JAMES RUSE AGRICULTURAL HIGH SCHOOL
YEAR 12 MATHEMATICS EXTENSION I
TRIAL EXAM 2004

Marks

QUESTION 1

(a) Find $\frac{d}{dx}\left(\ln\left(5+e^{x}\right)\right)$ **2**

(b) Find $\int \frac{19\,dx}{4+8x^2}$ **2**

(c) Evaluate $\int_6^{22} x\sqrt{x+3}\,dx$ using the substitution $u^2 = x+3$ **4**

(d) Solve for x : $\frac{x+1}{x-3} \geq 2$ **2**

(e) Six identical yellow discs and four identical blue discs are placed in a straight line.

(i) How many arrangements are possible ? **1**

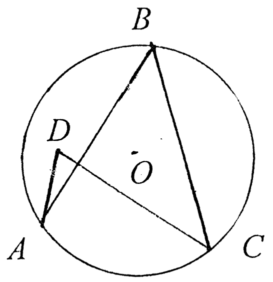
(ii) Find the probability that all the blue discs are together. **1**

QUESTION 2 (START A NEW PAGE)

(a) Find the acute angle (to nearest degree) between the lines : **2**

$$y = \frac{3x}{8} - \frac{7}{8} \quad \text{and} \quad 2x + y - 5 = 0$$

(b) Points A,B and C lie on the circumference of a circle with centre O, and point D lies inside the circle with $\angle ABC = 17^0$ and $\angle ADC = 34^0$. **3**



Prove $ADOC$ is a cyclic quadrilateral.

(c) Find $\int \frac{4x-1}{\sqrt{9-x^2}}\,dx$ **2**

(d) Evaluate $\int_0^1 \left(1+x^2\right)^4 dx$ **3**

(e) Find $\frac{d}{dx}\left(\cos^{-1}\left(2\cos^2 x - 1\right)\right)$ in simplest terms for $\{\ 0 \leq x \leq \frac{\pi}{2}\ \}$. **2**

QUESTION 3 (START A NEW PAGE)

(a)(i) On the same x-y axes graph the functions $y = f(x)$ and $y = f^{-1}(x)$ if $f(x) = e^{x} + e^{2x}$. **3**
Show all the y intercepts and asymptotes.

(ii) Find the equation of the inverse function $f^{-1}(x)$ if $f(x) = e^{x} + e^{2x}$ stating the domain and range of $f^{-1}(x)$. **4**

(b) If α is a multiple root of $P(x)=0$ then $P'(\alpha)=0$. **5**

Factorise $P(x) = 12x^3 - 16x^2 + 7x - 1$ if $P(x)$ has multiple zeros.

QUESTION 4 (START A NEW PAGE)

(a) A particle moves in a straight line.

The displacement function x metres in terms of time t seconds is given by :
$x(t) = 6\sin 2t - 6\cos 2t$

(i) Show that the displacement function can be written in the form : 2

$x(t) = R\sin(2t - \alpha)$ where $R>0$ and $\{ 0 < \alpha < 2\pi \}$.

State the exact values of R and α .

(ii) Graph the displacement function $x(t)$ for $\{ 0 < t < 2\pi \}$. 2

(iii) Show that the motion is Simple Harmonic Motion. 2

(iv) Find the expression v^2 in terms of displacement x if v is the velocity of the particle. 2

(v) Find the first time the particle is 2 metres from the centre of motion. 2

(b) Find the constant term in the expression $x^3\left(x^2 + \frac{2}{x}\right)^6$ 2

QUESTION 5

(a) A man has a loan of \$ 15800 with monthly reducible interest of 8% p.a. 5

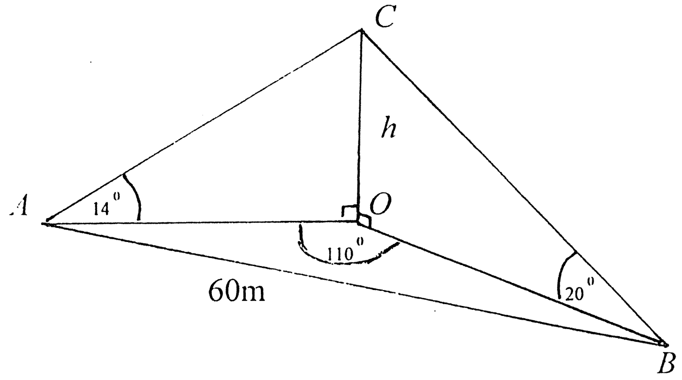
If the repayments are \$1250 per month , find the number of payments to

repay all the loan.

(b) Prove by induction for all positive integers n: 4

$$\frac{5}{6} + \frac{1}{4} + \dots\dots\dots\dots + \frac{n+4}{n(n+1)(n+2)} = \frac{3}{2} - \frac{n+3}{(n+1)(n+2)}$$

(c) 3



A vertical tower shown above has angles of elevation from A and B of 14^0 and 20^0 respectively.

If the distance AB is 60 metres and $\angle AOB = 110^0$, find the height h of the tower to the nearest metre.

QUESTION 6

(a) A bowman fires an arrow with an initial velocity of 50 m/s from 1.5 metres above ground to a target 80 metres away.
The bullseye of the target is 0.3 metres in diameter, and the centre of the bullseye is 1 metre above ground.

(i) Show that the trajectory equation for the flight of the arrow is given by : **3**

$y = x\tan\alpha - \frac{x^2}{500}\left(1+\tan^2\alpha\right)+1.5$ where α is the initial angle of elevation of the arrow,

the acceleration due to gravity g is $10 m/s^2$ and the Origin is at ground level .

(ii) Find the range of values of α (to the nearest second) for the arrow to hit the bullseye. **5**

(b) The bowman has a probability of $\frac{3}{5}$ of hitting the bullseye.

(i) Find the probability of hitting the bullseye exactly 7 times from 13 trials. **1**

(ii) By comparing the terms of $\left(\frac{3}{5}+\frac{2}{5}\right)^{13}$ find the most likely outcome of hitting the bullseye from 13 trials. **3**

QUESTION 7

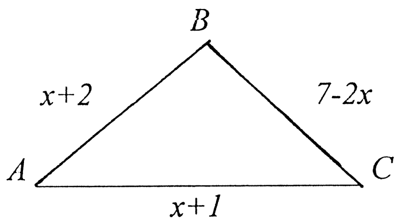
(a) The rate of growth of a population N over t years is given by : $\frac{dN}{dt} = -k\left(N-700\right)$.

(i) Show $N = 700 + Ae^{-kt}$ satisfies $\frac{dN}{dt} = -k\left(N-700\right)$ where A and k are constants. **1**

(ii) The population has decreased from an initial population of 8300 to 5100 in 5 years. **3**

Find the population at the end of the next 5 years.

(b) Triangle ABC is shown .



(i) Show that the domain of x for the triangle to exist is given by { $1<x<3$ }. **2**

(ii) The area A of a triangle with sides a, b and c is given by : **2**

$$A = \sqrt{s(s-a)(s-b)(s-c)} \quad \text{where } s = \frac{1}{2}(a+b+c)$$

Show that the expression for the area A of the triangle ABC in terms of x is given by :

$$A = \sqrt{10\left(x^3 - 8x^2 + 19x - 12\right)}$$

(iii) Find the value of x that gives the maximum area of ΔABC . **4**

END OF EXAM

12 TRIAL EXAM 2004

1 (a) $\frac{d}{dx}\ln(5+e^x) = \frac{e^x}{5+e^x}$

(b) $\int \frac{19\,dx}{4+8x^2} = \frac{19}{8}\int \frac{dx}{x^2+\frac{1}{2}}$

$= \frac{19}{8}\sqrt{2}\,\tan^{-1}\sqrt{2}x + C$

(c) $\int_6^{22} x\sqrt{x+3}\,dx$

$u^2 = x+3$

$2u\,du = dx$

$x = 22 \quad u = 5$

$x = 6 \quad u = 3$

$\int_3^5 (u^2-3)u.2u\,du$

$2\int_3^5 u^2(u^2-3)\,du$

$2\int_3^5 u^4-3u^2\,du$

$2\left[\frac{u^5}{5}-u^3\right]_3^5$

$2\left[625-125-\left(\frac{243}{5}-27\right)\right]$

$956\frac{4}{5}$

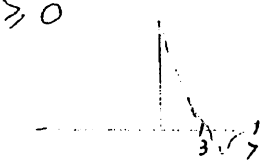
(d) $\frac{x+1}{x-3} \geq 2 \qquad x \neq 3.$

$(x-3)(x+1) \geq 2(x-3)^2$

$(x-3)[x+1-2(x-3)] \geq 0$

$(x-3)(-x+7) \geq 0$

$(x-3)(x-7) \leq 0$



Soln $\{3 < x \leq 7\}$

(e) (i) $\frac{10!}{6!\,4!} = 210$

(ii) Probability $= \frac{7}{210}$

$= \frac{1}{30}$

2 (a) $m_1 = \frac{3}{8} \qquad m_2 = -2.$

$\tan\theta = \left|\frac{m_1-m_2}{1+m_1m_2}\right|$

$= \left|\frac{\frac{3}{8}+2}{1-\frac{3}{8}.2}\right|$

$= \left|\frac{19}{2}\right|$

$\theta = 84^\circ$

(iv) $\angle AOC = 2\angle ABC$ (Angle at the Centre of a circle is twice the angle at the circumference standing on the same arc.)

$= 2 \times 17$

$= 34^\circ$

But $\angle AOC = \angle ADC = 34^\circ$

$\therefore$ ADOC is cyclic (If an interval subtends equal angles at two points on the same side of it then the endpoints of the interval and the two points are concyclic.)

d) $\int \frac{4x-1}{\sqrt{9-x^2}}\,dx = \int \left(4x(9-x^2)^{-1/2} - \frac{1}{\sqrt{9-x^2}}\right)dx$

$= -4\sqrt{9-x^2} - \sin^{-1}\frac{x}{3} + C$

) $\int_0^1 (1+x^2)^4\,dx = \int_0^1 (1+4x^2+6x^4+4x^6+x^8)\,dx$

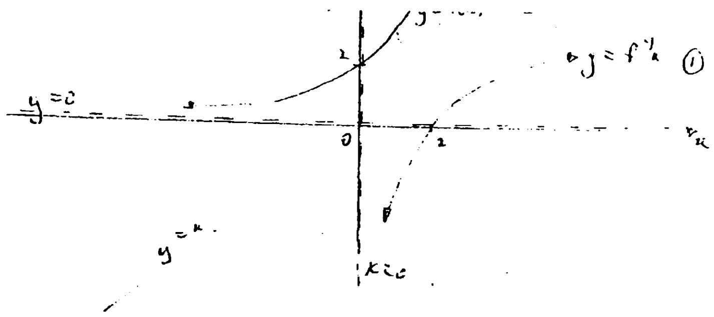
$= \left[x + \frac{4}{3}x^3 + \frac{6}{5}x^5 + \frac{4}{7}x^7 + \frac{x^9}{9}\right]_0^1$

$= 4\frac{68}{315}$

(e) $\frac{d}{dx}\cos^{-1}(2\cos^2 x - 1) = \frac{d}{dx}\cos^{-1}(\cos 2x)$ $\{0 < x < \frac{\pi}{2}\}$

$= \frac{d}{dx}2x$

(i)



(ii) $x = e^y + e^{2y}$

$(e^y)^2 + e^y - x = 0$

$e^y = \frac{-1 \pm \sqrt{1+4x}}{2}$

$y = \ln\left[\frac{\sqrt{1+4x} - 1}{2}\right]$ ① Only as $e^y > 0$ ①

Domain $\{x > 0\}$ ①

Range : defined for all y ①

(iii) $P(x) = 12x^3 - 16x^2 + 7x - 1$

$P'(x) = 36x^2 - 32x + 7$ ①

$P'(x) = 0$

$\therefore 36x^2 - 32x + 7 = 0$

$(2x-1)(18x-7) = 0$

$x = \frac{1}{2}$ or $x = \frac{7}{18}$ ②

$P(\frac{1}{2}) = 12(\frac{1}{2})^3 - 16(\frac{1}{2})^2 + \frac{7}{2} - 1$

$= 0$

$\therefore 2x-1$ is a factor of $P(x)$ ①

$\therefore P(x) = (2x-1)^2 Q(x)$

$= (2x-1)^2(3x-1)$ ①

(i) $x(t) = 6\sin 2t - 6\cos 2t$

$R\sin(2t-\alpha) = R\cos\alpha\sin 2t - R\sin\alpha\cos 2t$

$\therefore R\sin\alpha = 6$ $\quad R>0 \quad \left.\begin{matrix}\sin\alpha>0\\ \cos\alpha>0\end{matrix}\right\}\{0<\alpha<\frac{\pi}{2}\}$

$R\cos\alpha = 6$

$R = \sqrt{6^2+6^2}$

$= 6\sqrt{2}$

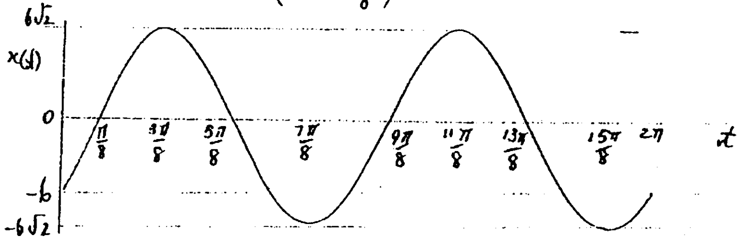
$\tan\alpha = 1$

$\alpha = \frac{\pi}{4}$

$\therefore x(t) = 6\sqrt{2}\sin\left(2t-\frac{\pi}{4}\right)$

$= 6\sqrt{2}\sin 2\left(t-\frac{\pi}{8}\right)$

)



)

$\dot{x}(t) = 12\sqrt{2}\cos\left(2t-\frac{\pi}{4}\right)$

$\ddot{x}(t) = -24\sqrt{2}\sin\left(2t-\frac{\pi}{4}\right)$

$= -4\left[6\sqrt{2}\sin\left(2t-\frac{\pi}{4}\right)\right]$

$\ddot{x} = -4x$

which is of the form $\ddot{x} = -n^2(x-\delta)$

$\therefore$ motion SHM $\quad n=2 \quad \delta=0$

$v^2 = 288\cos^2\left(2t-\frac{\pi}{4}\right)$

$= 288\left(1-\sin^2\left(2t-\frac{\pi}{4}\right)\right)$

$= 288\left(1-\left(\frac{x}{6\sqrt{2}}\right)^2\right)$

$v^2 = 288 - 4x^2$

(v) $-2 = 6\sqrt{2}\sin\left(2t-\frac{\pi}{4}\right)$

$\sin\left(2t-\frac{\pi}{4}\right) = -\frac{1}{3\sqrt{2}}$

$2t-\frac{\pi}{4} = -0.24$

$t = \frac{1}{2}\left[\frac{\pi}{4}-0.24\right]$

$t = 0.27$ seconds.

(b) $x^3\left[x^2+\frac{2}{x}\right]^6$

$T_{r+1} = x^3\binom{6}{r}\left(x^2\right)^{6-r}\left(\frac{2}{x}\right)^r$

$= \binom{6}{r}x^{3+12-2r-r}\cdot 2^r$

$= \binom{6}{r}2^r x^{15-3r}$

Constant term $r=5$.

$\therefore T_6 = \binom{6}{5}2^5$

$= 192$

5 (a) monthly interest $= \frac{8}{1200} = \frac{1}{150}$

Amount owing end 1st Payment $= 15800\left[1+\frac{1}{150}\right] - 1250$

$= 15800\cdot\frac{151}{150} - 1250$

Amount owing end 2nd Payment $= \left[15800\left(\frac{151}{150}\right)-1250\right]\frac{151}{150} - 1250$

$= 15800\left(\frac{151}{150}\right)^2 - 1250\left[1+\frac{151}{150}\right]$

Amount owing end 3rd Payment $= \left[15800\left(\frac{151}{150}\right)^2 - 1250\left(1+\frac{151}{150}\right)\right]\frac{151}{150} - 12$

$$\therefore \quad 0 = 15800\left(\frac{151}{150}\right)^n - 1250\left[1 + \frac{151}{150} + \left(\frac{151}{150}\right)^2 + \cdots \left(\frac{151}{150}\right)^{n-1}\right]$$

$$15800\left(\frac{151}{150}\right)^n = 1250\left[\frac{1\cdot\left(\left(\frac{151}{150}\right)^n - 1\right)}{\frac{151}{150} - 1}\right]$$ ①

$$= 187500\left[\left(\frac{151}{150}\right)^n - 1\right]$$

$$\therefore (187500 - 15800)\left(\frac{151}{150}\right)^n = 187500$$

$$\left(\frac{151}{150}\right)^n = \frac{187500}{171700}$$

$$n = \frac{\ln\left(\frac{187500}{171700}\right)}{\ln\left(\frac{151}{150}\right)}$$

$n = 13.24$

i.e. $n = 14$ payments ①

b) $\frac{5}{6} + \frac{1}{4} + \cdots \frac{n+4}{n(n+1)(n+2)} = \frac{3}{2} - \frac{n+3}{(n+1)(n+2)}$

Step 1 $n=1$ $S_1 = \frac{3}{2} - \frac{4}{2\cdot 3}$

$= \frac{9-4}{6}$

$= \frac{5}{6}$

$\therefore T_1 = S_1$

$\therefore$ True for $n=1$

... statement is true for n=k

i.e. $\frac{5}{6} + \frac{1}{4} + \cdots \frac{k+4}{k(k+1)(k+2)} = \frac{3}{2} - \frac{k+3}{(k+1)(k+2)}$

To prove statement is true for $n=k+1$

i.e. $\frac{5}{6} + \frac{1}{4} + \cdots + \frac{k+5}{(k+1)(k+2)(k+3)} = \frac{3}{2} - \frac{k+4}{(k+2)(k+3)}$

Now

$\frac{5}{6} + \frac{1}{4} + \cdots \frac{k+4}{k(k+1)(k+2)} + \frac{k+5}{(k+1)(k+2)(k+3)}$

$= \frac{3}{2} - \frac{k+3}{(k+1)(k+2)} + \frac{k+5}{(k+1)(k+2)(k+3)}$ By assumption

$= \frac{3}{2} + \frac{k+5 - (k+3)^2}{(k+1)(k+2)(k+3)}$

$= \frac{3}{2} + \frac{-k^2 - 5k - 4}{(k+1)(k+2)(k+3)}$

$= \frac{3}{2} - \frac{(k+1)(k+4)}{(k+1)(k+2)(k+3)}$

$= \frac{3}{2} - \frac{k+4}{(k+2)(k+3)}$

$\therefore$ If statement is true for $n=k$ it is also true for $n=k+1$

Step 3 Since statement is true for $n=1$ it also true for $n=1+1=2$, $n=2+1=3$, and so on for all positive integers n.

$$\alpha = \frac{400 - \sqrt{160000 - 4 \times 64 \times 66.15}}{128}$$

$$\alpha = 80.681^{\circ} \text{ or } 8.853^{\circ}$$
$$= 80^{\circ}40'53'' \quad 8^{\circ}51'11''$$

$$y = 1.15$$

$$1.15 = 80 \tan\alpha - \frac{64}{5}\left(1 + \tan^2\alpha\right) + 1.5$$

$$64 \tan^2\alpha - 400 \tan\alpha + 62.25 = 0$$

$$\tan\alpha = \frac{400 \pm \sqrt{400^2 - 4 \times 64 \times 62.25}}{128}$$

$$\alpha = 80.675^{\circ} \text{ or } 9.074^{\circ}$$
$$= 80^{\circ}40'30'' \quad 9^{\circ}4'26''$$

$\therefore$ Range $\{8^{\circ}51'11'' < \alpha < 9^{\circ}4'26''\}$

OR $\{80^{\circ}40'30'' < \alpha < 80^{\circ}40'53''\}$

$$= \frac{1716 \cdot 3^7 \, 2^6}{5^{13}}$$

OR 0.20

(ii) $T_{r+1} = \binom{13}{r} p^{13-r} q^{r}$ for $(p+q)^{13}$

$$\frac{T_{r+1}}{T_r} = \frac{\binom{13}{r} p^{13-r} q^{r}}{\binom{13}{r-1} p^{14-r} q^{r-1}}$$

$$= \frac{13!}{r!\,(13-r)!} \cdot \frac{(r-1)!\,(14-r)!}{13!} \cdot \frac{q}{p}$$

$$= \frac{14-r}{r} \cdot \frac{q}{p}$$

$$= \frac{2(14-r)}{3r}$$

$$\frac{T_{r+1}}{T_r} > 1$$

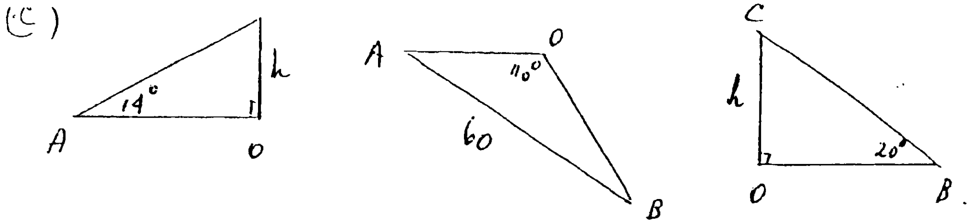
$$\therefore \frac{2(14-r)}{3r} > 1$$

$$5r < 28$$

$$r < 5.6$$

i.e. Most likely $r = 5 \Rightarrow 8$ times from 13 to hit bullseye.

(c)



Now $\frac{h}{AO} = \tan 14^\circ$ And $\frac{h}{OB} = \tan 20^\circ$

$AO = \frac{h}{\tan 14^\circ}$ $OB = \frac{h}{\tan 20^\circ}$

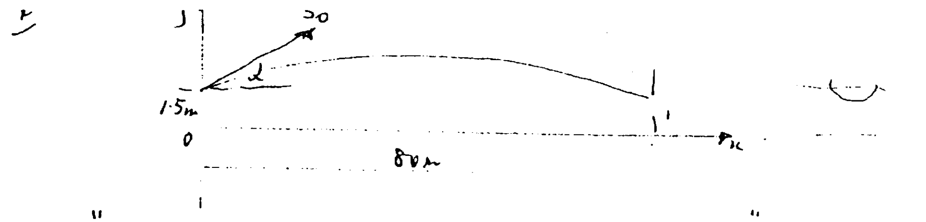
But $AB^2 = AO^2 + BO^2 - 2\,AO\,BO\cos 110^\circ$ (cosine rule)

$$60^2 = \frac{h^2}{\tan^2 14^\circ} + \frac{h^2}{\tan^2 20^\circ} - \frac{2h^2\cos 110^\circ}{\tan 14^\circ \tan 20^\circ}$$

$$h^2 = \frac{60^2 \tan^2 14^\circ \tan^2 20^\circ}{\tan^2 14^\circ + \tan^2 20^\circ - 2\cos 110^\circ \tan 14^\circ \tan 20^\circ}$$

$h = 10.75$

$\therefore$ height tower $= 11$m (nearest m).



$m\ddot{y} = -mg$

$\ddot{y} = -10$

$\dot{y} = -10t + c$

$t = 0 \quad \dot{y} = 50\sin\alpha \Rightarrow c = 50\sin\alpha$

$\therefore \dot{y} = -10t + 50\sin\alpha$

$y = -5t^2 + 50t\sin\alpha + c$

$t = 0 \quad y = 1.5 \Rightarrow c = 0$

$\therefore y = -5t^2 + 50t\sin\alpha + 1.5$

$m\ddot{x} = 0$

$\ddot{x} = 0$

$\dot{x} = 50\cos\alpha$

$x = 50t\cos\alpha$.

$t = 0\ x = 0 \Rightarrow c$

$\therefore x = 50t\cos\alpha$

$t = \frac{x}{50\cos\alpha}$

For trajectory

$$y = -5\left[\frac{x}{50\cos\alpha}\right]^2 + 50\sin\alpha \cdot \frac{x}{50\cos\alpha} + 1.5$$

$$= \frac{-5}{2500}x^2\sec^2\alpha + x\tan\alpha + 1.5$$

$$y = x\tan\alpha - \frac{x^2}{500}(1 + \tan^2\alpha) + 1.5.$$

(ii) Bullseye is $\left\{1 - \frac{0.3}{2} \le y \le 1 + \frac{0.3}{2}\right\}$

$\{0.85 \le y \le 1.15\}$

$\therefore y = 0.85 \quad x = 80$

$\therefore 0.85 = 80\tan\alpha - \frac{64}{5}(1 + \tan^2\alpha) + 1.5$

$\frac{64}{5}\tan^2\alpha - 80\tan\alpha + \frac{64}{5} - 0.65 = 0$

$64\tan^2\alpha - 400\tan\alpha + 60.75 = 0$

(i) $N = 700 + Ae^{-kt}$

$\frac{dN}{dt} = -Ake^{-kt}$ But $Ae^{-kt} = N - 700.$

$\therefore \frac{dN}{dt} = -k\left[N - 700\right]$

(ii) $t = 0$ $N = 8300$

$\therefore 8300 = 700 + A$

$A = 7600.$

$\therefore N = 700 + 7600e^{-kt}$

$t = 5$ $N = 5100$

$\therefore 5100 = 700 + 7600e^{-5k}$

$k = \frac{1}{5}\ln\frac{7600}{4400}$

$k = \frac{1}{5}\ln\left(\frac{19}{11}\right)$

$t = 10$ $N = 700 + 7600e^{-2\ln\left(\frac{19}{11}\right)}$

$N = 3247$

)(i) Sum two sides triangle > third side

$(x+1) + x+2 > 7-2x$ $x+2+7-2x > x+1$ or $x+1+7-2x > x+2$

$4x > 4$ $-2x > -8$ $-2x > -6$

$x > 1$ $x < 4$ $x < 3.$

$\therefore$ Domain $\{1 < x < 3\}$

$s = \frac{1}{2}\left[x+1 + x+2 + 7-2x\right]$

$= \frac{1}{2} \cdot 10$

$= 5$

$\therefore A = \sqrt{5\,(5-(x+1))(5-(x+2))(5-(7-2x))}$

$= \sqrt{5\,(4-x)(3-x)(2x-2)}$

$= \sqrt{10\,(x^2-7x+12)(x-1)}$

$A = \sqrt{10\,(x^3-8x^2+19x-12)}$

(iii) $\frac{dA}{dx} = \frac{\frac{1}{2}\cdot 10\,[3x^2-16x+19]}{\sqrt{10(x^3-8x^2+19x-12)}}$

$= \frac{5\,[3x^2-16x+19]}{\sqrt{10\,(x^3-8x^2+19x-12)}}$ (1)

For maximum area $\frac{dA}{dx} = 0$

$\therefore 3x^2 - 16x + 19 = 0$

$x = \frac{16 \pm \sqrt{256 - 4.3.19}}{6}$

$= \frac{16 \pm \sqrt{28}}{6}$ (1)

$= \frac{8-\sqrt{7}}{3}$ or $\frac{8+\sqrt{7}}{3}$

But domain x is $\{1 < x < 3\}$

$\therefore x = \frac{8-\sqrt{7}}{3}$ only.

For nature of turning point test $\frac{dA}{dx}$ since (

A and $\frac{dA}{dx}$ are continuous in domain $\{1 < x < 3\}$

A	1.7	$\frac{8-\sqrt{7}}{3}$	1.8
$\frac{dA}{dx}$	0.51	0	-0.09

1

$\therefore$ there is an maximum at $x = \frac{8-\sqrt{7}}{3}$

but since there is only one turning point in the domain $\{1 < x < 3\}$ then $x = \frac{8\sqrt{7}}{3}$ is an absolute maximum.