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GESTA NON VERBA

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION 2005

MATHEMATICS EXTENSION 1

Time Allowed – 2 Hours
(Plus 5 minutes Reading Time)

All questions may be attempted

All questions are of equal value

Department of Education approved calculators are permitted

In every question, show all necessary working

Marks may not be awarded for careless or badly arranged work

No grid paper is to be used unless provided with the examination paper

The answers to all questions are to be returned in separate bundles clearly labeled Question 1, Question 2, etc. Each question must show your Candidate Number.

Question 1. **[Start a new page]** — **Marks**

a) If $P(x) = x^3 - 2x^2 + ax + 4$ is divisible by $(x+2)$, what is the value of a? **1**

b) i) Find $\dfrac{d}{dx}\ln(\cos 2x)$ **1**

ii) Hence evaluate exactly $\displaystyle\int_0^{\frac{\pi}{6}} \tan 2x \; dx$ **2**

c) Find i) $\displaystyle\int \frac{e^{3x}\,dx}{2+e^{3x}}$ **1**

ii) $\displaystyle\int \frac{dx}{\sqrt{9-4x^2}}$ **2**

d) Find the acute angle between the straight lines $y = \sqrt{3}x + 2$ and $x = 2$. **2**

e) Solve: $x + 2 < \dfrac{4}{x-1} \quad (x \neq 1)$ **3**

Question 2. **[Start a new page]** — **Marks**

a) By making the substitution $u = \sqrt{x}$, evaluate exactly $\displaystyle\int_0^{\frac{\pi^2}{16}} \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$ **3**

b) i) Sketch the graph of the curve $y = 3\sin^{-1}(x/2)$, clearly indicating the domain and range. **2**

ii) Find the area enclosed between the curve $y = 3\sin^{-1}(x/2)$, the line $y = (3\pi/2)$ and the positive y axis. **2**

c) The polynomial equation $3x^3 - 2x^2 + 3x - 4 = 0$ has roots α, β and γ. Find the exact value of $\dfrac{1}{\alpha\beta} + \dfrac{1}{\beta\gamma} + \dfrac{1}{\gamma\alpha}$. **2**

d) The letters of the word ***MOUSE*** are to be rearranged.

i) How many arrangements are there which start with the letter *M* and end with the letter *E*? **1**

ii) How many arrangements are there in which the vowels are grouped together? (A vowel is one of the letters ***A, E, I, O, U***) **1**

iii) How would your answers to parts (i) and (ii) change if the given word had been ***MOOSE*** instead of ***MOUSE***? **1**

Question 3. **[Start a new page]** **Marks**

a) Find the general solution (in radian form) of the equation $\cos 2x = \cos x$ 3

b) i) At the distinct points $P(2at, at^2)$ and $Q(2au, au^2)$ on the parabola $4ay = x^2$, the tangents are drawn. You may assume, without proof, that the equation of the tangent at P is $y = tx - at^2$. Show that the tangents from P and Q intersect at the point $(a(u+t), aut)$. 2

ii) From the point $R\ (a, -6a)$, two tangents are drawn to the parabola $4ay = x^2$. If the points of contact of these tangents are P and Q, show that the triangle PQR is isosceles. 3

c) Suppose that $(5+2x)^{12} = \sum_{k=0}^{12} a_k x^k$.

i) Using the Binomial Theorem, write an expression for a_k. 2

ii) Show that $\dfrac{a_{k+1}}{a_k} = \dfrac{24-2k}{5k+5}$ 2

Question 4. **[Start a new page]** **Marks**

a) i) Sketch the function $y = f(x)$ where $f(x) = (x-2)^2 - 4$, clearly showing all intercepts on the axes. (Use the same scale on both axes) 2

ii) What is the largest positive domain of f for which $f(x)$ has a continuous inverse $f^{-1}(x)$? 1

iii) Sketch the graph of $f^{-1}(x)$ on the same axes as (i). 1

b) A particle moves along the x axis according to the equation

$$x = 6\sin 2t - 2\sqrt{3}\cos 2t\ .$$

i) Express x in the form $R\sin(2t - \alpha)$ where $R>0$ and $0 \le \alpha \le \pi/2$. 2

ii) Prove that the particle moves in simple harmonic motion 1

c) A, B and C are three sequential points on a straight line on horizontal ground. A vertical flagpole PQ is situated close by the line (but its base P is not on the line). The angles of elevation of the top of the flagpole from A, B and C are $\tan^{-1}\frac{1}{4}$, $\tan^{-1}\frac{1}{2}$ and $\tan^{-1}\frac{1}{3}$ respectively. If $AB = 90$m and $BC = 30$m, find the height of the flagpole. 5

Question 5. **[Start a new page]** **Marks**

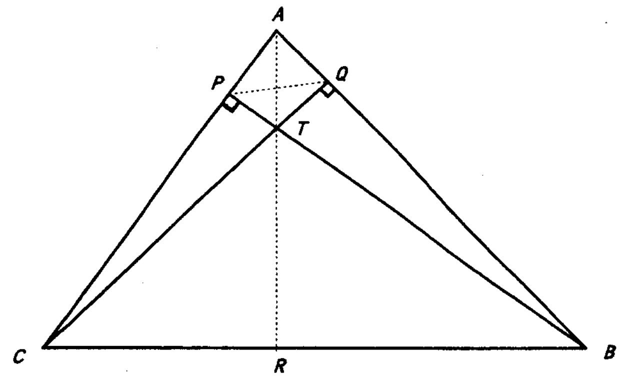
a) A particle is moving along the x-axis. Its velocity, v m/s at position x metres is given by 2

$$v = \sqrt{5x - x^2}\ .$$

Find the acceleration of the particle when $x = 2$.

b) Prove by induction that, for any positive integer n, the product $(n+1)(n+2)...(n+n)$ is always a multiple of 2^n but never a multiple of 2^{n+1}. 5

c)



In the diagram, CQ and BP are altitudes of the triangle ABC. The lines CQ and BP intersect at T, and AT is produced to meet CB at R.

i) Prove that $\angle TAQ = \angle QCB$. 3

ii) Prove that $AR \perp CB$. 2

Question 6. **[Start a new page]** **Marks**

a) Cane sugar, when placed in water, converts into dextrose at a rate which is proportional to the amount of unconverted material remaining. That is, if M grams is the amount of material converted after t minutes, then

$$dM/dt = k(S - M)$$

where S grams is the initial amount of cane sugar and k is a constant.

i) Show that $M = S + Ae^{-kt}$ satisfies the equation, where A is a constant. **1**

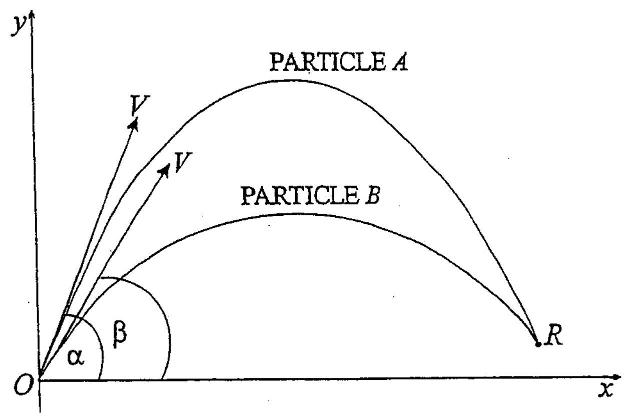
ii) If a certain amount of cane sugar is placed in water at time $t = 0$ and 40% of it has been converted after 10 minutes, show that the value of k is $\frac{1}{10}\log_e(\frac{5}{3})$. **2**

iii) How long will it take, to the nearest minute, for 99% of the cane sugar to be converted into dextrose. **2**

(Question 6 is continued on the next page)

Question 6 (Continued) **Marks**

b)



The diagram above shows two particles A and B projected from the origin. Particle A is projected with initial velocity V m/s at an angle α and particle B is projected T seconds later with the same initial velocity V m/s but at an angle of β. The particles collide at the point R.

i) You may assume that the equation of the path of A is given by

$$y = -\frac{gx^2}{2V^2}\sec^2\alpha + x\tan\alpha$$

Write down the equation of the path of B. **1**

Show that the x-coordinate of the collision point R is given by

$$x = \frac{2V^2\cos\alpha\cos\beta}{g\sin(\alpha+\beta)}$$ **3**

ii) You may assume that the horizontal displacement of A after t seconds is given by

$$x = Vt\cos\alpha$$

(α) Write down the equation for the horizontal displacement of B (Remember that B is projected T seconds after A) **1**

(β) Show that, for the collision to take place, the value of T is given by

$$T = \frac{2V(\cos\beta - \cos\alpha)}{g\sin(\alpha+\beta)}$$ **2**

Question 7. **[Start a new page]**

Marks

a) It is known that 5% of all gear boxes made in Factory A are faulty whereas 7% of gear boxes made in Factory B are faulty. If 20 gear boxes are bought, 10 from each factory, what is the probability that exactly two are faulty? **4**

b) i) By rotating the circle $x^2 + y^2 = r^2$ about the x axis between appropriate limits, show that the volume V of a spherical cap of height h, as shown in Figure 1, is given by

$$V = \frac{\pi h^2}{3}(3r - h) \qquad (0 \le h \le 2r)$$

3

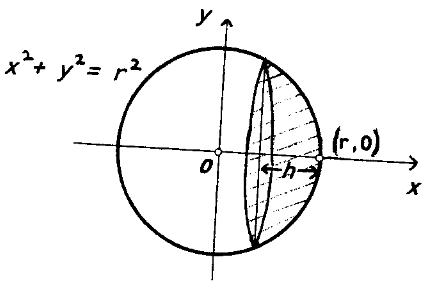


Figure 1

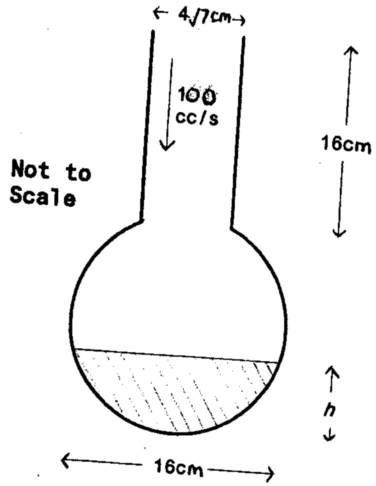


Figure 2

A chemical flask is modelled by surmounting an open cylinder on a thin spherical shell (with a matching circular opening at the top). See Figure 2.

ii) The body of the flask is of radius 8 cm. The neck has radius $2\sqrt{7}$ cm. and height 16 cm. Show that the total height of the flask is 30 cm. **1**

iii) Water is poured into the flask at a constant rate of 100 cm^3/sec. If h is the depth of the water in the flask, use the result from part (i) to find an expression (in terms of h) for the rate at which the water level rises in the spherical portion of the flask. **2**

iv) Find this rate at the instant when the water level reaches the base of the cylinder and hence, or otherwise, calculate how long it will take (from that point in time) to overflow the flask. Give your answer to the nearest second. **2**

THIS IS THE END OF THE EXAMINATION

JRAHS Ext^n 1 TRIAL 2005 ANSWERS

1) a) By Factor theorem $P(-2)=0$

$\therefore (-2)^3 - 2(-2)^2 + a(-2) + 4 = 0$

$-12 - 2a = 0$

$\underline{a = -6}$ [1]

b) i) $\frac{d}{dx}\ln(\cos 2x) = \frac{-2\sin 2x}{\cos 2x}$ [1]

$\left(= -2\tan 2x\right)$

ii) $\int_0^{\pi/6} \tan 2x = \left[-\frac{1}{2}\ln(\cos 2x)\right]_0^{\pi/6}$

$= -\frac{1}{2}\ln\left(\cos\frac{\pi}{3}\right) + \frac{1}{2}\ln 1$

$= -\frac{1}{2}\ln\left(\frac{1}{2}\right)$ [2]

$= \underline{\ln\sqrt{2}}$ or $\underline{\frac{1}{2}\ln 2}$

c) i) $\int \frac{e^{3x}\,dx}{2+e^{3x}} = \frac{1}{3}\ln(2+e^{3x}) + c$

where c is an undetermined constant. [1]

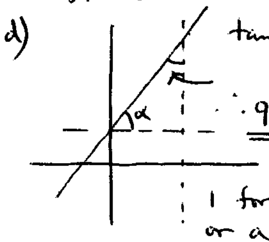
ii) $\int \frac{dx}{\sqrt{9-4x^2}} = \frac{1}{2}\int \frac{dx}{\sqrt{9/4 - x^2}}$

$= \frac{1}{2}\sin^{-1}\left(\frac{2x}{3}\right) + c$ [2]

$\left[\text{or } \frac{1}{2}\cos^{-1}\left(\frac{2x}{3}\right) + c\right]$

where c is an undetermined constant

d)



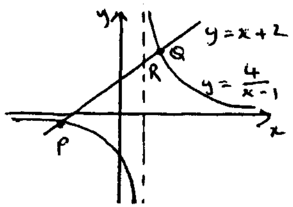
$\tan\alpha = \sqrt{3}$

$\alpha = 60^\circ\ (\pi/3)$

$\underline{90 - \alpha = 30^\circ}\ (\pi/6)$ [2]

1 for correct notion or a correct formula.

e)



From the diagram, $x + 2 < \frac{4}{x-1}$

for values of x less than that at P and for values between R (non-inclusive) and Q.

P, Q found by solving $(x+2)(x-1) = 4$

$x^2 + x - 6 = 0$

$(x+3)(x-2) = 0$

P is $(-3, -1)$ Q is $(2, 4)$ [3]

Soln. is $\underline{\{x : x < -3\} \cup \{x : 1 < x < 2\}}$

2) a) Let $u = x^{1/2}$

$\frac{du}{dx} = \frac{1}{2x^{1/2}} = \frac{1}{2u}$

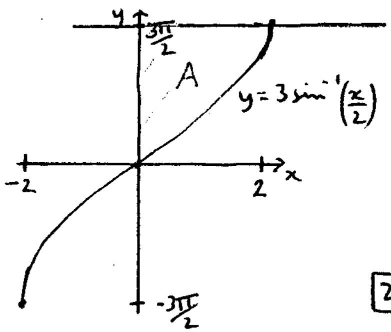
$\therefore \int_0^{\pi^2/16} \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx = \int_0^{\pi/4} 2\sin u\,du$

$= \left[-2\cos u\right]_0^{\pi/4}$

$= \left[\frac{-2}{\sqrt{2}} + 2\right]$

$= \underline{2 - \sqrt{2}}$ [3]

b) i)



[2]

ii) Area $A = \int_0^{3\pi/2} x\,dy$

$= \int_0^{3\pi/2} 2\sin\frac{y}{3}\,dy$

$= \left[-6\cos\frac{y}{3}\right]_0^{3\pi/2}$

$= \underline{6}$ sq units. [2]

2 c) $\alpha + \beta + \gamma = 2/3$

$(\alpha\beta + \beta\gamma + \gamma\alpha = 1)$

$\alpha\beta\gamma = 4/3$

$\frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha} = \frac{\alpha + \beta + \gamma}{\alpha\beta\gamma}$

$= \frac{2/3}{4/3} = \underline{\frac{1}{2}}$ [2]

d) i) $3! = \underline{6}$ [1]

ii) Initially treat vowels as 1 unit → 3!

But vowels can be arranged in 3! ways.

$\therefore$ Total $= 3!\,3! = \underline{36}$ [1]

iii) Both answers will be divided by 2. [1]

3) a) $\cos 2x - \cos x = 0$

$2\cos^2 x - \cos x - 1 = 0$

$(2\cos x + 1)(\cos x - 1) = 0$

$\underline{\cos x = -\frac{1}{2} \text{ or } \cos x = 1}$

But $\cos\pi/3 = -\frac{1}{2}$ and $\cos 0 = 1$

$\therefore \underline{x = 2n\pi \pm \frac{2\pi}{3} \text{ or } 2m\pi}$ [3]

when $n = \ldots -2, -1, 0, 1, 2, \ldots$

$m = \ldots -2, -1, 0, 1, 2, \ldots$

b) (i) Tangent at P is $y = tx - at^2$

$\therefore$ Tangent at Q is $y = ux - au^2$

Solve these equations

Subtract $0 = x(t-u) - a(t^2 - u^2)$

$\underline{a(t+u) = x}$

Sub into first equation

$y = at(t+u) - at^2$

$\underline{y = atu}$ [2]

ii) Referring to part 1,

$u + t = 1$

$ut = -6$

Solving, $u = 3$, $t = -2$

(or vice-versa)

$\therefore P$ is $(-4a, 4a)$

and Q is $(6a, 9a)$

R is $(a, -6a)$

$PQ = \sqrt{25a^2 + 100a^2} = a5\sqrt{5}$

$PR = \sqrt{25a^2 + 100a^2} = a5\sqrt{5}$.

$\therefore \underline{\triangle PQR \text{ is isosceles}}$ [3]

c) i) $(5+2x)^{12} = 5^{12} + {}^{12}C_1 5^{11}(2x)^1 + \ldots$

$= \sum_0^{12} a_k x^k$ [2]

where $a_k = \underline{{}^{12}C_k\, 2^k\, 5^{12-k}}$

ii) $\frac{a_{k+1}}{a_k} = \frac{{}^{12}C_{k+1}\,2^{k+1}\,5^{12-k-1}}{{}^{12}C_k\,2^k\,5^{12-k}}$

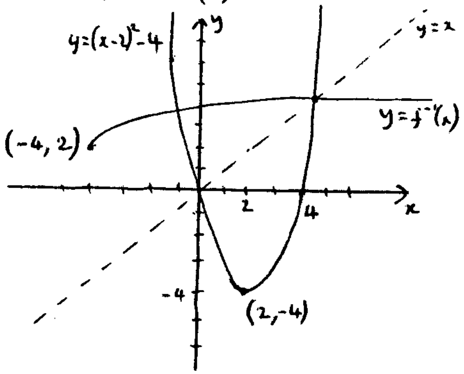
$= \frac{\frac{12!}{(k+1)!\,(12-(k+1))!} \times 2}{\frac{12!}{k!\,(12-k)!} \times 5}$

$= \frac{k!\,(12-k)! \times 2}{(k+1)!\,(12-(k+1))! \times 5}$

$= \frac{2\,(12-k)}{5\,(k+1)}$

$= \underline{\frac{24 - 2k}{5k + 5}}$ [2]

4) a) i) This is a parabola with vertex at $(2,-4)$



ii) Largest domain will be $\{x : x \geq 2\}$

iii) See diagram. [4]

b) i) Let $x = R\sin(2t-\alpha)$

$= R\sin 2t\cos\alpha - R\cos 2t\sin\alpha$

$\therefore R\sin\alpha = 2\sqrt{3}$, $R\cos\alpha = 6$ } comparing coefficients

Divide equations

$\tan\alpha = \frac{1}{\sqrt{3}}$

$\alpha = \pi/6$

Square and add equations

$R^2 = 12 + 36$

$R = \sqrt{48} = 4\sqrt{3}$

$\therefore x = 4\sqrt{3}\sin(2t - \pi/6)$ [2]

ii) Using this form of x (the other works just as well)

$\dot{x} = 8\sqrt{3}\cos(2t-\pi/6)$

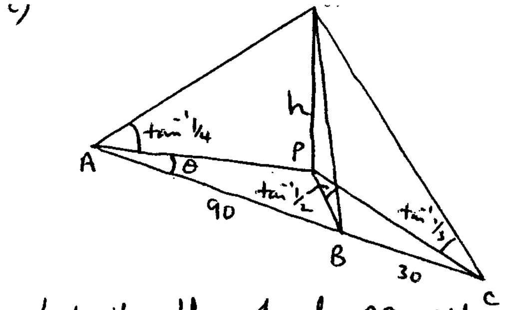
$\ddot{x} = -16\sqrt{3}\sin(2t-\pi/6)$

$\ddot{x} = -4x$

$\ddot{x} = -n^2x$ where $n=2$ [1]

Thus the particle moves in S.H.M.

c)



Let the flagpole be PQ with height h as in the diagram.

$\frac{h}{AP} = \frac{1}{4}$ (Normal trig on ΔAPQ)

$\therefore AP = 4h$

Similarly $BP = 2h$, $CP = 3h$.

Solve for $\cos PAB$ in ΔPAB and PAC:

$\cos\theta = \frac{(4h)^2+(90)^2-(2h)^2}{2.4h.90}$ (ΔPAB)

$= \frac{(4h)^2+(120)^2-(3h)^2}{2.4h.120}$ (ΔPAC)

$\therefore \frac{16h^2+8100-4h^2}{3} = \frac{16h^2+14400-9h^2}{4}$

$4(12h^2+8100) = 3(7h^2+14400)$

$27h^2 = 10800$

$h^2 = 400$

$h = 20$ [5]

$\therefore$ Flagpole is 20m in height.

5) a) Acceleration $= \frac{d}{dx}\left(\frac{v^2}{2}\right)$

$= \frac{d}{dx}\left(\frac{5x}{2} - \frac{x^2}{2}\right)$

$= \frac{5}{2} - x$ [2]

At $x=2$, acc^n is $\frac{1}{2}\ ms^{-2}$

5) b) Assume that, for some pos^{ve} integer k, that

$(k+1)(k+2)\ldots(k+k) = 2^kM$

Where M is an odd integer (since product NOT divisible by 2^{k+1})

Now consider the product for $k+1$

$((k+1)+1)((k+1)+2)\ldots((k+1)+(k+1))$

$= (k+2)(k+3)\ldots 2k(2k+1)(2k+2)$

$= 2(k+1)(k+2)\ldots(k+k)(2k+1)$

$= 2.2^kM.(2k+1)$ from assumption

$= 2^{k+1}M(2k+1)$

This is divisible by 2^{k+1} but NOT by 2^{k+2} since both M and $(2k+1)$ are odd.

Thus, if true for k, the result is also true for $k+1$.

But, if $k=1$, $(1+1)=2$ is divisible by 2^1 but not by 2^2.

Since true for $k=1$, it will be true for $k=2$, hence for $k=3$ etc.

Thus proved that result true for all positive integers. [5]

c) i) CPQB is a cyclic quadrilateral because the angle subtended at P by the interval CB is equal to the angle CB subtends at Q.

Also PAQT is a cyclic quadrilateral because the opposite angles are supplementary - witness the $90°$ angles at $\angle APT$ and $\angle AQT$

$\angle TAQ = \angle TPQ$ (Angles at the circumference from the chord TQ of cyclic quad^l PAQT are equal)

$\angle TPQ = \angle QCB$ (Angles at the circumference from the chord BQ of cyclic quad^l CPQB are equal)

$\therefore \angle TAQ = \angle QCB$ [3]

ii) Consider the sums of the angles in ΔAQT and ΔTCR.

$\angle TAQ = \angle TCR$ (Proved above)

$\angle ATQ = \angle CTR$ (Vertically opposite angles are equal)

$\therefore \angle TCR = \angle TQA$ (Sum of the angles of each triangle adds to $180°$)

But $\angle TQA = 90°$

$\therefore \angle TCR = 90°$ [2]

$\therefore AR \perp CB$.

6) i) LHS $= \frac{d}{dt}(S + Ae^{-kt})$

$= -kAe^{-kt}$

RHS $= k(S-(S+Ae^{-kt}))$

$= k(-Ae^{-kt})$

$= -kAe^{-kt}$ [1]

LHS = RHS

$\therefore M = S + Ae^{-kt}$ satisfies the equation.

ii) When $t=0$, $M=0 \Rightarrow A=-S$

When $t=10$, $M=0.4S$

$\therefore 0.4S = S(1-e^{-10k})$

$\therefore 0.6 = e^{-10k}$

$5/3 = e^{10k}$

$\frac{1}{10}\ln(5/3) = k$ [2]

iii) Find t so that $M = 0.99S$

$0.99 = 1 - e^{-kt}$ (Dividing by S)

$e^{kt} = 100$

$t = 10\ln 100/\ln(5/3)$ [2]

$t \doteq 90$ minutes

$$y = -\frac{gx^2}{2V^2}\sec^2\beta + x\tan\beta \quad \boxed{1}$$

Collision will occur when y values same and x values same.

$$-\frac{gx^2}{2V^2}\sec^2\beta + x\tan\beta = -\frac{gx^2}{2V^2}\sec^2\alpha + x\tan\alpha$$

$$\frac{gx^2}{2V^2}(\sec^2\alpha - \sec^2\beta) = x(\tan\alpha - \tan\beta)$$

Neglect the zero at $x=0$ (start point)

$$x = \frac{2V^2(\tan\alpha - \tan\beta)}{g\ (\sec^2\alpha - \sec^2\beta)}$$

$$= \frac{2V^2(\tan\alpha - \tan\beta)}{g\ (\tan^2\alpha - \tan^2\beta)} \quad (\sec^2\alpha = 1+\tan^2\alpha)$$

$$= \frac{2V^2}{g(\tan\alpha + \tan\beta)} \quad \boxed{3}$$

$$= \frac{2V^2\cos\alpha\cos\beta}{g\ (2\sin\alpha\cos\beta)} = \frac{2V^2\cos\alpha\cos\beta}{g\sin(\alpha+\beta)}$$

ii) For particle B, $x_B = V(t-T)\cos\beta$ $\boxed{1}$
(t measured from when A fired)

Collision occurs when both x values are equal, at some time t, to the form from part (ii)

$$\therefore\ Vt\cos\alpha = \frac{2V^2\cos\alpha\cos\beta}{g\sin(\alpha+\beta)} \Rightarrow t = \frac{2V\cos\beta}{g\sin(\alpha+\beta)}$$

so $$V(t-T)\cos\beta = \frac{2V^2\cos\alpha\cos\beta}{g\sin(\alpha+\beta)} \Rightarrow t-T = \frac{2V\cos\alpha}{g\sin(\alpha+\beta)}$$

$$\therefore T = \{t-(t-T)\} = \frac{2V(\cos\beta - \cos\alpha)}{g\sin(\alpha+\beta)} \quad \boxed{2}$$

7) a) This can be achieved in three ways, the individual probabilities of which must be added.

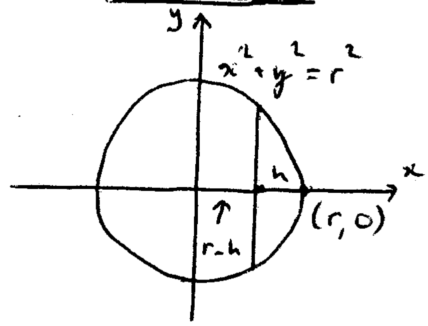
Prob (2 from A, 0 from B) $= {}^{10}C_2(0.05)^2(0.95)^8(0.93)^{10}$

or (1 from A, 1 from B) $= {}^{10}C_1(0.05)^1(0.95)^9\ {}^{10}C_1(0.07)^1(0.93)^9$

or (0 from A, 2 from B) $= (0.95)^{10}\cdot{}^{10}C_2(0.07)^2(0.93)^8$

$\therefore$ Total Prob $= 0.036 + 0.115 + 0.074$
$= 0.225$ approx. $\boxed{4}$

b) i)



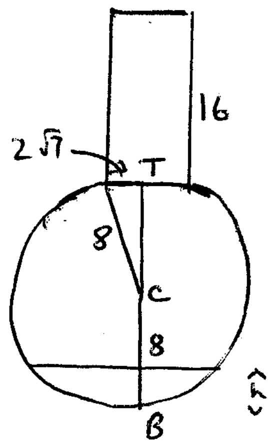
$$V\Omega = \pi\int_{r-h}^{r} y^2\,dx$$

$$= \pi\int_{r-h}^{r} r^2 - x^2\,dx = \pi\left[r^2x - \frac{x^3}{3}\right]_{r-h}^{r}$$

$$= \pi\left(\frac{2r^3}{3} - \left\{r^2(r-h) - \frac{(r-h)^3}{3}\right\}\right)$$

$$= \frac{\pi h^2}{3}(3r-h) \quad \boxed{3}$$

7 b) ii)



$$CT = \sqrt{64-28} = 6\text{ cm}$$

$\therefore$ Total height $= 8+6+16$
$= 30$ cms. $\boxed{1}$

iii) $$V = 8\pi h^2 - \frac{\pi h^3}{3}$$

$$\therefore \frac{dV}{dt} = 16\pi h - \pi h^2\ \frac{dh}{dt}$$

But $\frac{dV}{dt} = 100$

$$\therefore \frac{dh}{dt} = \frac{100}{\pi h(16-h)} \text{ cm/sec} \quad \boxed{2}$$

iv) When $h = 14$, $\frac{dh}{dt} = \frac{100}{28\pi}$ $\boxed{1}$
$= 1.14$ cm/sec

At this rate it will take
$\frac{16}{1.14}$ secs $\doteqdot$ 14 seconds $\boxed{1}$