QUESTION 1

(a) Find the perpendicular distance from the point (1,4) to the line $4y = 3x - 2$.

(b) Fully factorise $2x^3 - 128$.

(c) Differentiate with respect to x :

(i) $y = \sin 2x$ (ii) $y = \log_e \sqrt{\dfrac{2x-1}{3x+2}}$

(d) Find the remainder when the polynomial $P(x) = x^4 - 2x^3 - 3$ is divided by ($x - 2$).

(e) Evaluate: $\displaystyle\int_{\sqrt{2}}^{\sqrt{3}} \frac{dx}{\sqrt{4-x^2}}$

QUESTION 2 (Start a new page)

(a) A fair six faced die with faces numbered 1,2,3,4,5,6 is tossed seven times. What is the probability that a "6" occurs on exactly two of the seven tosses.

(b) A circular plate of radius R cm. is heated so that its area expands at a constant rate of 5 cm^2 per minute. At what rate is its radius increasing when R = 10 cm.

(c) Use the substitution $u = \tan x$ to evaluate:

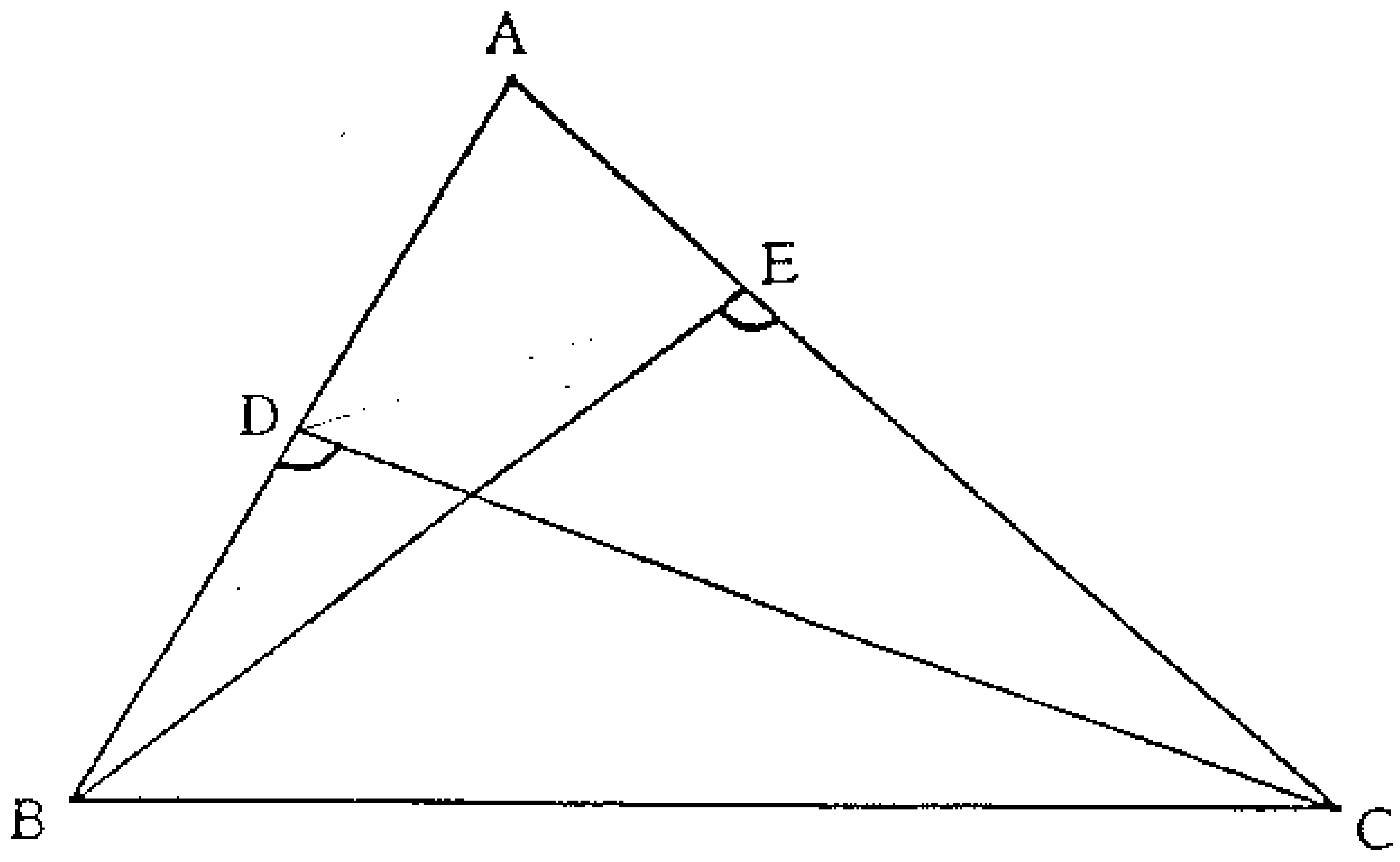
$$\int_0^{\frac{\pi}{3}} (\sec x \ \tan x)^2 \, dx$$

(d) Prove that $\dfrac{\cos 2\theta}{\cos\theta - \sin\theta} = \cos\theta + \sin\theta$.

(e) Find all solutions to: $\dfrac{1}{x(2-x)} < 0$.

QUESTION 3 (Start a new page)

(a) In triangle ABC given, D lies on AB, E lies on AC and $\angle BDC = \angle BEC$.
Copy the diagram onto your answer sheet and prove that $\angle ADE = \angle BCE$ giving all reasons.



(b) An object moves x metres along a straight line after t seconds in simple harmonic motion, with equation of motion $x(t) = 2 + 3\sin t + 4\cos t$. Find:

(i) its amplitude.

(ii) the time it takes to travel 100 metres.

(c) Prove by mathematical induction that:

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots\cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$$

where n is a positive integer.

QUESTION 4 (Start a new page)

(a) (i) Neatly sketch the curve $y = 2\sin^{-1}\left(\frac{x}{3}\right)$.

of a b=-1

(ii) Find the volume generated when the area between the curve $y = 2\sin^{-1}\left(\frac{x}{3}\right)$ and the y–axis is rotated one revolution about the positive y–axis.

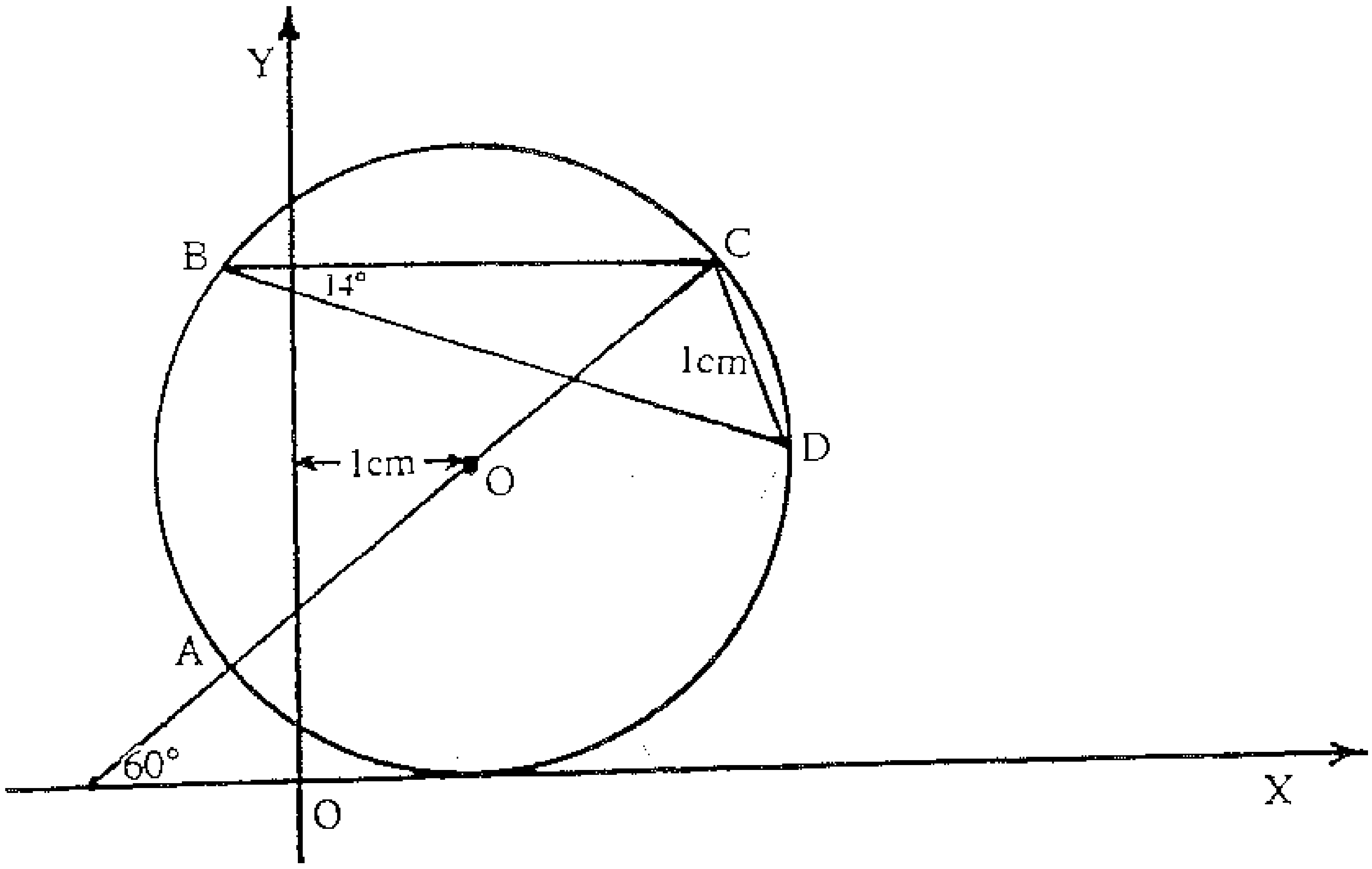
(b) The gradient function of a curve is given by $\frac{dy}{dx} = 1 + y$, and the curve passes through the point (1, 2). Find the equation of the curve and state its RANGE.

(c) Find all values of θ for which $\cos^2\theta = \sin\theta\cos\theta$.

QUESTION 5 (**Start a new page**)

(a) A function is defined by $x = \sin y$ for $\frac{\pi}{2} \le y \le \pi$. Find $\frac{dy}{dx}$ in terms of x.

(b) A circle is tangential to the x–axis and has its centre 1cm. right of the y–axis. The diameter AC is inclined at 60° to the x–axis. The line segment CD is 1cm. long and $\angle$ CBD = 14°. (diagram not to scale)



COPY THE DIAGRAM ONTO YOUR ANSWER SHEET

(i) Show that the diameter of the circle is 4 cm. (to the nearest centimetre).

(ii) Find the equation of the circle.

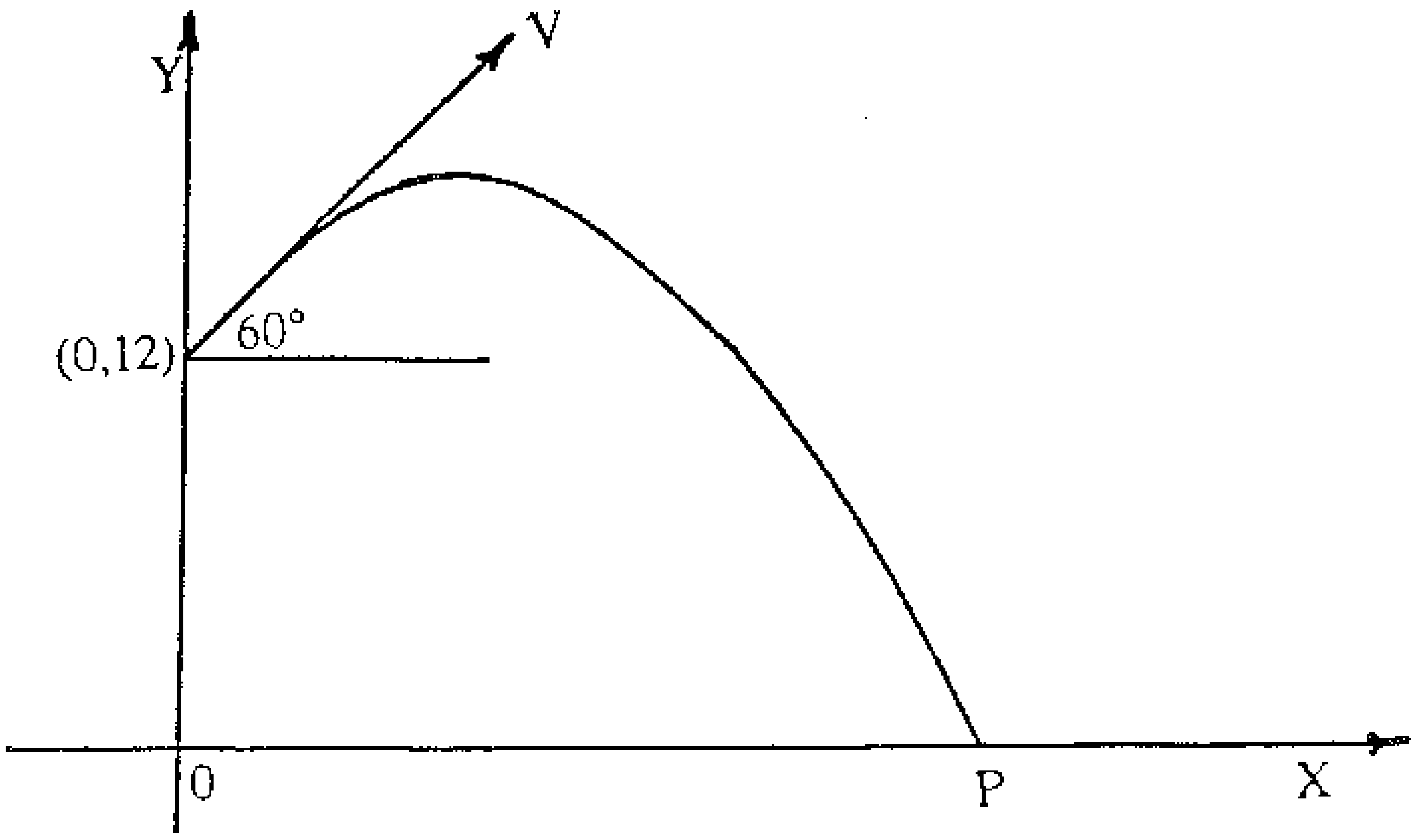
(iii) The point E lies on the circumference of the circle between A and D. Find $\angle$ AED giving all reasons.

(iv) The circle is now rolled along the x–axis such that the centre is displaced by 3cm. in the positive direction. With the circle in this position, how far is the point C above the x–axis.

QUESTION 6 (Start a new page)

(a) A particle is projected from a point (0,12) at an angle of 60° to the horizontal with a velocity of 50 metres/second. The equations of motion are:

$$\ddot{x} = 0 \qquad \text{and} \qquad \ddot{y} = -g \quad (\text{ take } g = 10 \text{ m/s}^2)$$



Using integration, find the velocity of the particle at the point P where it strikes the x–axis.

(b) Let each different arrangement of all the letters of PROPRIETY be called a word.

(i) How many words are possible.

(ii) In how many of these words will the letters ETY be together.

(iii) A PR is now deleted. The letters of the remaining word PROIETY are now shuffled and four letters drawn out at random. What is the probability that they form the word POET.

(c) The acceleration of a particle moving x metres along a straight line after t seconds is given by:

$$\frac{d^2x}{dt^2} = 10x - 4x^3$$

When $t = 0$, $v = 0$ and $x = \sqrt{5}$ metres.

(i) Find an expression for v^2 as a function of x.

(ii) Briefly describe the motion of the particle.

QUESTION 7 (Start a new page)

(a) Ross and his wife Evelyn both work for the C.S.I.R.O. and each earns a salary of \$48,000 annually (ie; 52 weeks). They each contribute $6\frac{1}{2}\%$ of their annual salary to a superannuation fund which earns interest at a rate of 5% per annum compounded fortnighly.

Ross contributes to the fund for ten years, but Evelyn decides to withdraw from the fund at the end of eight years and invest her lump sum payment for the next two years at an interest rate of 12% per annum compounded monthly.

They agree that the difference in their lump sum payments at the end of ten years will pay for a holiday for their daughter Sherry.

(i) Calculate the lump sum Ross will receive at the end of ten years.

(ii) Calculate Evelyn's lump sum payment at the end of ten years, and who will pay for Sherry's holiday.

(b) (i) Neatly sketch the curve $f(x) = \dfrac{1 - x^2}{1 + x^2}$, clearly showing all x,y–intercepts and asymptotes.

(ii) Differentiate $y = \cos^{-1}\left(\dfrac{1 - x^2}{1 + x^2}\right)$ with respect to x.

(iii) Using (i) and (ii) or otherwise, neatly graph $y = \cos^{-1}\left(\dfrac{1 - x^2}{1 + x^2}\right)$.

END of PAPER

TD

SOLUTIONS TO 3/4 UNIT TRIAL H.S.C. 2000

Question 1

(a) $d = \left|\dfrac{3.1+-4\times4+-2}{\sqrt{3^2+4^2}}\right|$ 1

$= \left|-\dfrac{15}{5}\right|$ (2)

$\therefore \underline{d = 3}$ 1

(b) $2x^3 - 128$

$= 2(x^3 - 64)$ 1

$= \underline{2(x-4)(x^2+4x+16)}$ 1 (2)

(c) (i) $\underline{\dfrac{dy}{dx} = 2\cos 2x}$ (1)

(ii) $y = \dfrac{1}{2}\left[\log_e(2x-1) - \log_e(3x+2)\right]$

$\therefore \dfrac{dy}{dx} = \dfrac{1}{2}\left[\dfrac{2}{2x-1} - \dfrac{3}{3x+2}\right]$ 1

$= \underline{\dfrac{7}{2(2x-1)(3x+2)}}$ 1 (2)

(d) If $(x-2)$ is a factor of $P(x)$ then $P(2) = 0$. 1

Now $P(2) = 2^4 - 2(2^3) - 3$

$= -3$

$\therefore$ \underline{remainder is -3} 1 (2)

(e) $\displaystyle\int_{\sqrt2}^{\sqrt3} \frac{dx}{\sqrt{4-x^2}} = \left[\sin^{-1}\left(\frac{x}{2}\right)\right]_{\sqrt2}^{\sqrt3}$ 1

$= \sin^{-1}\dfrac{\sqrt3}{2} - \sin^{-1}\sqrt2$ 1 (3)

$= \dfrac{\pi}{3} - \dfrac{\pi}{4} = \underline{\dfrac{\pi}{12}}$ 1

/12

Question 2

(a) $P = \binom{7}{5}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^5$ 1

$= \dfrac{21857}{93312}$ 1 (2)

$\therefore \underline{P \doteqdot 0.2344}$

(b) $A = \pi R^2$

$\dfrac{dA}{dt} = 5\ \text{cm}^2/\text{min}$.

$\dfrac{dR}{dt} = \ ?$

$\dfrac{dA}{dR} = 2\pi R$

Now $\dfrac{dR}{dt} = \dfrac{dR}{dA}\cdot\dfrac{dA}{dt}$ 1

$= \dfrac{1}{2\pi R}\cdot 5$ 1 (3)

when $R = 10$

$\dfrac{dA}{dt} = \dfrac{5}{2\pi\times10} = \dfrac{1}{4\pi}$ cm/s. 1

$\underline{\doteqdot 0.08\ \text{cm/s}}$

(c) $\displaystyle\int_0^{\frac{\pi}{3}} (\sec x \tan x)^2\,dx$ $\quad u = \tan x$ 1

$\therefore du = \sec^2 x\,dx$

$= \displaystyle\int_0^{\frac{\pi}{3}} \sec^2 x \tan^2 x\,dx$

$= \displaystyle\int_0^{\sqrt3} u^2\,du$ 1

$= \left.\dfrac{u^3}{3}\right]_0^{\sqrt3}$ 1 (3)

$= \underline{\sqrt3}$

(d) Prove $\dfrac{\cos 2\theta}{\cos\theta - \sin\theta} = \cos\theta + \sin\theta$

$LHS = \dfrac{\cos 2\theta}{\cos\theta - \sin\theta}$

$= \dfrac{\cos^2\theta - \sin^2\theta}{\cos\theta - \sin\theta}$ 1

$= \dfrac{(\cos\theta - \sin\theta)(\cos\theta + \sin\theta)}{(\cos\theta - \sin\theta)}$ (2)

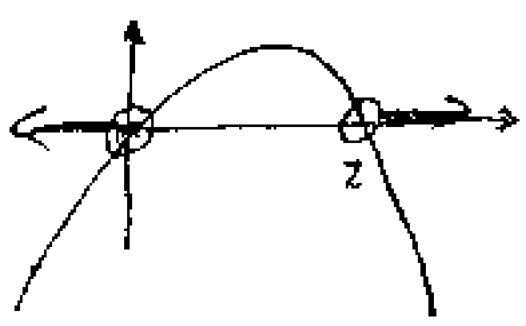
$= \cos\theta + \sin\theta$ 1

$= RHS$

(e) Since $\dfrac{1}{x(2-x)} < 0$

$\therefore \dfrac{x^2(2-x)^2}{x(2-x)} < 0$ 1 (2)

$\therefore x(2-x) < 0$ 1



$\therefore \underline{x < 0,\ x > 2}$

/12

Question 3

(a) $B\hat{D}C = B\hat{E}C$ (given)

Since equal angles at D, E are subtended on same side of interval BC, $\therefore$ the points B, D, E, C are concyclic. (4)

$\therefore A\hat{D}E = B\hat{C}E$ (Exterior angle of cyclic quad BDEC equal to interior remote angle)

(b)(i) $x(t) = 2 + 3\sin t + 4\cos t$

Let $3\sin t + 4\cos t \equiv A\sin(t+\alpha)$

$A = 5, \quad \alpha = \tan^{-1}\frac{4}{3}$

$\therefore x(t) = 2 + 5\sin(t+\alpha)$ (2)

$\therefore$ Amplitude is 5.

(ii) Time taken for 1 oscillation is 2π secs. Since amplitude is 5, $\therefore$ it moves 20 metres in 2π secs.

$\therefore$ it takes 10π secs to move 100 metres. (2)

(c) Prove $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$

Test for $n=1$:

$LHS = RHS = \frac{1}{2}$ $\therefore$ proven true for $n=1$.

Assume true for $n=K$:

$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{K}{(K+1)!} = 1 - \frac{1}{(K+1)!}$

Prove true for $n = K+1$:

RTP that $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{K}{(K+1)!} + \frac{(K+1)}{(K+2)!} = 1 - \frac{1}{(K+2)!}$

Now $LHS = 1 - \frac{1}{(K+1)!} + \frac{(K+1)}{(K+2)!}$ by assumption

$= 1 - \frac{1}{(K+1)!}\left[1 - \frac{(K+1)}{(K+2)}\right]$ (4)

$= 1 - \frac{1}{(K+1)!}\left[\frac{1}{(K+2)}\right]$

$= 1 - \frac{1}{(K+2)!}$

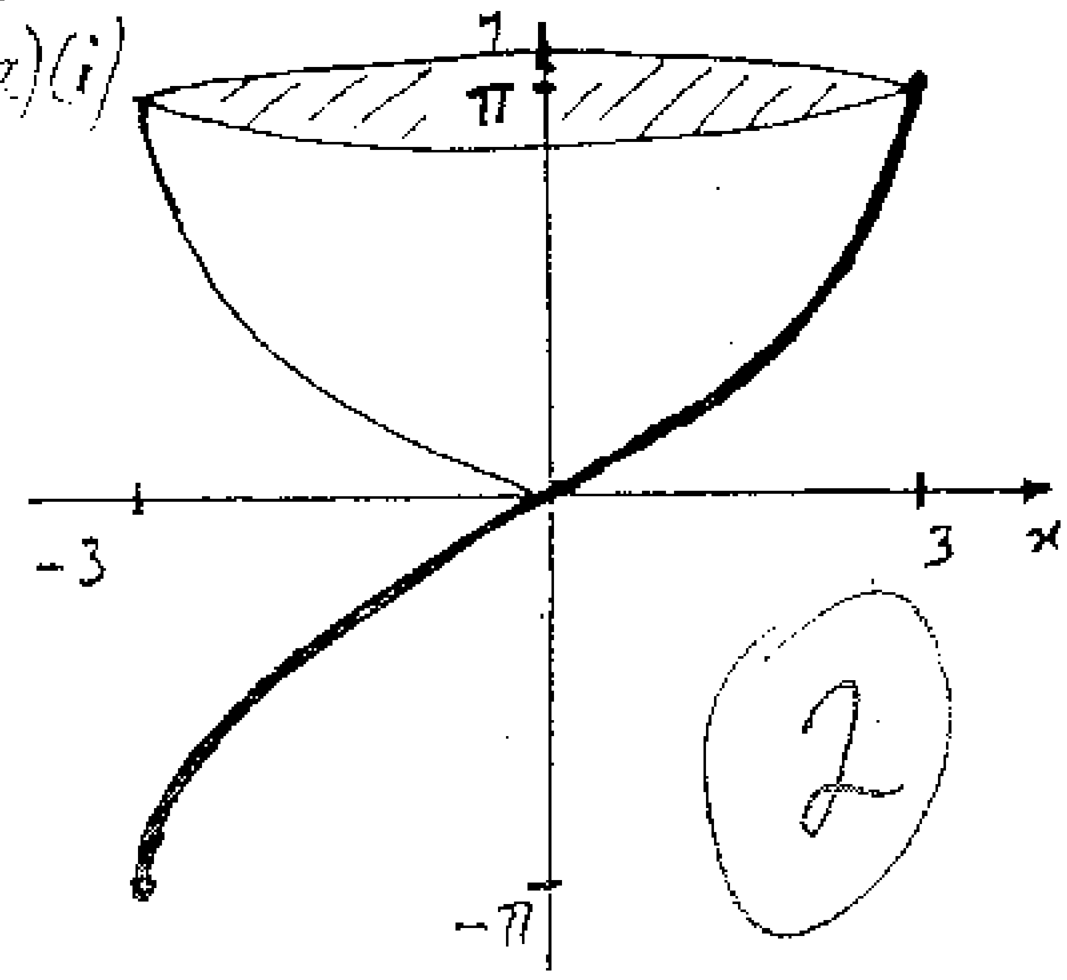
$= RHS$

Hence proven true for $n = 1, 2, 3, \ldots$ by mathematical induction.

12

Question 4

(a)(i)



(2)

(ii) Now $V=\pi\int_0^{\pi} x^2\,dy$ $\qquad \frac{y}{2}=\sin^{-1}\left(\frac{x}{3}\right)$

$\therefore \frac{x}{3}=\sin\frac{y}{2}$ 1

$\therefore x^2 = 9\sin^2\left(\frac{y}{2}\right)$

$\therefore V = 9\pi\int_0^{\pi}\sin^2\left(\frac{y}{2}\right)dy$ 1

$= 18\pi\int_0^{\pi}\sin^2\left(\frac{y}{2}\right)d\left(\frac{y}{2}\right)$

$= 9\pi\int_0^{\pi}\left[1-\cos 2\left(\frac{y}{2}\right)\right]d\left(\frac{y}{2}\right)$ 1

$= 9\pi\left[\frac{y}{2} - 2\sin y\right]_0^{\pi}$

$\therefore V = \frac{9\pi^2}{2}$ units^3 1

(4)

(b) $\frac{dy}{dx} = 1+y$

$\therefore \frac{dy}{1+y} = dx$

$\therefore \ln(1+y) = x + C$

when $x=1$, $y=2$ $\therefore C=\ln 3 - 1$

$\ln(1+y) = x + \ln 3 - 1$

$\therefore \ln\frac{(1+y)}{3} = x-1$

$\therefore \frac{1+y}{3} = e^{x-1}$ 1

$\therefore y = 3e^{x-1} - 1$ and RANGE is $\{y: y > -1\}$

[Note: Since $\frac{dy}{dx} = y+1$, then $y = -1 + Ae^{x}$ by definition of exponential growth. At $(1,2)$ we find $A = 3e^{-1}$ and $y = 3e^{x-1} - 1$]

(3)

(c) Solve $\cos^2\theta = \sin\theta\cos\theta$

Now $\cos^2\theta - \sin\theta\cos\theta = 0$

$\therefore \cos\theta(\cos\theta - \sin\theta) = 0$ 1

$\therefore \cos\theta = 0$ or $\tan\theta = 1$

$\therefore \theta = 2n\pi \pm \cos^{-1}0$ or $\theta = n\pi + \tan^{-1}1$

$\therefore \theta = 2n\pi \pm \frac{\pi}{2}$ 1 or $n\pi + \frac{\pi}{4}$ 1 (3)

12

Question 5

(a) $x = \sin y$

$\therefore \frac{dx}{dy} = \cos y$

$\therefore \frac{dy}{dx} = \frac{1}{\cos y}$ 1

Now $x^2 = \sin^2 y = 1 - \cos^2 y$

$\therefore \cos^2 y = 1 - x^2$ 1

$\therefore \cos y = \pm\sqrt{1-x^2}$

But $\frac{\pi}{2} \le y \le \pi$ for $0 \le x \le 1$

$\therefore \frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$ 1

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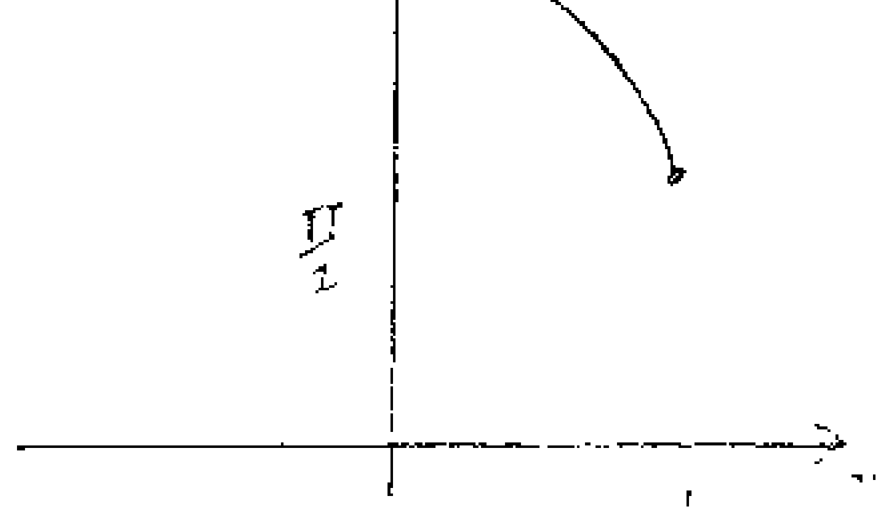
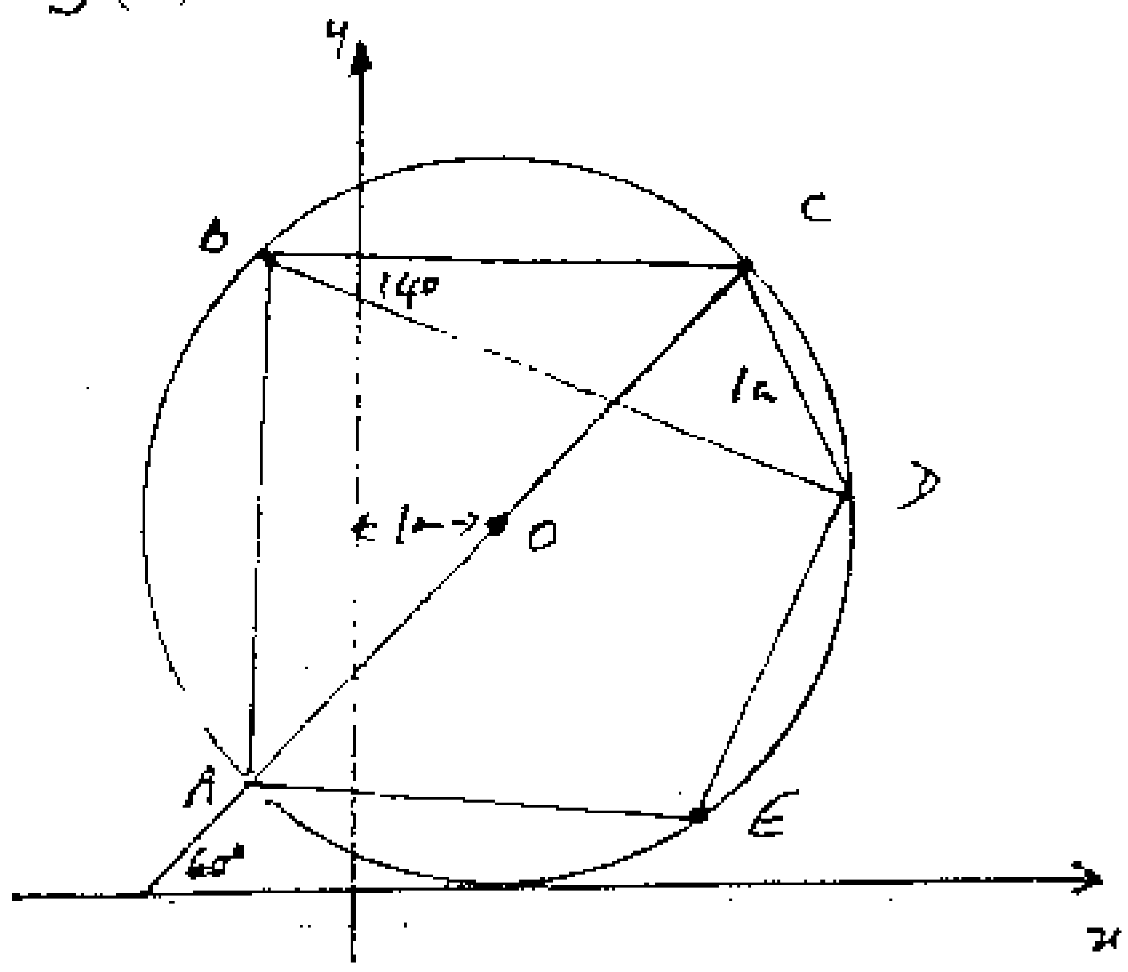


Graph of $x = \sin y$

(4)

5(b)



(i) Since $\sin 14° = \frac{1}{2R}$ 1

$\therefore R = \frac{1}{2\sin 14°} \doteqdot 2$ cm.

$\therefore$ Diameter is approx. $\underline{4 \text{ cm}}$ 1 (2)

(ii) Equation of the circle is:

$(x-1)^2 + (y-2)^2 = 4$

ie; $\underline{x^2+y^2-2x-4y+1=0}$ (1)

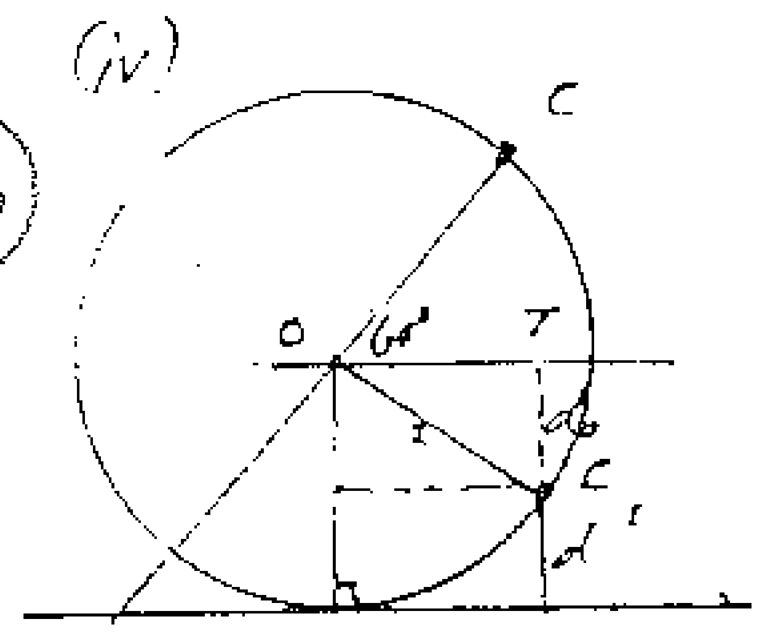
(iii) Join AB, AE, DE

$A\hat{B}C = 90°$ (Angle in semi-circle) 1

$\therefore A\hat{B}D = 76°$ (by subtraction) 1 (3)

$\therefore A\hat{E}D = 104°$ (Opposite angles of cyclic quad. ABDE supplementary) 1

(iv)



Since $l = R\theta$

$\therefore 3 = 2\theta \quad \therefore \theta = \frac{3}{2}^c$

$\therefore \theta \doteqdot 85°57'$

$\therefore T\hat{O}C_1 \doteqdot 25°57'$ 1

Now $\sin 25°57' = \frac{d_0}{2}$

$\therefore d_0 \doteqdot 0.875$ cm

$\therefore$ distance required is $(2 - 0.875)$ cm

ie; $d \doteqdot \underline{1.12 \text{ cm}}$. 1 (2)

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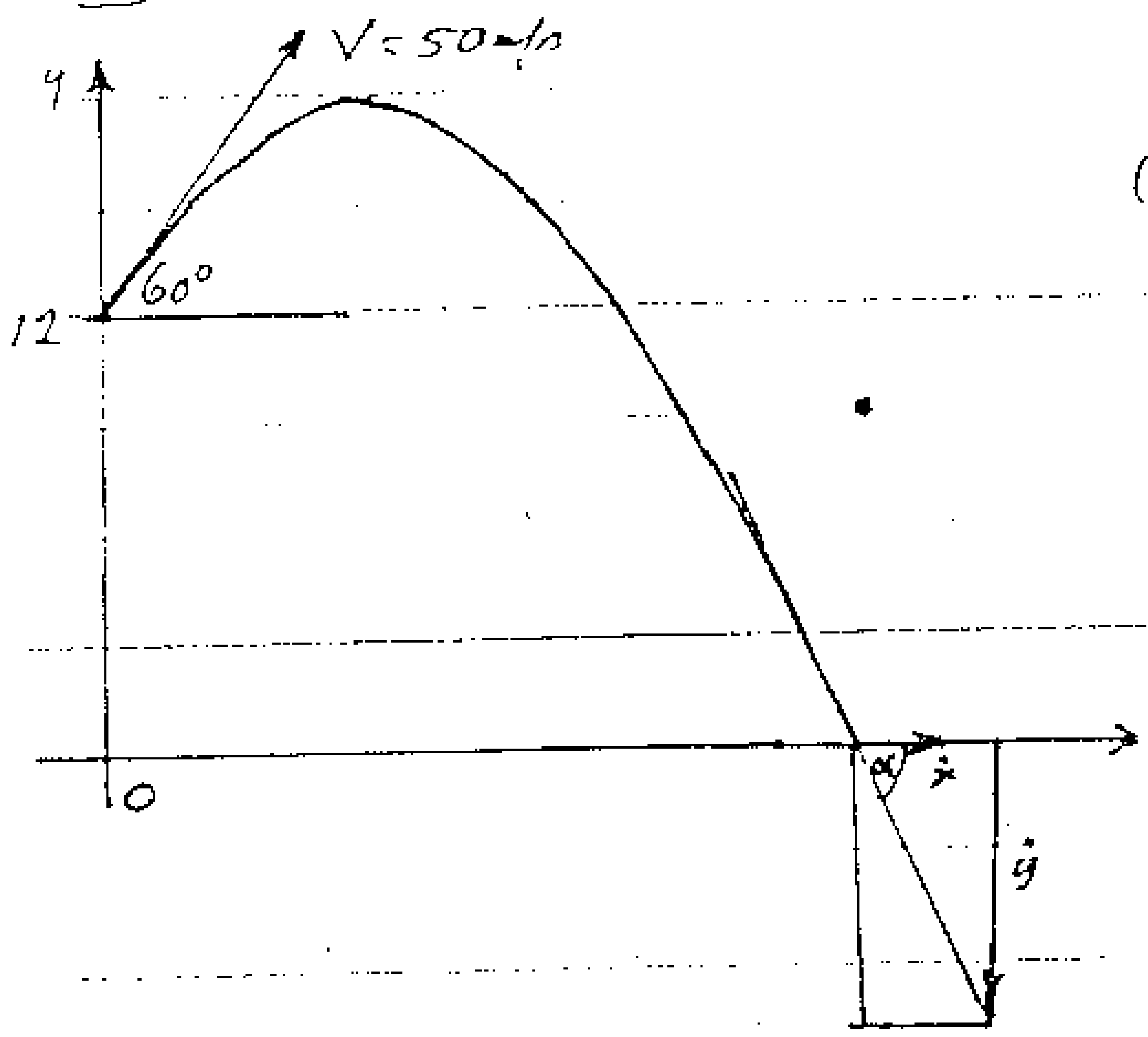
Question 6



(a)

$\ddot{x} = 0 \qquad \ddot{y} = -g$

$\dot{x} = C_1 = V\cos 60° \qquad \dot{y} = -gt + V\sin 60°$

$\therefore \dot{x} = 50\cos 60 \qquad = -10t + 50 \cdot \frac{\sqrt{3}}{2}$

$\dot{x} = 25 \qquad \therefore \dot{y} = -10t + 25\sqrt{3}$

$x = 25t + C_3 \qquad y = -5t^2 + 25\sqrt{3}\,t + C_4$

When $t = 0$, $C_3 = 0$, $C_4 = 12$

$\therefore x = 25t$ 1 $\qquad y = -5t^2 + 25\sqrt{3}t + 12$ 1

When $y = 0$ $5t^2 - 25\sqrt{3}t - 12 = 0$

$t = \frac{25\sqrt{3} \pm \sqrt{(25\sqrt{3})^2 + 240}}{10} \quad (t > 0)$

$\therefore t \doteqdot 8.93$

$\therefore \dot{x} = 25, \quad \dot{y} \doteqdot -46$

Now $v = \sqrt{v_x^2 + v_y^2} = \sqrt{\dot{x}^2 + \dot{y}^2}$

$= \sqrt{25^2 + (46)^2}$

$\therefore v \doteqdot 52.35$ m/s (5)

$\tan\alpha = \frac{\dot{y}}{\dot{x}}$

$= -\frac{46}{25}$

$\therefore \underline{\alpha = 61°29'}$ 1

$\therefore$ Velocity of particle is 52.35 m/s at an angle of 61°29' to horizontal or 118°31'

6(b) PROPRIETY (PPRROIETY)

(i) $\frac{9!}{2!2!} = 90720$ (1) (ii) $\frac{7!3!}{2!2!} = 7560$ (1) (iii) $P = \frac{4!}{7.6.5.4} = \frac{1}{35}$ (1)

(c)(i) $\ddot{x} = v\frac{dv}{dx} = 10x - 4x^3$

$\therefore v\,dv = (10x - 4x^3)\,dx$

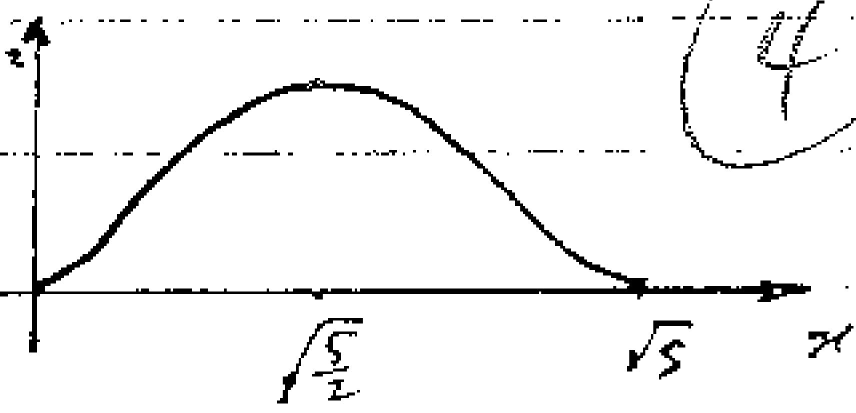
$\therefore \frac{1}{2}v^2 = 5x^2 - x^4 + C$ 1

When $x = \sqrt{5}$, $v = 0$

$\therefore 0 = 25 - 25 + C \quad \therefore C = 0$ 1

$\therefore v^2 = 2x^2(5 - x^2)$ 1

(ii) The particle starts at $x = \sqrt{5}$ m and moves towards 0 with increasing speed. After $x = \sqrt{\frac{5}{2}}$ m it continues towards 0 decreasing its speed indefinitely. 1 (4)



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Question 7

(a) (i)

	B	E
1	120 (ie; $\frac{4800 \times 0.065}{26}$)	$120 + \frac{1}{26}\cdot\frac{5}{100}\cdot 120 \left[= 120\left(\frac{2605}{2600}\right)\right]$
2	$120\left(\frac{2605}{2600}\right) + 120$	$120\left(\frac{2605}{2600}\right)^2 + 120\left(\frac{2605}{2600}\right)$ 1

At the end of 10 years (260 fortnights), the investment A, is worth

$$A = 120\left[\left(\frac{2605}{2600}\right) + \left(\frac{2605}{2600}\right)^2 + \cdots + \left(\frac{2605}{2600}\right)^{260}\right]$$

$$= 120 \times \left(\frac{2605}{2600}\right)\frac{\left[\left(\frac{2605}{2600}\right)^{260} - 1\right]}{\left(\frac{5}{2600}\right)}$$ 1

$\therefore A = \$40{,}508$ 1 which is the amount Ross will receive. (3)

(ii) At the end of 8 years (208 fortnights), the investment B is

$$B = 120 \times \left(\frac{2605}{2600}\right)\frac{\left[\left(\frac{2605}{2600}\right)^{208} - 1\right]}{\left(\frac{5}{2600}\right)}$$

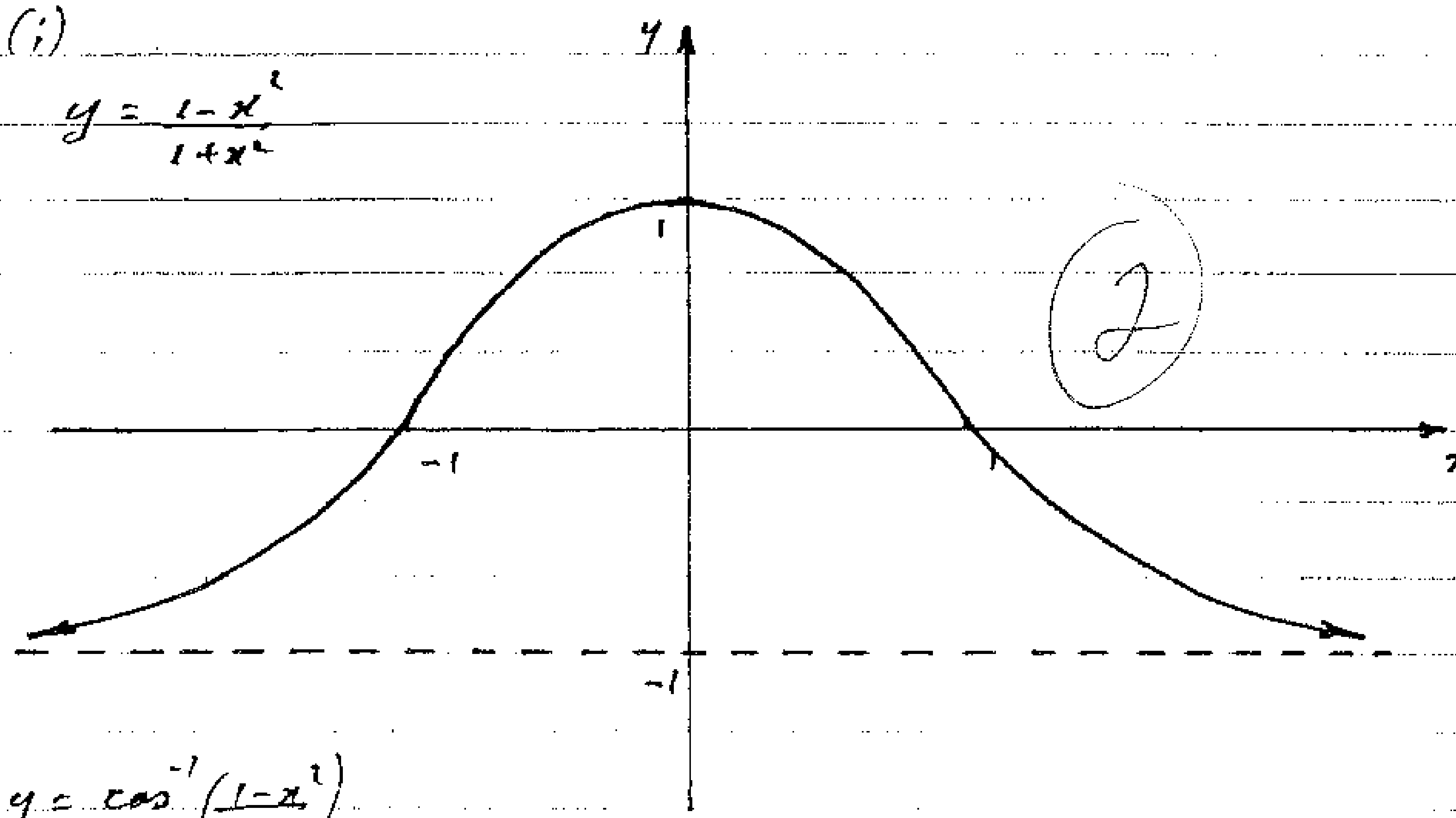
$\therefore B = \$30713$ 1

For the next 2 years, at 12% PA, compounded monthly, Evelyn will receive $\$30{,}713\,(1.01)^{24}$

$= \$38{,}997$ 1 (3)

The difference is $\$40{,}508 - \$38{,}997 = \$1511$ in favour of ROSS $\therefore$ he pays for Sherry's holiday.

7(b) (i) $y = \dfrac{1-x^2}{1+x^2}$



(2)

(ii) $y = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-\left(\frac{1-x^2}{1+x^2}\right)^2}} \times \left[\frac{(1+x^2)\cdot -2x - (1-x^2)\cdot 2x}{(1+x^2)^2}\right]$$ 1

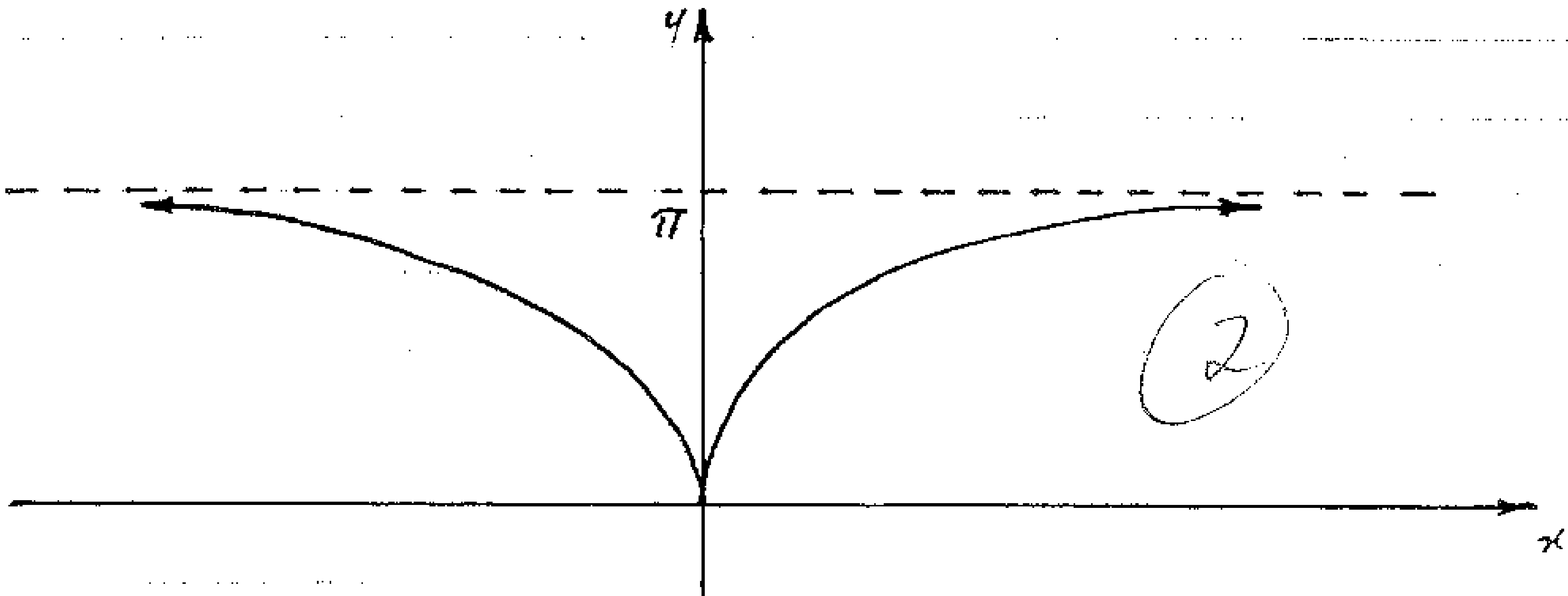
$$= -\frac{(1+x^2)}{\sqrt{(1+x^2)^2-(1-x^2)^2}}\left[\frac{-2x-\cancel{2x^3}-2x+\cancel{2x^3}}{(1+x^2)^2}\right]$$

$$= \frac{(1+x^2)}{\sqrt{4x^2}} \cdot \frac{4x}{(1+x^2)^2}$$

(2)

$$\therefore \frac{dy}{dx} = \frac{2x}{|x|(1+x^2)}$$ 1

(iii)



(2)

Since $f(-x) = f(x)$ $\therefore$ $f(x)$ is EVEN

For $x > 0$, $\frac{dy}{dx} > 0$ and for $x < 0$, $\frac{dy}{dx} < 0$

As $x \to \pm\infty$, $f(x) \to \pi$.

12