TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION 2003

MATHEMATICS

EXTENSION I

Time Allowed – 2 Hours
(Plus 5 minutes reading time)

All questions may be attempted

All questions are of equal value

In every question, show all necessary working

Marks may not be awarded for careless or badly arranged work

Standard integral tables are provided for your convenience.
Approved silent calculators may be used.

The answers to all questions are to be returned in separate bundles clearly labelled Question 1, Question 2 , etc. Each bundle must show your candidate number.

QUESTION 1 [12 Marks] **Marks**

(a) Differentiate the following:

(i) $f(x) = \cos^{-1} 2x$ 1

(ii) $y = \ln(\tan^{-1} x)$ 2

(b) Find $\int \cos^2 2x \, dx$. 3

(c) Find $\lim_{x \to 0} \frac{x^2}{2 - 2\cos 2x}$ 3

(d) $\int_0^1 \frac{dx}{x^2 + 3} = a\pi$. Find the exact value of a. 3

QUESTION 2 [12 Marks] **Start a new page**

(a) (i) Graph accurately the curve $y = \frac{2}{x-1}$. 3

(ii) Hence, solve $\frac{2}{x-1} \geq -1$ 2

(b) The interval PQ has endpoints P(2, 3) and Q(–3, 5). 2
Find the coordinates of the point T, which divides the interval PQ *externally* in the ratio 3 : 1.

(c) Find the general solution of $\tan 3\theta = 1$ 2

(d) A particle is moving in simple harmonic motion. Its displacement, x metres, at time, t seconds, is given by $x = 3\sin\left(2t - \frac{\pi}{4}\right)$.

(i) Find the period of the motion. 1

(ii) Find the velocity of the particle when $t = 0$. 2

QUESTION 3 [12 Marks] Start a new page. Marks

(a) A particle is moving along the x – axis. Its velocity, v m/s at position x metres is given by 2

$$v = \sqrt{3x - x^2}$$

Find the acceleration of the particle when $x = 2$.

(b) $Q(x) = x^3 + ax^2 + 2x + b$. Given that $Q(x)$ has a factor of $(x + 3)$ and when $Q(x)$ is divided by $(x - 1)$ the remainder is 4. 3

Find the values of a and b.

(c)

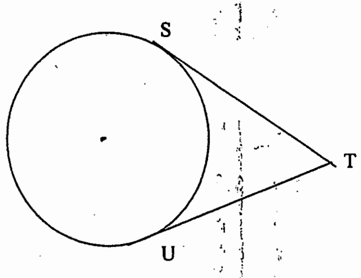

Diagram not to scale

S and U are points on a circle. The tangents to the circle at S and U meet at T. R is point on the circle so that the chord SR is parallel to UT.

(i) Draw a neat sketch showing the given information. 1

(ii) Prove that SU = UR. 3

(d) Find the ratio of the 5th term to the 8th term in the expansion $(2x + 3)^{10}$ when $x = \frac{1}{2}$. 3

QUESTION 4 [12 Marks] Start a new page Marks

(a) Using the substitution $x = 1 - u^2$, find $\int \frac{x\,dx}{\sqrt{1-x}}$. 3

(b) Consider the function $f(x) = \frac{1}{2}\sin^{-1} x$.

(i) State the domain and range of the function. 2

(ii) Find the area of the region bounded by the curve, the x – axis and the line $x = 1$. 3

(c) Show that the constant term in the expansion $\left(\frac{3x^2}{2} - \frac{1}{3x}\right)^9$ is $\frac{{}^9C_6}{6^3}$. 4

Question 5 [12 Marks] Start a new page. Marks

(a) Solve for $0 \le \theta \le \pi$, $\cos\theta + 3\sin\frac{\theta}{2} - 2 = 0$. 3

(b) Homer Simpson borrows $15 000 at 11.95 % per annum reducible interest, calculated monthly. The loan is to be repaid in 60 monthly instalments of $333.30 at the end of each month.

The amount A_n, of the loan remaining after n months is given by

$A_n = MR^n - \$333.30\left(\frac{R^n - 1}{R - 1}\right)$, where M is the principle amount borrowed.

(i) Find the exact value of R. 1

(ii) After 2 years, Homer inherits $1500 and wishes to pay this towards his loan. By how many months is the term of his loan reduced, by paying this extra amount? 3

Question 5 continued overleaf.

Question 5 continued

(c)

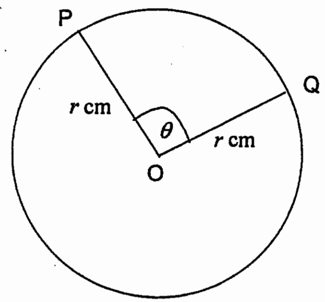

Diagram not to scale

A sector with centre O and radius r cm, is bounded by radii OP, OQ and arc PQ. $\angle$ POQ is θ radians.

(i) Given that r and θ vary in such a way that the area of the sector POQ is always equal to 50 cm^2, show that $\theta = \dfrac{100}{r^2}$. **2**

(ii) Given also that the radius is increasing at a constant rate of 0.5 cm/s, find the rate at which the angle POQ is decreasing when $r = 10$ cm. **3**

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Question 6 [12 Marks] Start a new page.

(a) (i) Find the equation of the tangent to the parabola $x = 2at$, $y = at^2$ at the point P where $t = p$. **1**

(ii) If Q is the point where $t = q$, and O is the origin, show that if OQ is parallel to the tangent, then $q = 2p$. **1**

(iii) If M is the midpoint of PQ, find the equation of the locus of M as P and Q vary along the parabola such that OQ remains parallel to the tangent at P. **4**

(b) Using the principle of mathematical induction, prove that **4**

$$ln[(n+2)!] > n + 2, \text{ for } n \geq 4$$

(c) At a referendum, 30% of parents were in favour of a new uniform logo. An S.R.C. member approached 8 parents chosen at random. **2**

Find the probability that from this group, exactly 3 parents voted in favour of the logo.

QUESTION 7 [12 Marks] Start a new page

Marks

(a) All the letters of the word **ENCUMBRANCE** are arranged in a line. **4**

Find the total number of arrangements, which contain all the vowels *in alphabetical order* but separated by at least one consonant.

Question 7 continued overleaf.

Question 7 continued.

(b) A golf ball is lying at point P, at the middle of a sand bunker, which is surrounded by level ground. The point A is at the edge of the bunker and the line AB lies on the level ground. The bunker is 8 metres wide and 1 metre deep.

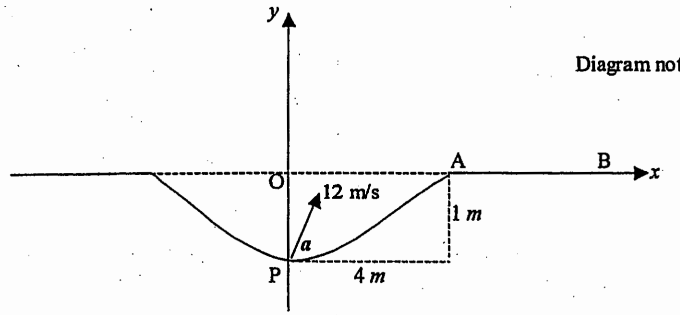


Diagram not to scale

The ball is hit towards A with an initial speed of 12 m/s and angle of elevation α.
(You may assume that the acceleration due to gravity is 10 m/s^2)

The golf ball's trajectory at time t seconds after being hit may be defined by the equations $x = (12\cos\alpha)t$ and $y = -5t^2 + (12\sin\alpha)t - 1$, where x and y are the horizontal and vertical displacements, in metres, of the ball from the origin O, shown in the diagram.

(i) If $\alpha = 30^0$, how far to the right of A will the ball land? (Give your answer correct to 0.1m) **4**

(ii) Find the range of values of α, to the nearest degree, at which the ball must be hit so that it will land to the right of A. **4**

~ End of Exam~

☺

QUESTION 1 [12 Marks]

(a) (i) $f'(x) = \dfrac{-2}{\sqrt{1-4x^2}}$ [1mk]

(ii) $y' = \dfrac{1}{(1+x^2)(\tan^{-1} x)}$ [2mk]

(b) $\int \cos^2 2x \, dx$

$= \frac{1}{2}\int 1+\cos 4x \, dx$ [1mk]

$= \begin{cases} \frac{1}{2}[\frac{\sin 4x}{4} + x] + c \\ \frac{\sin 4x}{8} + \frac{x}{2} + c \end{cases}$ [2mk]

(c) $\lim\limits_{x\to 0} \dfrac{x^2}{2-2\cos 2x}$

$= \frac{1}{2}\lim\limits_{x\to 0}\dfrac{x^2}{1-\cos 2x}$ [1mk]

$= \frac{1}{2}\lim\limits_{x\to 0}\dfrac{x^2}{2\sin^2 x}$ [1mk]

$= \frac{1}{4}\lim\limits_{x\to 0}\dfrac{x}{\sin x}\cdot\dfrac{x}{\sin x}$ [1mk]

$= \frac{1}{4}$

(d) $\int_0^1 \dfrac{dx}{x^2+3} = a\pi$

$\int_0^1 \dfrac{dx}{x^2+3}$

LHS $= \dfrac{1}{\sqrt{3}}\left[\tan^{-1}\dfrac{x}{\sqrt{3}}\right]_0^1$ [1mk]

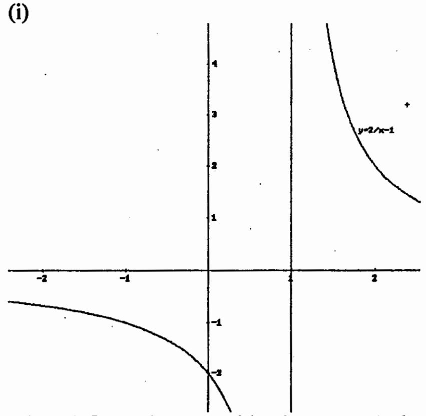
$= \dfrac{1}{\sqrt{3}}\left(\dfrac{\pi}{6}\right)$

$= \dfrac{\pi}{6\sqrt{3}}$ [1mk]

$\therefore a = \dfrac{1}{6\sqrt{3}}$

Question 2 [12 Marks]

(a) (i)



[Graph 3mks] – 1mk curve with y intercept, 1mk scale, 1mk asymptote

(ii) $x \le -1$ or $x > 1$ [2mk]

(b) P(2, 3)   Q(-3, 5) ratio 3 : – 1

$T = \left(\dfrac{3(-3)-1(2)}{3-1}, \dfrac{3(5)-1(3)}{3-1}\right)$

$= \left(\dfrac{-11}{2}, 6\right)$ [2mk]

(c) $\tan 3\theta = 1$

$3\theta = -\pi + \frac{\pi}{4},\ \frac{\pi}{4},\ \pi + \frac{\pi}{4}$..... [1mk]

$\therefore 3\theta = n\pi + \frac{\pi}{4}$

$\therefore \theta = \frac{n\pi}{3} + \frac{\pi}{12}$ [1mk]

(d) (i) Period $= \pi$ secs [1mk]

(ii) $\dot{x} = 6\cos\left(2t - \frac{\pi}{4}\right)$ [1mk]

when $t = 0$

$\dot{x} = 3\sqrt{2}$ [1mk]

$\therefore$ the velocity is $3\sqrt{2}$ m/s

Question 3 [12 Marks]

(a)

$a = \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$

$= \dfrac{d}{dx}\left(\dfrac{1}{2}(3x - x^2)\right)$ [1mk]

$= \dfrac{1}{2}(3-2x)$

$\therefore$ when $x = 2$, $a = -\dfrac{1}{2}$ [1mk]

$\therefore$ the acceleration is $-\dfrac{1}{2}$ m/s^2

(b) $P(1) = 3 + a + b = 4$ [1mk]

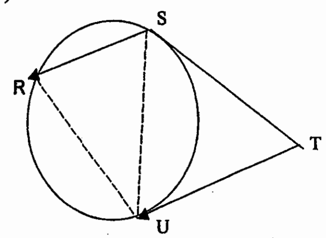
$P(-3) = 9a + b - 33 = 0$ [1mk]

Solve simultaneously

$a + b = 1$ and $9a + b = 33$

$\therefore a = 4$ and $b = -3$ [1mk]

(c) (i)



[1mk]

(ii) Join chords SU and UR

$\angle RSU = \angle SUT$ (alternate angles, RS||UT)

$\angle SUT = \angle SRU$ (Angle in alternate segment)

$\therefore \angle SRU = \angle RSA$

$\therefore$ RU = SU (sides opposite equal angles are equal) [3mks]

(d) $\dfrac{T_5}{T_8} = \dfrac{{}^{10}C_4(2x)^4(3^6)}{{}^{10}C_7(2x)^7(3^3)}$ [1mk]

$= \dfrac{210\times 3^3}{120\times(2x)^3}$ [1mk]

$\therefore$ when $x = ½$

$\dfrac{T_5}{T_8} = \dfrac{210\times(3^3)}{120\times(1)^3} = \dfrac{189}{4}$ [1mk]

or can have $\dfrac{7}{108}$ if the terms are swapped around.

Question 4 [12 Marks]

(a) $\int \dfrac{x\,dx}{\sqrt{1-x}}$ $x = 1-u^2$

$dx = -2u\,du$ [1mk]

$= \int \dfrac{1-u^2}{u}\cdot -2u\,du$

$= -2\int 1-u^2 du$

$= -2(u - \dfrac{u^3}{3}) + c$ [1mk]

$= -2\sqrt{1-x}\left(\dfrac{2+x}{3}\right) + c$ [1mk]

(b) (i) Domain : $-1 \le x \le 1$ [1mk]

Range: $-\dfrac{\pi}{4} \le f(x) \le \dfrac{\pi}{4}$ [1mk]

(ii) Area $= \int_0^1 \dfrac{1}{2}\sin^{-1} x\,dx$

$= \dfrac{\pi}{4} - \int_0^{\frac{\pi}{4}} \sin 2y\,dy$ [1mk]

$= \dfrac{\pi}{4} - \left[\dfrac{1}{2}\cos 2y\right]_0^{\frac{\pi}{4}}$ [1mk]

$= \dfrac{\pi}{4} - \left[\dfrac{1}{2}\cos\left(\dfrac{\pi}{2}\right) - \dfrac{1}{2}\cos 0\right]$ [1mk]

$= (\dfrac{\pi}{4} - \dfrac{1}{2})$ sq units

(c) $\left(\dfrac{3x^2}{2}-\dfrac{1}{3x}\right)^9$

$$T_{k+1}={}^9C_k\left(\frac{3x^2}{2}\right)^{9-k}\left(-\frac{1}{3x}\right)^k$$

$$\therefore \quad = {}^9C_k\left(\frac{3}{2}\right)^{9-k}\left(\frac{-1}{3}\right)^k x^{-k}\cdot x^{18-2k} \quad \text{[2mk]}$$

$\therefore$ constant term occurs when

$18-3k=0 \quad \therefore k=6$ [1mk]

$\therefore T_7$ is the constant term.

i.e. ${}^9C_6\left(\dfrac{3}{2}\right)^3\left(\dfrac{-1}{3}\right)^6=\dfrac{{}^9C_6}{6^3}$ [1mk]

Question 5 [12 Marks]

(a)

$\cos\theta+3\sin\dfrac{\theta}{2}-2=0$

$1-2\sin^2\dfrac{\theta}{2}+3\sin\dfrac{\theta}{2}-2=0$ [1mk]

$2\sin^2\dfrac{\theta}{2}-3\sin\dfrac{\theta}{2}+1=0$

$\left(2\sin\dfrac{\theta}{2}-1\right)\left(\sin\dfrac{\theta}{2}-1\right)=0$

$\therefore \sin\dfrac{\theta}{2}=\dfrac{1}{2}$ or $\sin\dfrac{\theta}{2}=1$ [1mk]

$\therefore \dfrac{\theta}{2}=\dfrac{\pi}{6}$, or $\dfrac{\theta}{2}=\dfrac{\pi}{2}$

$\therefore \theta=\dfrac{\pi}{3}$ or π [1mk]

(b) (i) $R=\dfrac{121195}{120000}$ or $1\dfrac{239}{24000}$ [1mk]

(ii) $A_{24}=15000\left(\dfrac{121195}{120000}\right)^{24}-333.30\left[\dfrac{\left(\dfrac{121195}{120000}\right)^{24}-1}{\dfrac{1195}{120000}}\right]$

$\therefore A_{24}=\$10041.37$

$\therefore$ Total remaining after
2 years = \$8541.37 [1mk]

$\therefore$ let $A_n=0$ to find n

$$\therefore\ 8541.37\left(\frac{121195}{120000}\right)^n=333.30\left[\frac{\left(\frac{121195}{120000}\right)^n-1}{\frac{1195}{120000}}\right]$$

$$-24928.09\left(\frac{121195}{120000}\right)^n=-33469.47$$

$$\frac{121195^n}{120000}=1.3426408$$

$$\therefore n=\frac{\ln 1.3426408}{\ln\frac{121195}{120000}}$$

$n=30$ [1mk]

$\therefore$ Homer's term is reduced by 6 months. [1mk]

(c) (i) Area $=50=\dfrac{1}{2}r^2\theta$ [1mk]

$\therefore\ \theta=\dfrac{100}{r^2}$ [1mk]

(ii) $\dfrac{d\theta}{dt}=\dfrac{d\theta}{dr}\cdot\dfrac{dr}{dt}$ [1mk]

i.e. $\dfrac{d\theta}{dt}=-\dfrac{200}{r^3}\cdot\dfrac{1}{2}$ [1mk]

$\therefore$ when $r=10$

$\dfrac{d\theta}{dt}=-\dfrac{2}{10}\times\dfrac{1}{2}=-\dfrac{1}{10}$ [1mk]

$\therefore \angle POQ$ is decreasing at a rate of 0.1 m/s.

Question 6 [12 Marks]

(a) (i) Eqn. of tangent: $y=px-ap^2$ [1mk]

(ii) Gradient of OQ $=\dfrac{aq^2}{2aq}=\dfrac{q}{2}$

For OQ $\parallel$ to tangent in part (i)

Then $p=\dfrac{q}{2}$. $\therefore q=2p$. [1mk]

(iii) MP $=\left(ap+aq,\dfrac{a(p^2+q^2)}{2}\right)$ [1mk]

$\therefore x=a(p+q)$ and $y=\dfrac{a\left(p^2+q^2\right)}{2}$ [1mk]

i.e. $x=3ap$ ..(1) and $y=\dfrac{5ap^2}{2}$ ..(2) [1mk]

From (1) we get $p=\dfrac{x}{3a}$, sub into (2)

We get $y=\dfrac{a}{2}\left(\dfrac{5x^2}{9a^2}\right)$

$\therefore y=\dfrac{5x^2}{18a}$ [1mk]

(b) Test $n=4$

LHS $=ln(6!)\cong 6.579$

RHS $=6$

$\therefore$ LHS > RHS

$\therefore$ True for $n=4$. [1mk]

Assume true for some integer $n=k$ where $k\geq 4$,
i.e. $ln[(k+2)!]>k+2$

Prove true for $n=k+1$
[i.e.RTP $ln[(k+3)!]>k+3$]

LHS $=ln[((k+1)+2)!]$
$=ln\,[(k+3)!]$
$=ln\,[(k+3)(k+2)!]$
$=ln(k+3)+ln\,[(k+2)!]$ [1mk]
$>ln\,(k+3)+k+2$ (using assumption)
$>1+k+2$ since $\ln(k+3)\geq\ln 7\cong 1\cdot 9..>1$ for $k\geq 4$ [1mk]

$\therefore ln[(k+3)!]>k+3$

$\therefore$ Since it is true for $n=4$, and assuming it's true for $n=k$, and proven true for $n=k+1$, it is true for $n=5, 6, 7$ $\therefore$ true for all values of $n\geq 4$. [1mk]

NB. Other approaches possible.

Eg. Consider $\ln[(k+3)!]-k-3$

$=\ln[(k+3)(k+2)!]-k-3$
$=\ln[(k+3)]+\ln[(k+2)!]-k-3$
$>\ln(k+3)+k+2-k-3=\ln(k+3)-1$
>0

since $\ln(k+3)-1\geq\ln 7-1\cong 1\cdot 9-1>0$ for $k\geq 4$

etc

....

(c) $P(X=3)={}^8C_3\left(0.3^3\right)\left(0.7^5\right)$ [1mk]

$=0.254$ [1mk]

Question 7 [12 Marks]

(a) Vowels: AEEU
Consonants: NNCCMBR

Choose 7 consonants & 1 vowel in 8! Ways.
CCCV$_1$C CCC _ _

Place the 2[nd] vowel, V_2, anywhere in line (but not next to V_1) – 7 spaces to choose from.

Place 3[rd] Vowel, V_3, 6 spaces to choose from.

Place 4[th] Vowel, V_4, 5 choices to choose from.

$\therefore$ Number of ways of separating vowels & consonants considering all different = $8!\times7\times6\times5$

but have 2 E's, 2 C's, 2 N's

$\therefore$ No. of arrangements =

$$\frac{8!\times7\times6\times5}{2!2!2!}=1\,058\,400$$

No. of ways of arranging AEEU = $\frac{4!}{2!}=12$

$\therefore$ No. of arrangements with vowels in order

$$=\frac{1058400}{12}=88\,200$$

*** Can also map out each letter to get the same answer.**

(b) (i) $y=-5t^2+(12\sin\alpha)t-1$

$0=-5t^2+12\sin(30^0)t-1$

$0=5t^2-6t+1$

$0=(5t-1)(t-1)$

$\therefore t=\frac{1}{5}$ or 1 [1mk]

Must take $t=1$ as want second time when $y=0$. [1mk]

$\therefore x=12\cos 30^0\ (1)$

$=6\sqrt{3}$ [1mk]

$\therefore$ distance to right of A is $(6\sqrt{3}-4)$ metres
(approx. 6.39 m) [1mk]

(ii) $x=(12\cos\alpha)t\ \ldots(1)$
$y=-5t^2+(12\sin\alpha)t-1\ldots(2)$

From (1) $t=\dfrac{x}{12\cos\alpha}$ sub into (2)

$$y=-5\left(\frac{x^2}{144\cos^2\alpha}\right)+12\sin\alpha\frac{x}{12\cos\alpha}-1$$

$$=\frac{-5x^2}{144}\left(1+\tan^2\alpha\right)+x\tan\alpha-1 \qquad [1mk]$$

when $x=4, y>0$

$$\therefore\ \frac{-5(4)^2}{144}\left(1+\tan^2\alpha\right)+4\tan\alpha-1>0 \qquad [1mk]$$

i.e. $-5(1+\tan^2\alpha)+36\text{n}\alpha-9>0$

$-5-5\tan^2\alpha+36\tan\alpha-9>0$

i.e. $5\tan^2\alpha-36\tan\alpha+14>0$ [1mk]

Using Quadratic Formula for equality :

$$\tan\alpha=\frac{36\pm\sqrt{36^2-4(5)(14)}}{10}$$

$$=\frac{36\pm\sqrt{1016}}{10}$$

$\therefore\ \alpha=80°37'$ or $22°25'$

$\therefore\ 22°25'<\alpha<80°37'$ [1mk]