TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION 2002

MATHEMATICS

EXTENSION I

Time Allowed – 2 Hours
(Plus 5 minutes reading time)

All questions may be attempted

All questions are of equal value

In every question, show all necessary working

Marks may not be awarded for careless or badly arranged work

Standard integral tables are included with the examination paper.
Approved silent calculators may be used.

The answers to all questions are to be returned in separate bundles clearly labelled Question 1, Question 2 , etc. Each bundle must show your candidate number.

Question 1:

(a) Find the acute angle between the lines

$$2x + y = 17 \quad \text{and} \quad 3x - y = 3$$ 2

(b) Differentiate $y = \tan^{-1}\sqrt{2x^2 - 1}$ **3**

(c) Evaluate $\displaystyle\int_0^3 \frac{y}{\sqrt{y+1}}\,dy$, using the substitution $y = u^2 - 1$ **3**

(d) Eight identical coins show 3 heads and 5 tails.

- (i) In how many ways can they be arranged in a straight line? **1**
- (ii) What is the probability that all the tails will be together? **1**

(e) Solve for x : $\dfrac{2x-3}{x-2} \geq 1$ **2**

Question 2: (START A NEW PAGE)

(a)

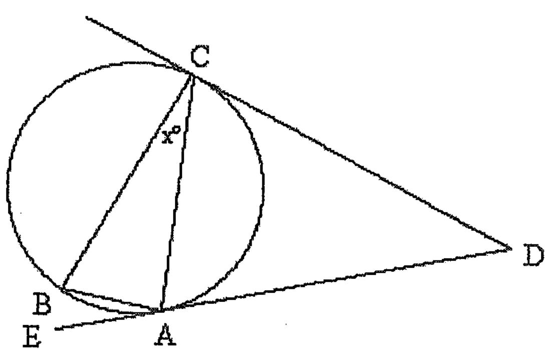


Diagram not to scale **4**

AD and CD are tangents to a circle. B is a point on the circle such that $\angle CBA$ and $\angle CDA$ are equal and are each
both double $\angle BCA$. Prove that b is a diameter of the circle.

(b) The roots of the equation $9x^2 + 6x + 1 = 4kx$ where k is a real constant, are α and β. Show that the equation with roots $\dfrac{1}{\alpha}$ and $\dfrac{1}{\beta}$ is **4**

$$x^2 + 6x + 9 = 4kx$$

(c) Prove by Mathematical Induction that

$$1\times 2^0 + 2\times 2^1 + 3\times 2^2 + \ldots + n\times 2^{n-1} = 1 + (n-1)2^n$$ **4**

for all integers $n \geq 1$.

Question 3: **(START A NEW PAGE)**

(a) The angle of elevation of a tower PQ of height h metres ~~at~~ from a point A due east of it is 15^0. From another point B, the bearing of the tower is 032^0T and the angle of elevation is 13^0. The points A and B are 500 metres apart and on the same level as the base Q of the tower.

(i) Draw a neat sketch showing all the information on your diagram **1**

(ii) Show that $\angle AQB = 122^0$. **1**

(iii) Calculate the height of the tower PQ to the nearest metre. **2**

(b) The speed v m/s of a particle moving in a straight line is given by

$$v^2 = 64 - 16x - 8x^2$$

where the displacement from a fixed point O is x metres.

(i) Find an expression for the acceleration and show the motion is simple harmonic. **2**

(ii) Find the period of the motion **1**

(iii) Find the amplitude of the motion **1**

(c) (i) Find the largest possible domain for which $f(x) = \sin^{-1}(2x+1)$ defines a function **1**

(ii) Hence find and sketch $f^{-1}(x)$, stating its domain and range. **3**

Question 4: (START A NEW PAGE)

(a) N is the number of kangaroos in a certain population at time t years. The population size N satisfies the equation

$$\frac{dN}{dt} = -k(N-500), \text{ for some constant } k.$$

(i) Verify that $N = 500 + Ae^{-kt}$ with A constant, is a solution of the equation **1**

(ii) Initially, there are 3500 kangaroos but after 3 years there are only 3300 left. Find the values of A and k. **2**

(iii) Find when the number of kangaroos begins to fall below 2300 **2**

(iv) Sketch the graph of the population size against time **2**

(b) An urn contains 6 cards numbered 1, 2, 3, 4, 5, 6. One card is drawn at random and a second card is drawn without the first card being replaced. Find the probability that: -

(i) the second number is 3 **1**

(ii) the larger number is 5 **2**

(iii) the larger number is even **2**

Question 5: (START A NEW PAGE)

(a) At an air show, a Harrier Jump Jet leaves the ground 200 metres from an observer and rises vertically at the rate of 25 m/sec. At what rate is the observer's angle of elevation of the aircraft changing when the jet is 500 metres above the ground? **3**

Question 5 continued over page.....

(b) A chord joining the points $P(2p, p^2)$ and $Q(2q, q^2)$ on the parabola $x^2 = 4y$ passes through the point (0,-1)

(i) Find the coordinates of M, the midpoint of PQ, as a function of m, the gradient of the chord **3**

(ii) Show that the cartesian equation of the locus of M is

$$x^2 = 2(y+1) \text{ for } |x| \geq 2.$$ **2**

(c) (i) Express $\sin x + \sqrt{3}\cos x$ in the form $A\cos(x+\alpha)$. **2**

(iii) Hence solve $\sin x + \sqrt{3}\cos x = 1$ for $0 \leq x \leq 2\pi$. **2**

Question 6: (START A NEW PAGE)

(a) The deck of a ship was $1{\cdot}4\,m$ below the level of a wharf at low tide and $0{\cdot}6\,m$ above wharf level at high tide. Low tide was at 8:24 am and hightide at 2:40pm. If tide's motion is simple harmonic, find the first time after low tide that the deck was level with the wharf. **4**

(b) Steven borrows \$50 000 to pay for a new car. He plans to repay the loan by making 60 equal monthly instalments. Interest is charged at the rate of $0{\cdot}6$ % per month on the balance owing.

(i) Show that immediately after making two monthly instalments of \$$P$, the balance owing is given by

$$\$(50\,601{\cdot}80 - 2{\cdot}006P)$$ **2**

(ii) Calculate the value of each monthly instalment **2**

(c) A particle is projected with an initial velocity of 60 *m/s* at an angle of 45° to the horizontal. (use $g = 10ms^{-2}$)

(i) Calculate the greatest height reached by the particle. **3**

(ii) What is the speed of the particle at the greatest height? **1**

Question 7: **(START A NEW PAGE)**

(a) In a box, there are 10 black counters (each marked with the digit "2") and 5 white counters (each marked with digits "3"). 4 counters are withdrawn one at a time, the first being replaced before the second is drawn. Find the probability that

(i) 2 blacks and 2 white counters are drawn in any order **2**

(ii) The sum of digits on the counters drawn is greater than 9 **3**

(b) (i) Show that $(1+x)^m(1-\frac{1}{x})^m = (x-\frac{1}{x})^m$ **1**

(ii) By considering the term(s) independent of x in the expansion of the result from part (b) (i), justify the result: **3**

$$\binom{2002}{0}^2 - \binom{2002}{1}^2 + \binom{2002}{2}^2 - \ldots + \binom{2002}{2002}^2 = -1\binom{2002}{1001}$$

(iii) Hence, or otherwise, show that: **3**

$$\sum_{k=0}^{1001}(-1)^k\binom{2002}{k}^2 = -\frac{1}{2}\binom{2002}{1001}\left[1+\binom{2002}{1001}\right].$$

END OF PAPER

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

TRIAL H.S.C. EXTENSION 1 2002

Qu1:

(a) $2x+y=17 \quad m_1=-2$

$3x-y=3 \quad m_2=3$

$\tan\theta = \left|\frac{-2-3}{1+-2\times3}\right|$

$=1$

$\therefore \theta = 45^\circ$ ②

(b) $y=\tan^{-1}\sqrt{2x^2-1}$

$\frac{dy}{dx} = \frac{1}{(2x^2-1)+1} \times \frac{4x}{2\sqrt{2x^2-1}}$

$= \frac{1}{2x^2} \times \frac{2x}{\sqrt{2x^2-1}}$

$= \frac{1}{x\sqrt{2x^2-1}}$ ③

(c) $\int_0^3 \frac{y}{\sqrt{y+1}}\,dy = \int_1^2 \frac{u^2-1}{u}\cdot 2u\,du$ $\qquad y=u^2-1$

$=\int_1^2 (2u^2-2)\,du$ $\qquad \frac{dy}{du}=2u$

$=\left[\frac{2u^3}{3}-2u\right]_1^2$ $\qquad dy=2u\,du$

$=2\frac{2}{3}$ $\qquad$ When $y=0$ $u=1$, When $y=3$ $u=2$ ③

(d)(i) No. of ways $= \frac{8!}{5!\,3!}$

$=56$ ①

(ii) P(all tails tog.) $= \frac{4}{56}$

$= \frac{1}{14}$

TTTTTHHH

HTTTTTHH ①

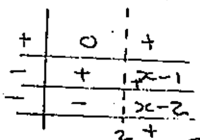
HHTTTTTH

HHHTTTTT

(e) $\frac{2x-3}{x-2} \geq 1$

$\frac{2x-3}{x-2} - \frac{x-2}{x-2} \geq 0$

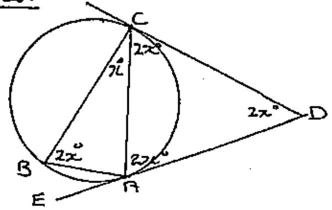
$\frac{x-1}{x-2} \geq 0$



$\therefore x \leq 1$ and $x > 2$ ②

Qu2:

(a)



$\angle CBA = \angle CDA = 2x^\circ$ (given)

$\angle DAC = \angle CBA$ (angle between a tangent & a chord equals angle in the alternate segment)

Similarly

$\angle DCA = \angle CBA = 2x^\circ$ ①

$\therefore$ In $\triangle CDA$ $\quad 2x+2x+2x=180^\circ$ (angle sum $\triangle$)

$\therefore x = 30^\circ$ ①

In $\triangle BAC$

$x^\circ + 2x^\circ + \angle BAC = 180^\circ$

$30^\circ + 60^\circ + \angle BAC = 180^\circ$ ①

$\therefore \angle BAC = 90^\circ$

$\therefore$ BC is a diameter (angle in semi-circle is 90°) ①

(b) $9x^2+6x+1=4kx$

$9x^2+(6-4k)x+1=0$

$\alpha+\beta = \frac{4k-6}{9}$ ①

$\alpha\beta = \frac{1}{9}$

$\frac{1}{\alpha}+\frac{1}{\beta} = \frac{4k-6}{1} = \frac{-b}{a}$ ①

$\frac{1}{\alpha}\times\frac{1}{\beta} = \frac{9}{1} = \frac{c}{a}$

$\therefore a=1$

$b=6-4k$ ①

$c=9$

$\therefore$ Eqn. is $x^2+(6-4k)x+9=0$ ①

$x^2+6x+9 = 4kx$

(c) Let $P(n)$ be proposition

$1\times2^0+2\times2^1+3\times2^2+\ldots+n\times2^{n-1} = 1+(n-1)2^n$

Step 1: For $P(1)$

$1\times2^0 = 1+(1-1)2^0$

$1=1$ ①

$\therefore P(1)$ is true

Step 2: Assume that $P(k)$ is true for some integer $k\geq1$

i.e. $P(k)$: $1\times2^0+2\times2^1+3\times2^2+\ldots+k\times2^{k-1} = 1+(k-1)2^k$

and RTS $P(k+1)$ is true ①

Proof: For $P(k+1)$

$1\times2^0+2\times2^1+3\times2^2+\ldots+k\times2^{k-1}+(k+1)2^k$

$=1+(k-1)2^k+(k+1)2^k$

$=1+((k-1)+(k+1))2^k$

$=1+2k\cdot2^k$ ②

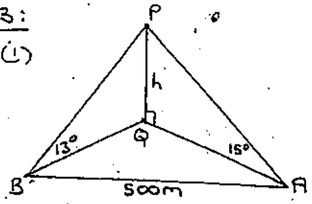
$=1+k\cdot2^{k+1}$

$\therefore P(k+1)$ is true

Step 3: If the result is true for $P(1)$, assumed true for $P(k)$ and proven true for $P(k+1)$ then it is true for all positive integral values of n.

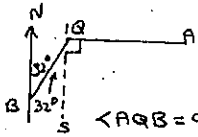
Qu3:

(a)(i)



①

(ii)



$\angle AQB = 90^\circ+32^\circ$ ①

$\angle AQB = 122^\circ$ as $QA \perp QS$ & $\angle NBQ = 32^\circ$

(iii) In $\triangle APQ$ $\qquad$ In $\triangle PQB$

$h = AQ\cot15^\circ$ $\qquad h = BQ\cot13^\circ$

In $\triangle ABQ$

$500^2 = h^2\cot^2 13^\circ + h^2\cot^2 15^\circ - 2\cdot h^2\cot13^\circ\cot15^\circ\cos122^\circ$

$h^2 = \frac{500^2}{\cot^2 13^\circ+\cot^2 15^\circ+2\cot13^\circ\cot15^\circ\cos58^\circ}$

$h = \frac{500}{\sqrt{\cot^2 13^\circ+\cot^2 15^\circ+2\cot13^\circ\cot15^\circ\cos58^\circ}}$ ②

$\therefore h = 71$m (nearest m)

(b) $v^2 = 64-16x-8x^2$

(i) $\frac{1}{2}v^2 = 32-8x-4x^2$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -8-8x$

$\therefore \ddot{x} = -8(x+1)$ ②

$\therefore$ Motion is SHM centre at $x=-1$

(ii) Period $= \frac{2\pi}{2\sqrt{2}} = \frac{\pi}{\sqrt{2}}$ sec $\qquad$ as $n=2\sqrt{2}$ ①

(iii) For motion to exist $v^2 \geq 0$

$8(4+x)(2-x) \geq 0$

$-4 \leq x \leq 2$ ①

$\therefore$ Amplitude $=3$m

(c)(i) D: $-1 \leq 2x+1 \leq 1$

$-2 \leq 2x \leq 0$

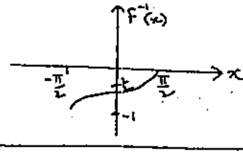
$-1 \leq x \leq 0$ ①

(ii) $x = \sin^{-1}(2y+1)$

$y = \frac{1}{2}(\sin x - 1)$

$D_{f^{-1}}$ $= -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ ②

$R_{f^{-1}}$ $= -1 \leq y \leq 0$



①

Qu4:

(a)(i) $N = 500+Ae^{-kt}$

$\frac{dN}{dt} = -kAe^{-kt}$

$\therefore \frac{dN}{dt} = -k(N-500)$ as $Ae^{-kt} = N-500$

(ii) $A=3000$ when $t=0$

When $t=3$ $\quad N=3300$

$3300 = 500+3000e^{-3k}$

$\ln\frac{14}{15} = -3k$

$k = -\frac{1}{3}\ln\frac{14}{15}$

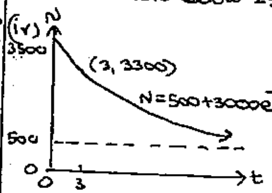
(iii) $500+3000e^{-kt} \leq 2300$

$e^{-kt} \leq \frac{1800}{3000}$

$\therefore t \geq 22\cdot21$ yrs

$\therefore$ No. falls below 2300 when $t \doteqdot 22$

(iv)



②

(b)(i) Sample space $= {}^6P_2 = 30$

No. of favourable events $=5$

i.e. (1,3)(2,3)(4,3)(5,3)(6,3)

P(2nd. no. is 3) $= \frac{5}{30}$ ①

$= \frac{1}{6}$

(ii) Sample space $= {}^6C_2 = 15$

No. of favourable events $=4$

P(larger no. is a 5) $= \frac{4}{15}$ ②

(iii) $n(S) = {}^6C_2 = 15$

5 has 2 larger even no's.

3 " 4 " " "

1 " 6 " " "

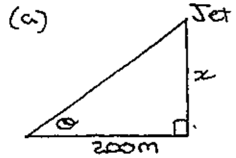
9 $\quad \therefore n(E)=9$

P(larger no. even) $= \frac{9}{15}$ ②

$= \frac{3}{5}$

Qu5:

(a)



Given $\frac{dh}{dt} = 25$

$\tan\theta = \frac{h}{200}$ (1)

$\theta = \tan^{-1}\frac{h}{200}$

$\frac{d\theta}{dh} = \frac{200}{200^2 + x^2}$

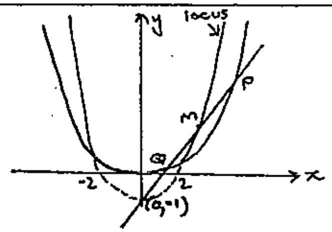
$\left[\frac{d\theta}{dh}\right]_{h=500} = \frac{200}{290000}$ (1)

Now $\frac{d\theta}{dt} = \frac{d\theta}{dh} \times \frac{dh}{dt}$

$= \frac{2}{2900} \times 25$

$= \frac{1}{58}$ rads/sec. (1)

(b)



(i) $y + 1 = mx$ .... (1)

$x^2 = 4y$ .... (2)

$\therefore x^2 = 4(mx - 1)$ on subst.

$x^2 - 4mx + 4 = 0$

$\frac{x_p + x_q}{2} = x_m$ where x_p, x_q are roots

$2x_m = x_p + x_q = \alpha + \beta = 4m$

$\therefore x_m = 2m$ .... (3)

$y_m = m(2m) - 1$ subst. in (1)

$M(2m, 2m^2 - 1)$ .... (3)

(ii) For locus $x^2 - 4mx + 4 = 0$

$16m^2 - 16 \geq 0$

$|m| \geq 1$ (1)

Subst. in (3)

$|x| \geq 2$

$x = 2m$ .... (4)

$y = 2m^2 - 1$ .... (5)

Square (4) Sub in (5) i.e. $x^2 = 2(2m^2)$

$\therefore y + 1 = \frac{x^2}{2}$

$x^2 = 2(y + 1)$ (1)

(c)(i) $\sin x + \sqrt{3}\cos x = A\cos(x + \alpha)$

$A = \sqrt{(1)^2 + (\sqrt{3})^2}$

$\therefore A = 2$

$\sin x + \sqrt{3}\cos x = 2\cos x\cos\alpha - 2\sin x\sin\alpha$

$\therefore \sin\alpha = -\frac{1}{2}$ 3rd & 4th quad.

$\cos\alpha = \frac{\sqrt{3}}{2}$ 1st & 4th quad.

$\tan\alpha = -\frac{1}{\sqrt{3}}$

$\therefore \sqrt{3}\sin x + \cos x = 2\cos\left(x - \frac{\pi}{6}\right)$ (2)

OR $= 2\cos\left(x + \frac{11\pi}{6}\right)$

(ii) $\sin x + \sqrt{3}\cos x = 1$ $0 \leq x \leq 2\pi$

$2\cos\left(x - \frac{\pi}{6}\right) = 1$

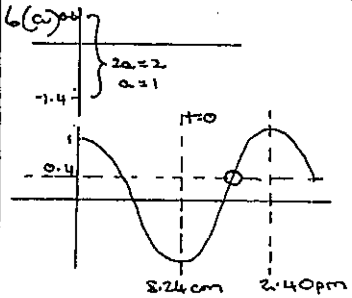
$\cos\left(x - \frac{\pi}{6}\right) = \frac{1}{2}$

$x - \frac{\pi}{6} = \frac{\pi}{3} \Rightarrow x = \frac{\pi}{2}$

$x - \frac{\pi}{6} = \frac{5\pi}{3} \Rightarrow x = \frac{11\pi}{6}$ (2)

$x = \frac{\pi}{2}, \frac{11\pi}{6}$

6(a)



(4)

$\frac{T}{2} = 6\text{hrs } 16\text{min} = 6\frac{4}{15}$

$\frac{T}{2} = \frac{2\pi}{2n}$

$n = \frac{15\pi}{94}$

$\therefore 0.4 = -\cos\frac{15\pi}{94}t$

$\therefore t = \frac{94}{15\pi}\left[2n\pi \pm \cos^{-1}\left(-\frac{2}{5}\right)\right]$

$= 3$hrs 57 min

$\therefore$ First time after low tide deck is level with wharf is 12.21pm.

(b)(i) Money owing after 1st payment

$= \$50000 \times 1.006 - P$

Money owing after 2nd payment

$= \$50000 \times 1.006^2 - P(1 + 1.006)$

Balance $= \$50601.80 - 2.006P$ (2)

(ii) $50000 \times 1.006^{60} - \frac{P(1.006^{60} - 1)}{0.006} = 0$

$\therefore P = \frac{50000 \times 1.006^{60}}{\frac{1.006^{60} - 1}{0.006}}$

$\therefore P = \$994.78$

6(c)(i) $\ddot{y} = -10$

$\dot{y} = -10t + 30\sqrt{2}$ sub $t = 0$ $\dot{y} = 60\sin 45° = 30\sqrt{2}$

$y = 30\sqrt{2}t - 5t^2$

Greatest height is when $\dot{y} = 0$

$-10t + 30\sqrt{2} = 0$

$t = 3\sqrt{2}$ sec

Subst. in y for greatest height

$y = 30\sqrt{2}(3\sqrt{2}) - 5(3\sqrt{2})^2$

$y = 90$m (3)

(ii) At greatest height entire speed is horizontal

$\dot{x} = 60\cos 45°$

$= 30\sqrt{2}$ m/s (1)

Qu7:

(a) 4 chosen (2B, 2W) $= \frac{4!}{2!2!}$ ways

(i) $P(2B, 2W) = 6\left(\frac{10}{15}\right)^2\left(\frac{5}{15}\right)^2$

$= 6\left(\frac{4}{9}\right)\left(\frac{1}{9}\right)$ (2)

$= \frac{8}{27}$

(ii) Listing

$P(2W, 2B) = 6\left(\frac{10}{15}\right)^2\left(\frac{5}{15}\right)^2$

$P(3W, 1B) = \frac{4!}{3!}\left(\frac{10}{15}\right)\left(\frac{5}{15}\right)^3$

$P(4W) = \frac{4!}{4!}\left(\frac{5}{15}\right)^4$

$P(\text{sum} > 9) = \frac{5^2}{15^4}(6 \times 10^2 + 4 \times 10 \times 5 + 25)$

$= \frac{11}{27}$ (3)

(b)(i) $(1+x)^m\left(1 - \frac{1}{x}\right)^m = \left[(1+x)\left(1 - \frac{1}{x}\right)\right]^m$

$= \left[1 - \frac{1}{x} + x - 1\right]^m$

$= \left[x - \frac{1}{x}\right]^m$ (1)

(ii) Letting $m = 2002$

LHS $= (1+x)^{2002}\left(1 - \frac{1}{x}\right)^{2002}$

$= \left[\binom{2002}{0} + \binom{2002}{1}x + \cdots \binom{2002}{r}x^r + \cdots \binom{2002}{2002}x^{2002}\right]$

$\times \left[\binom{2002}{0} - \binom{2002}{1}\frac{1}{x} + \cdots + (-1)^r\binom{2002}{r}\frac{1}{x^r} + \cdots + \binom{2002}{2002}\frac{1}{x^{2002}}\right]$ (1)

Coeff. of x^0 in LHS is

$\binom{2002}{0} \times \binom{2002}{0} + \binom{2002}{1} \times -\binom{2002}{1} + \cdots + (-1)^r\binom{2002}{r}^2 + \cdots + \binom{2002}{2002}^2$ (1)

i.e. $\binom{2002}{0}^2 - \binom{2002}{1}^2 + \binom{2002}{2}^2 - \cdots + (-1)^r\binom{2002}{r}^2 + \cdots + \binom{2002}{2002}^2$

RHS $= \left(x - \frac{1}{x}\right)^{2002}$

General term is $\binom{2002}{r}x^{2002-r}\left(\frac{-1}{x}\right)^r$

$= (-1)^r\binom{2002}{r}x^{2002-2r}$

$\therefore$ coeff. of x^0 occurs when $2002 - 2r = 0$

i.e. $r = 1001$ (1)

$\therefore$ coeff. is $(-1)^{1001}\binom{2002}{1001} = -1\binom{2002}{1001}$

$\therefore$ LHS = RHS

(iii) L.H.S. $= \binom{2002}{0}^2 - \binom{2002}{1}^2 + \binom{2002}{2}^2 - \cdots + \binom{2002}{1000}^2 - \binom{2002}{1001}^2 + \binom{2002}{1002}^2 - \cdots + \binom{2002}{2001}^2 + \binom{2002}{2002}^2$

$= 2\left[\binom{2002}{0}^2 - \binom{2002}{1}^2 + \binom{2002}{2}^2 - \cdots + \binom{2002}{1000}^2\right] - \binom{2002}{1001}^2$ as ${}^nC_r = {}^nC_{n-r}$ (1)

$= 2\left[\binom{2002}{0}^2 - \binom{2002}{1}^2 + \binom{2002}{2}^2 - \cdots + \binom{2002}{1000}^2 - \binom{2002}{1001}^2\right] + \binom{2002}{1001}^2$ (1)

$\therefore -1\binom{2002}{1001} = 2\Sigma + \binom{2002}{1001}^2$

$2\Sigma = -1\binom{2002}{1001} - \binom{2002}{1001}^2$

$\therefore \sum_{k=0}^{1001}(-1)^k\binom{2002}{k}^2 = -\frac{1}{2}\binom{2002}{1001}\left[1 + \binom{2002}{1001}\right]$ (1)

(63)

2002 HSI Q7 TRIAL DRAFT

7 (a) 10 Black ("2") 5 white ("3")

$P(B) = P(2) = \frac{10}{15} = \frac{2}{3}$

$P(W) = P(3) = \frac{1}{3}$ $\boxed{2}$

(b) (i) $P(2B\,2W) = \frac{4!}{2!2!} P(B)^2 P(W)^2 = {}^4C_2 (\frac{2}{3})^2(\frac{1}{3})^2$ (1) (1)

$= \frac{24}{81} = \frac{8}{27} = 0.296\ldots$

(ii)

B(2)	W(3)	Total	P(E)
4	0	8	$1\cdot(\frac{2}{3})^4 = \frac{16}{81}$
3	1	9	${}^4C_3(\frac{2}{3})^3(\frac{1}{3}) = \frac{32}{81}$
2	2	10 ✓	${}^4C_2(\frac{2}{3})^2(\frac{1}{3})^2 = \frac{24}{81}$ ✓
1	3	11 ✓	${}^4C_1(\frac{2}{3})(\frac{1}{3})^3 = \frac{8}{81}$ ✓
0	4	12 ✓	$1\cdot(\frac{1}{3})^4 = \frac{1}{81}$ ✓

$(\frac{2}{3}+\frac{1}{3})^4$

$P(X>9) = P(2B\,2W \text{ or } 1B\,3W \text{ or } 0B\,4W)$ (1)

$= \text{(i)} + 4\cdot\frac{2}{3}(\frac{1}{3})^3 + 1\cdot(\frac{1}{3})^4 = \frac{33}{81} = \frac{11}{27}$ (1) (1)

or $= 1 - P(X\le 9)$ $\boxed{3}$

$= 1 - P(4B\,0W \text{ or } 3B\,1W)$

$1 - \frac{48}{81} = 1 - \frac{16}{27} = \frac{11}{27} = 0.407\ldots$

(c) (i) $(1+x)^m(1-\frac{1}{x})^m = [(1+x)(1-\frac{1}{x})]^m$

$= [1 - \frac{1}{x} + x - 1]^m$ (1)

$= [x - \frac{1}{x}]^m$

(ii) $LHS = (1+x)^{2002}(1-\frac{1}{x})^{2002}$

$= [\binom{2002}{0} + \binom{2002}{1}x + \ldots + \binom{2002}{2002}x^{2002}][\binom{2002}{0} - \binom{2002}{1}\frac{1}{x} + \ldots + (-1)^k\binom{2002}{k}\frac{1}{x^{2002}}]$ (1)

coeff of x^0 (1)

$\binom{2002}{0}\times\binom{2002}{0} + \binom{2002}{1}x - \binom{2002}{1} + \ldots + \binom{2002}{r}(-1)^r\binom{2002}{r} + \ldots + \binom{2002}{2002}^2$

watch if they use m until last line when they substitute — need to have $(-1)^m\binom{2002}{2002}^2$

$RHS = (x-\frac{1}{x})^m = (x - \frac{1}{x})^{2002}$

$T_{r+1} = {}^mC_r\, x^{m-r}(-\frac{1}{x})^r = {}^mC_r(-1)^r x^{m-2r}$

$\text{}x^0 \equiv$ (1)

$\therefore m - 2r = 0 \quad \therefore r = \frac{m}{2} \quad (r = 1001)$ $\boxed{3}$

$\therefore = {}^mC_r(-1)^{m/2}$ (need to show not quote).

(iii) A few approaches

APPROACH I

$LHS = \sum_0^{1001} = \binom{2002}{0}^2 - \binom{2002}{1}^2 + \ldots + (-1)^r\binom{2002}{r}^2 + \ldots + \binom{2002}{1001}^2$

$= \sum_0^{2002} - [\sum_{1002}^{2002}(-1)^r\binom{2002}{r}^2]$

but $\binom{2002}{2002} = \binom{2002}{0}, \binom{2002}{2001} = \binom{2002}{1}, \ldots, \binom{n}{r} = \binom{n}{n-r}$ (1)

so $\sum_0^{1001} = \sum_0^{2002} - [\sum_{1002}(-1)^r\binom{2002}{2002-r}^2]$

$= -1\binom{2002}{1001} - [\binom{2002}{1000}^2 - \binom{2002}{999}^2 + \ldots + \binom{2002}{0}^2]$

$= -1\binom{2002}{1001} - [\binom{2002}{0}^2 - \binom{2002}{1}^2 + \binom{2002}{2}^2 - \ldots + \binom{2002}{1000}^2]$

$\quad - [\sum_0^{1001} + \binom{2002}{1001}^2]$ (1)

$= -1\binom{2002}{1001} - \sum - \binom{2002}{1001}^2$

$\therefore 2\sum = -1\binom{2002}{1001}[1 + \binom{2002}{1001}]$

so $\sum = -\frac{1}{2}\binom{2002}{1001}[1 + \binom{2002}{1001}]$

APPROACH II

$LHS = \sum = \binom{2002}{0}^2 - \binom{2002}{1}^2 + \binom{2002}{2}^2 - \ldots + \binom{2002}{1000}^2 - \binom{2002}{1001}^2 + \binom{2002}{1002}^2 - \ldots$
$+ \binom{2002}{2000}^2 - \binom{2002}{2001}^2 + \binom{2002}{2002}^2$

$\sum_0^{1001} = 2[\binom{2002}{0}^2 - \binom{2002}{1}^2 + \binom{2002}{2}^2 - \ldots + \binom{2002}{1000}^2] - \binom{2002}{1001}^2$ and (1) can do ${}^nC_r = {}^nC_{n-r}$

$\therefore = 2[\binom{2002}{0}^2 - \binom{2002}{1}^2 + \binom{2002}{2}^2 - \ldots + \binom{2002}{1000}^2 - \binom{2002}{1001}^2] + \binom{2002}{1001}^2$ (1) closing

so $-1\binom{2002}{1001} = 2\sum + \binom{2002}{1001}^2$ (1)

$\Rightarrow \sum_0^{1001} = -\frac{1}{2}\binom{2002}{1001}[1 + \binom{2002}{1001}]$ equal