}\sin 2y\right]_0^h$

$= \frac{\pi}{2}\left[(h - \frac{1}{2}\sin 2h) - 0\right]$

$= \frac{\pi}{4}\left[2h - \sin 2h\right]$ (2 marks)

(ii) $\frac{dV}{dt} = \frac{dV}{dh}\cdot\frac{dh}{dt}$

$V = \frac{\pi}{4}(2h - \sin 2h) \quad \therefore \frac{dV}{dh} = \frac{\pi}{4}(2 - 2\cos 2h)$

$= \frac{\pi}{2}(1-\cos 2h)$

$\therefore 2 = \frac{\pi}{2}(1-\cos 2h)\cdot\frac{dh}{dt}$

$\therefore \frac{dh}{dt} = \frac{4}{\pi(1-\cos 2h)}$ (rate at any depth)

when $h = \frac{\pi}{4}$, $\frac{dh}{dt} = \frac{4}{\pi(1-0)} = \frac{4}{\pi}$ cm/s (2 marks)

QUESTION 7

(a)(i) $\dot{y} = -gt + v\sin\alpha$

At maximum height, $\dot{y} = 0$ (vertical component) $\therefore gt = v\sin\alpha \quad \therefore t = \frac{v\sin\alpha}{g}$

Substitute in y, we get:

$\therefore y_{max} = -\frac{1}{2}g \times \left(\frac{v\sin\alpha}{g}\right)^2 + v\sin\alpha \times \frac{v\sin\alpha}{g}$

$\therefore 3h = -\frac{v^2\sin^2\alpha}{2g} + \frac{v^2\sin^2\alpha}{g}$

$\therefore 3h = \frac{v^2\sin^2\alpha}{2g} \quad \therefore v^2\sin^2\alpha = 6gh$

$\therefore v\sin\alpha = \sqrt{6gh}$ (since initial vertical component is positive). (2 marks)

(ii) Let $y = 0$ $\quad \therefore -\frac{1}{2}gt^2 + v\sin\alpha\, t = 0$

$\therefore t(-\frac{gt}{2} + v\sin\alpha) = 0 \quad \therefore t = 0$ (initial time) or $t = \frac{2v\sin\alpha}{g}$ (time to return to x-axis if it didn't strike plane at Q).

$\therefore d = v\cos\alpha \times \frac{2v\sin\alpha}{g} = v\cos\alpha \times \frac{2\sqrt{6gh}}{g}$

$\therefore v\cos\alpha = \frac{gd}{2\sqrt{6gh}}$ (2 marks)

(iii) $x = v\cos\alpha\, t \quad \therefore t = \frac{x}{v\cos\alpha} = \frac{2x\sqrt{6gh}}{gd}$

$\therefore y = -\frac{1}{2}g \times \left(\frac{2x\sqrt{6gh}}{gd}\right)^2 + \sqrt{6gh}\cdot\frac{2x\sqrt{6gh}}{gd}$

$\therefore y = -\frac{1}{2}g \times 4x^2 \times \frac{6gh}{g^2d^2} + \frac{12xgh}{gd}$

$\therefore y = -\frac{12x^2h}{d^2} + \frac{12xh}{d} = \frac{12xh}{d}(1 - \frac{x}{d})$

(2 marks)

(iv) The rocket will strike the plane at Q when $y = h$.

$\therefore h = \frac{12xh}{d}(1 - \frac{x}{d})$

$\therefore 1 = \frac{12x}{d}(1 - \frac{x}{d})$

$\therefore d = 12x - \frac{12x^2}{d}$

$\therefore d^2 - 12xd + 12x^2 = 0$

$\therefore 12x^2 - 12dx + d^2 = 0$

$\therefore x = \frac{12d \pm\sqrt{144d^2 - 48d^2}}{24}$

$\therefore x = \frac{3d \pm d\sqrt{6}}{6} \quad \therefore x_Q = \frac{3d + d\sqrt{6}}{6}$

Time taken by rocket to reach Q is:

$x = v\cos\alpha t$

$\therefore \frac{3d + d\sqrt{6}}{6} = 100(3+\sqrt{6})t$

$\therefore \frac{d(3+\sqrt{6})}{6} = 100(3+\sqrt{6})t$

$\therefore t = d/600$ (2 marks)

(v) The distance travelled by the plane from P to Q is:

$\frac{3d + d\sqrt{6}}{6} - \frac{3d - d\sqrt{6}}{6} = \frac{d\sqrt{6}}{3}$

Time taken for plane to travel from P to Q is the same time taken by rocket to reach Q. $\therefore t = d/600$

$\therefore u = \frac{d\sqrt{6}}{3} \times \frac{600}{d} = 200\sqrt{6}$ m/s (1 mark)

(b)(i) $(1+x)^{2n+1} = {}^{2n+1}C_0 + {}^{2n+1}C_1 x^1 + \ldots + {}^{2n+1}C_n x^n + {}^{2n+1}C_{n+1}x^{n+1} + \ldots + {}^{2n+1}C_{2n}x^{2n} + {}^{2n+1}C_{2n+1}x^{2n+1}$

For $x = 1$:

$2^{2n+1} = {}^{2n+1}C_0 + {}^{2n+1}C_1 + \ldots + {}^{2n+1}C_n + {}^{2n+1}C_{n+1} + \ldots + {}^{2n+1}C_{2n} + {}^{2n+1}C_{2n+1}$

Using ${}^nC_r = {}^nC_{n-r}$

$\therefore {}^{2n+1}C_n = {}^{2n+1}C_{n+1}$

Also, ${}^{2n+1}C_0 = {}^{2n+1}C_{2n+1}$

${}^{2n+1}C_1 = {}^{2n+1}C_{2n}$

$\therefore 2^{2n+1} = 2({}^{2n+1}C_0 + {}^{2n+1}C_1 + \ldots + {}^{2n+1}C_n)$

$\therefore 2^{2n} - 1 = {}^{2n+1}C_1 + {}^{2n+1}C_2 + \ldots + {}^{2n+1}C_n$

$\therefore (n-2)!(2^{2n}-1) = (n-2)!\,{}^{2n+1}C_1 + \ldots + (n-2)!\,{}^{2n+1}C_n$ (2 marks)

(ii) $(n-2)!\,{}^{2n+1}C_1 + \ldots + (n-2)!\,{}^{2n+1}C_n > 1000000$

$\therefore (n-2)!(2^{2n}-1) > 1000000$

By calculator, for $n=6$: 98280

for $n=7$: 1965960

$\therefore n=7$ is the smallest positive integer. (1 mark)