Name:	
Class:	

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION 2012

MATHEMATICS EXTENSION 1

General Instructions:

- Reading Time: 5 minutes.
- Working Time: 2 hours.
- Write in black or blue pen.

- Board approved calculators & templates may be used
- A Standard Integral Sheet is provided.

- In every question, show all necessary working
- Marks may not be awarded for careless or badly arranged working.

Total Marks 70

Section I: **10 marks**

- Attempt Question 1 – 10.
- Answer on the Multiple Choice answer sheet provided.
- Allow about 15 minutes for this section.

Section II: **60 Marks**

- Attempt Question 11 - 14
- Answer on blank paper unless otherwise instructed. Start a new page for each new question.
- Allow about 1 hours & 45 minutes for this section.

The answers to all questions are to be returned in separate *stapled* bundles clearly labelled Question 11, Question 12, etc. Each question must show your Candidate Number.

SECTION I MULTIPLE CHOICE (10 marks)

Attempt Question 1 – 10 (1 mark each)
Allow approximately 15 minutes for this section

1 A bowl of soup at temperature T°C, when placed in a cooler environment, loses heat according to the law $\frac{dT}{dt} = k(T - T_0)$ where t is the time elapsed in minutes and T_0 is the temperature of the environment in degrees Celsius. A bowl of soup at 96°C is left to stand in a room at a temperature of 18°C. After 3 minutes the soup cools down to 75°C. What is the value of k correct to 4 decimal places?

(A) −0.0784
(B) −0.0856
(C) −0.1046
(D) −0.1236

2 Which of the following is an expression for $\int \cos^2 2x \, dx$?

(A) $x - \frac{1}{4}\sin 4x + c$
(B) $x + \frac{1}{4}\sin 4x + c$
(C) $\frac{x}{2} - \frac{1}{8}\sin 4x + c$
(D) $\frac{x}{2} + \frac{1}{8}\sin 4x + c$

3 The velocity of a particle moving in a straight line is given by $v = 2x + 3$ where x metres is the distance from fixed point O and v is the velocity in metres per second. What is the acceleration of the particle when it is 4 metres from O?

(A) $a = 11\text{ms}^{-2}$
(B) $a = 19\text{ms}^{-2}$
(C) $a = 22\text{ms}^{-2}$
(D) $a = 23.5\text{ms}^{-2}$

4 Which of the following is an expression for $\int x\sqrt{1 - x^2}dx$?
Use the substitution $u = 1 - x^2$.

(A) $\frac{-(1-x^2)^3}{3} + c$
(B) $\frac{(1-x^2)^3}{3} + c$
(C) $\frac{-(1-x^2)^{\frac{3}{2}}}{3} + c$
(D) $\frac{(1-x^2)^{\frac{3}{2}}}{3} + c$

5 What are the solutions to the equation $e^{6x} - 7e^{3x} + 6 = 0$?

(A) $x = 1$ and $x = 6$

(B) $x = 0$ and $x = \frac{\ln 6}{2}$

(C) $x = 0$ and $x = \frac{\ln 6}{3}$

(D) $x = 1$ and $x = \frac{\ln 6}{2}$

6 A particle moving in a straight line obeys $v^2 = -x^2 + 2x + 8$ where x is its displacement from the origin in metres and v is its velocity in ms^{-1}. The motion is simple harmonic. What is the amplitude?

(A) 2π metres

(B) 3 metres

(C) 8 metres

(D) 9 metres

7 How many distinct permutations of the letters of the word 'ATTAINS' are possible in a straight line when the word begins and ends with the letter T?

(A) 60

(B) 120

(C) 360

(D) 1260

8 If $f(x) = e^{x+2}$ what is the inverse function $f^{-1}(x)$?

(A) $f^{-1}(x) = e^{y-2}$

(B) $f^{-1}(x) = e^{y+2}$

(C) $f^{-1}(x) = \log_e x - 2$

(D) $f^{-1}(x) = \log_e x + 2$

9 What is the coefficient of x^5 in the expansion of $(1 - 3x + 2x^3)(1 - 2x)^6$?

(A) -792

(B) -720

(C) 120

(D) 312

10 A die is tossed 3 times. What is the probability of 0 or 1 six turning up?

(A) $\frac{2}{27}$

(B) $\frac{25}{27}$

(C) $\frac{91}{216}$

(D) $\frac{125}{216}$

SECTION II EXTENDED RESPONSE (60 marks)

Total Marks is 60
Attempt Question 11 – 14.
Allow approximately 1 hour & 45 minutes for this section.

Answer all questions, starting each new question on a new sheet of paper with your **student ID number** in the top right hand corner and the question number on the left hand side of your paper. All necessary working must be shown in each and every question.

<u>QUESTION 11</u> *(15 Marks)* **Marks**

(a) Solve $\frac{4}{3x+1} < 5$. **2**

(b) If α, β and γ are the roots of the equation $x^3 + 2x^2 - 3x - 5 = 0$, **2**
find the value of $\frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\alpha\gamma}$.

(c) Use the substitution $x = u^2 + 1$ for $u > 0$ to evaluate the integral: **4**

$$\int_1^5 (x+1)\sqrt{x-1}\,dx$$

(d) A series is given by $1 + \frac{1-p}{p} + \left(\frac{1-p}{p}\right)^2 + \cdots$, where p is positive.

(i) Find the domain of p such that the series has a sum to infinity. **2**

(ii) Find this sum to infinity in terms of p. **1**

(e) Prove that the tangent to a parabola $x^2 = 4ay$ at a given point $P(2ap, ap^2)$ is equally inclined to the axis of the parabola and the focal chord through the point. **4**

QUESTION 12 *(15 Marks)* START A NEW PAGE **Marks**

(a) Solve: $x^3 + 2x^2 - 5x - 6 = 0$ **2**

(b) When the polynomial $P(x)$ is divided by $x^2 - 1$, the remainder is $3x + 1$. **2**
What is the remainder when $P(x)$ is divided by $x + 1$?

(c) In how many ways can 4 men and 4 women be arranged around a circular table if:

(i) All women sit together? **2**

(ii) All the men are in pairs separated by two pairs of women? **2**

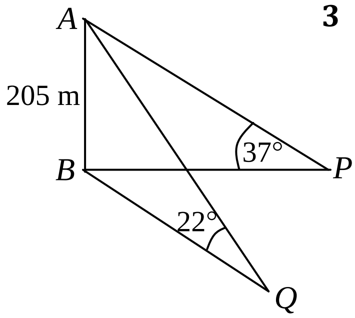
(d) Find the general solution to: $\cos 5\theta - \cos 2\theta = 0$ **2**

(e) A thin-walled cone-shaped cup is to hold 36π cm^3 of water when full. **5**
What dimensions will minimize the amount of material needed for the cup?
[You may make use of the formula $A = \pi rs$, where s is the slant height of a cone]

QUESTION 13 *(15 Marks)* START A NEW PAGE **Marks**

(a) A is 205 metres above the horizontal plane BPQ. AB is vertical. The angle of elevation of A from P is $37°$ and the angle of elevation of A from Q is $22°$. P is due East of B and Q is South $47°$ East from B. **3**

Calculate the distance from P to Q, to the nearest metre.



NOT TO SCALE

(b) Use mathematical induction to show: **4**

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\ldots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}$$

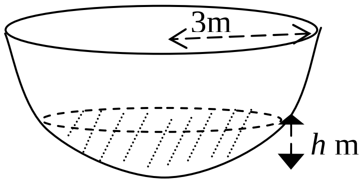
for $n \geq 2$ where n is an integer.

(c) A particle moves in SHM on a horizontal line and its acceleration is $\frac{d^2x}{dt^2} = 36 - 9x$, where x is the displacement after t seconds.

(i) Find the centre of its motion. **1**

(ii) If the particle is initially at rest at $x = 6$, find the amplitude. **1**

(d) A hemi-spherical bowl has a radius of 3m. Oil is poured in at a constant rate of $\frac{\pi}{3}$ m^3/min.



NOT TO SCALE

(i) Show that, when the depth of the oil is h metres, the volume of oil is: **2**

$$V = \frac{\pi}{3}(9h^2 - h^3)\text{m}^3$$

(ii) How deep is the oil after 8 minutes? **2**

(iii) At what rate is h increasing at this time? **2**

QUESTION 14 *(15 Marks)* START A NEW PAGE **Marks**

(a) A particle is moving in a straight line and its position x, in metres, from the origin O at time t seconds is given by

$$x = 3\cos 2t + 4\sin 2t + 2.$$

(i) Express $3\cos 2t + 4\sin 2t$ in the form **2**

$$R\cos(2t - \alpha) \quad \text{where } 0 < \alpha < \tfrac{\pi}{2} \text{ and } R > 0.$$

(ii) Prove that the particle is undergoing simple harmonic motion. **2**
Find the amplitude of the motion.

(iii) Find the maximum speed of the particle. **2**
When does the particle **first** reach this maximum speed?
Provide your answer to 2 decimal places.

(b) Given the binomial expansion of

$$(1 + x)^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \text{ and}$$

$$(1 + x)^{n+1} = b_0 + b_1 x + b_2 x^2 + \cdots + b_{n+1} x^{n+1}$$

(i) Find the relationship for co-efficient b_k in terms of a_r. **1**

(ii) Hence find the expression, in terms of n only, of: **3**

$$\frac{1}{a_0 a_1 \ldots a_n} \times (a_0 + a_1)(a_1 + a_2) \ldots (a_{n-1} + a_n) \qquad \text{for } n = 1, 2, 3 \ldots$$

(c) (i) Show that $\tan^{-1}(n + 1) - \tan^{-1}(n - 1) = \tan^{-1}\left(\frac{2}{n^2}\right)$ for $n \geq 1$. **2**

(ii) Hence or otherwise show that: **3**

$$\sum_{r=1}^{n} \tan^{-1}\left(\frac{2}{r^2}\right) = \tan^{-1}\left(\frac{2n + 1}{1 - n - n^2}\right) + \frac{3\pi}{4}$$

END OF PAPER

M. EXTENSION 1 TRIAL 2012

Section I

10 Marks
Attempt Question 1 – 10.
Allow approximately 15 minutes for this section.

Use the multiple choice answer sheet below to record your answers to Question 1 – 10.

Select the alternative: A, B, C or D that best answers the question.

Colour in the response oval completely.

Sample:

$2 + 4 = ?$ (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, draw a cross through the incorrect answer and colour in the new answer

ie A ● B ● C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate this by writing the word "*correct*" and draw an arrow as follows:

correct

A ● B ● C ○ D ○

Trial HSC Examination
Mathematics Extension 1, 2012

Multiple Choice Answer Sheet

Student id number: | | | | |

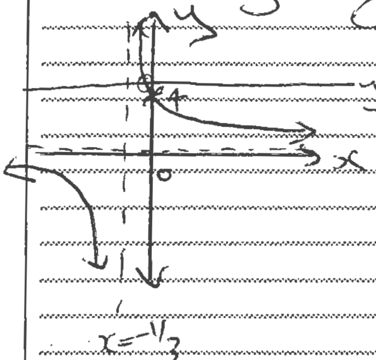
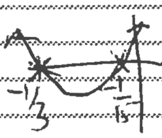
Completely colour in the response oval representing the most correct answer.

1	A	○	B	○	C	●	D	○
2	A	○	B	○	C	○	D	●
3	A	○	B	○	C	●	D	○
4	A	○	B	○	C	●	D	○
5	A	○	B	○	C	●	D	○
6	A	○	B	●	C	○	D	○
7	A	●	B	○	C	○	D	○
8	A	○	B	○	C	●	D	○
9	A	●	B	○	C	○	D	○
10	A	○	B	●	C	○	D	○

Mark: /10

1/5

MATHEMATICS Extension 1 : Question 11

Suggested Solutions	Marks	Marker's Comments
(a) $\frac{4}{3x+1} < 5$		
could have been done using a graph or by algebra.		
 pt. of intersection: $\frac{4}{3x+1} = 5$, $4 = 15x+5$, $-1 = 15x$, $x = -\frac{1}{15}$		* ½ mk off if the signs were wrong
$\therefore\ x > -\frac{1}{15}$ or $x < -\frac{1}{3}$		* If you only got $x > -\frac{1}{15}$, you scored one mk only.
or using algebra: $(3x+1)^2 \frac{4}{3x+1} < 5\,(3x+1)^2$		*A lot of students were confused with the signs!
METHOD II: $4(3x+1) < 5(9x^2+6x+1)$		
$12x+4 < 45x^2+30x+5$		
$-45x^2-18x-1 < 0$		
$45x^2+18x+1 > 0$		
$(15x+1)(3x+1) > 0$		
 $\therefore\ x < -\frac{1}{3}$ or $x > -\frac{1}{15}$		
(b) $\alpha+\beta+\gamma = -b/a = -2$	½	
$\alpha\beta\gamma = -d/a = 5$	½	
$\frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\alpha\gamma} = \frac{\alpha+\beta+\gamma}{\alpha\beta\gamma}$	½	
$= -\frac{2}{5}$	½	
(c) $x = u^2+1$		
$\frac{dx}{du} = 2u$, $dx = 2u\,du$	½	

2/5

MATHEMATICS Extension 1 : Question 11

Suggested Solutions	Marks	Marker's Comments
when $x=1$ $u=0$, $x=5$ $u=2$	½	
$\int_1^5 (x+1)\sqrt{x-1}\,dx = \int_0^2 (u^2+1+1)\sqrt{u^2+1-1}\,.2u\,du$		½ mk
$= \int_0^2 (u^2+2)u.2u\,du$		½
$= 2\int_0^2 (u^4+2u^2)\,du$		½
$= 2\left[\frac{u^5}{5}+\frac{2u^3}{3}\right]_0^2$		½
$= 2\left(\frac{32}{5}+\frac{2\times 8}{3}-0\right)$		½
$= \frac{64}{5}+\frac{32}{3}$		* Answers of "$\frac{64}{5}$" or "8" → 3 mks max.
$= \frac{352}{15}$		
$= 23\frac{7}{15}$	½	
(d) (i) For there to be a sum to infinity		
$\lvert r\rvert < 1$ ie $-1 < r < 1$ $\therefore -1 < \frac{1-p}{p} < 1$	1 mk	* lost 1 mk if they wrote $0<r<1$ or if they wrote $r<1 \Rightarrow$ 1 mk.
$-1 < \frac{1-p}{p}$, $-p < 1-p$, no soln. ½ mk or $\frac{1-p}{p} < 1$, $1-p < p$, $1 < 2p$, $\frac{1}{2} < p$, $\therefore p > \frac{1}{2}$	½ mk.	* lost one mark if they squared the absolute value expression or $\left\lvert\frac{1-p}{p}\right\rvert < 1$, $\frac{\lvert 1-p\rvert}{\lvert p\rvert} < 1$, $p \neq 0$, $\lvert 1-p\rvert < \lvert p\rvert$ etc
$p > 0$ (data) $\therefore p > \frac{1}{2}$		

3/5

MATHEMATICS Extension 1 : Question 11

Suggested Solutions	Marks	Marker's Comments

d) (ii) $S_\infty = \dfrac{a}{1-r}$

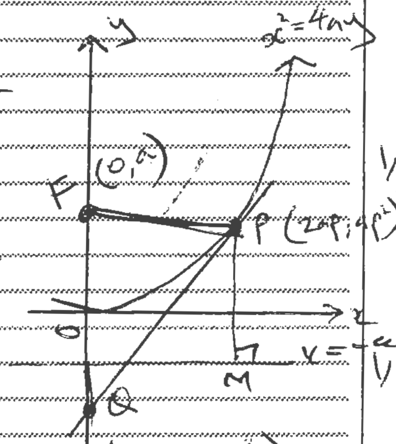
$= \dfrac{1}{1-\frac{1-p}{p}}$ — ½ — 0 mks if they got the formula wrong!!

$= \dfrac{p}{p-(1-p)}$

$= \dfrac{p}{p-1+p}$

$= \dfrac{p}{2p-1}, \quad p > \frac{1}{2}$ — ½

(e) METHOD ONE



when $x = 2ap$, $y = ap^2$

$\dfrac{dy}{dx} = p$ — ½

eqn. of tangent is:

$y - ap^2 = p(x - 2ap)$

$y = px - ap^2$ — ½

y intercept is Q, Q is $(0, -ap^2)$

$\therefore d_{FQ} = ap^2 + a$

$= a(p^2+1)$ — ½

$d_{FP} = \sqrt{(2ap-0)^2 + (ap^2-a)^2}$

$= \sqrt{4a^2p^2 + a^2p^4 - 2a^2p^2 + a^2}$ — ½

$d_{FP} = d_{PM}$ focus directrix defn

$FP = PM$
$= ap^2 - (-a)$
$= ap^2 + a$

$= \sqrt{a^2p^4 + 2a^2p^2 + a^2}$

$= a\sqrt{p^4 + 2p^2 + 1}$

$= a\sqrt{(p^2+1)^2}$

$= a(p^2+1)$ — ½

$\therefore d_{FP} = d_{FQ}$ (both $a(p^2+1)$) — ½

\\CALLISTO\StaffHome$\WOH\JRAH M Fac Admin\Assessment info\Suggested Mk solns template_V4_half Ls.doc

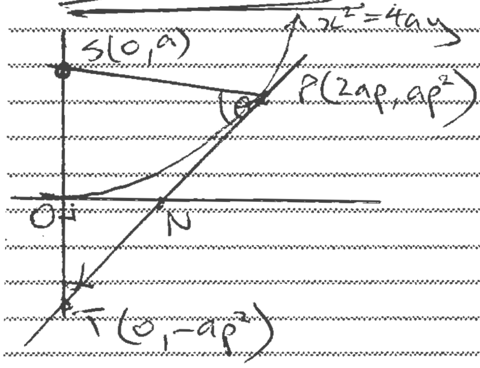
MATHEMATICS Extension 1 : Question 11

Suggested Solutions	Marks	Marker's Comments

$\therefore \angle FQP = \angle FPQ$ (equal angles are opposite equal sides in $\triangle FQP$) — $\frac{1}{2}$

$\therefore$ The tangent is equally inclined to the focal chord and the axis $\}\ \frac{1}{2}$

METHOD 2:



$\frac{1}{2}$ mk for a decent diagram with data marked on it!!

gradient of tangent $= p$

gradient of focal chord $= \dfrac{ap^2 - a}{2ap - 0}$

$= \dfrac{a(p^2-1)}{2ap}$

$= \dfrac{p^2-1}{2p}$

$\frac{1}{2}$ mk for both correct gradients

In $\triangle SPT$,

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$= \left|\frac{p - \left(\frac{p^2-1}{2p}\right)}{1 + \frac{p(p^2-1)}{2p}}\right| \quad \tfrac{1}{2}$$

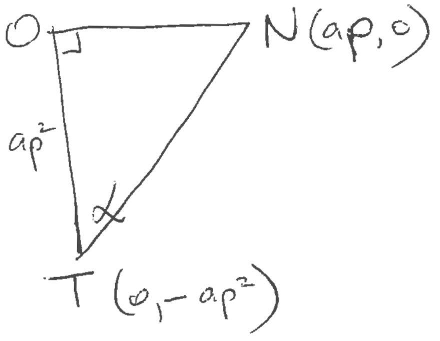
$$= \left|\frac{2p^2 - (p^2-1)}{2p + p(p^2-1)}\right|$$

$$= \left|\frac{p^2+1}{p^3 + 2p - p}\right|$$

$$= \left|\frac{p^2+1}{p^3+p}\right| \quad \tfrac{1}{2}$$

$$= \left|\frac{p^2+1}{p(p^2+1)}\right| = \frac{1}{p} \quad \tfrac{1}{2}$$

the gradient of the y axis is undefined



In $\triangle ONT$,

$$\tan\alpha = \frac{ap}{ap^2}$$

$$\therefore \tan\alpha = \frac{1}{p}$$

1mk

$$\therefore \tan\alpha = \tan\theta = \frac{1}{p}$$

conclusion ___________

½mk

* Fudging resulted in a maximum of 2mks.
* students were making statements without any justifications, no evidence $\therefore$ they lost marks!!!

MATHEMATICS Extension 1 : Question 12

Suggested Solutions	Marks	Marker's Comments

(a) Let $P(x) = x^3 + 2x^2 - 5x - 6$

possible integer/rational zeros $\pm1, \pm2, \pm3, \pm6$

TEST: $x-1$: $P(1) = 1+2-5-6 = -8 \neq 0$

$x+1$: $P(-1) = -1+2+5-6 = 0$ — $\frac{1}{2}$

$\therefore x+1$ is a factor // $x=-1$ is a root (Factor Thm)

$$\begin{array}{r} x^2 + x - 6 \\ x+1 \enclose{longdiv}{x^3 + 2x^2 - 5x - 6} \\ \underline{x^3 + x^2} \\ 0 \quad x^2 - 5x \\ \underline{x^2 + x} \\ 0 \quad -6x - 6 \\ \underline{-6x - 6} \end{array}$$

$$\begin{array}{r|rrrr} -1 & 1 & 2 & -5 & -6 \\ & & -1 & -1 & 6 \\ \hline & 1 & 1 & -6 & 0 \end{array}$$ — $\frac{1}{2}$

$\boxed{2}$

$\therefore (x+1)(x^2+x-6) = (x+1)(x+3)(x-2) = 0$ — $\frac{1}{2}$ — $\frac{1}{2}$ For either -3 or 2; $-\frac{1}{2}$ if $+1$ or $+3$

$\therefore$ $\underline{x = -3, -1 \text{ or } 2}$ — $\frac{1}{2}$

(b) $P(x) = A(x)\,Q(x) + R(x)$ — $\boxed{2}$

$= (x^2-1)\,Q(x) + 3x+1$ — $\frac{1}{2}$

$x+1$ $\quad P(-1) = 0 \times Q(-1) + 3(-1) + 1$ — $\frac{1}{2}$

$P(-1) = -2$ — $\frac{1}{2}$

$\therefore$ the remainder is -2 when divide by $x+1$ (Rem. Thm) — $\frac{1}{2}$

(c) (i)

4M together and 4W together

$n(E) = \overset{M}{4!} \times \overset{W}{4!} = (4!)^2$ — 1

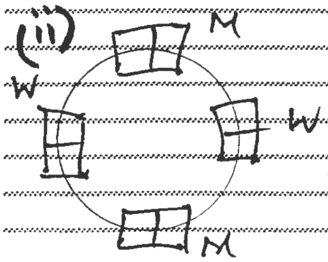
$= \underline{576 \text{ ways}}$ — (1)

$\frac{2!}{2!} \times 4! \times 4!$ $\quad \boxed{2}$

$\frac{5!}{5} \times 4!$

$(4W, M_1, M_2, M_3, M_4)$

(ii)



Method I : $(\overset{M_1 \text{ other } M}{2 \times 3!}) \times \overset{W}{4!}$

$n(E) = (2 \times 3!) \times 4! = \underline{288}$

1 For '$2 \times 3!$' or equiv.; $\frac{1}{2}$ $4!$ ✓; $\frac{1}{2}$ 288

Method II

$n(E) = \left(\frac{{}^4C_2}{2!} \times 2!\right) \times 4! = \underline{288}$ — $\boxed{2}$

1^{st} 2M other

METHOD III $\quad n(E) = 3 \times 2! \times 2! \times \overset{W}{4!} = \underline{288}$

who pair with M

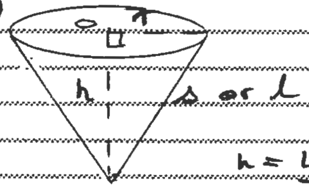
Method IV $\quad n(E) = 2 \times 2^4 \times 3 \times 3 = \underline{288}$

opposites in 2 — each group — W — M

Very UNCLEAR ARGUMENTS by students

144 576 common but ....

MATHEMATICS Extension 1 : Question 12

Suggested Solutions	Marks	Marker's Comments
(d) $\therefore \cos 5\theta = \cos 2\theta$		Expansions ☺ $\boxed{2}$
METHOD I METHOD II		
$5\theta = 2m\pi \pm 2\theta$ $2\theta = 2n\pi \pm 5\theta$		$\frac{1}{2}$
$\theta = \begin{cases} \frac{2m\pi}{7} \\ \frac{2m\pi}{3} \end{cases}, m \in \mathbb{Z}$ $\theta = \begin{cases} \frac{2n\pi}{7} \\ -\frac{2n\pi}{3} \end{cases}; n \in \mathbb{Z}$		$\frac{1}{2}$ each
(e)  $V = \frac{1}{3}\pi r^2 h \equiv 36\pi$		
$r^2 h = 108$		$108^2 = 11\,664$
$h = \frac{108}{r^2}$	$\frac{1}{2}$	
$s^2 = r^2 + h^2$ (Pyth. Thm)		$s^2 = \frac{108}{h} + h^2 = \frac{108 + h^3}{h}$
$s^2 = r^2 + \frac{108^2}{r^4} = \frac{r^6 + 108^2}{r^4}$		$s = \sqrt{h^2 + \frac{108}{h}} = \frac{\sqrt{h^3 + 108}}{\sqrt{h}}$
$s = \frac{\sqrt{r^6 + 108^2}}{r^2}$	$\frac{1}{2}$	
Now $A(r) = \pi r s$	$(\frac{1}{2})$	
$= \pi r \frac{\sqrt{r^6 + 108^2}}{r^2}$		$A = \pi\sqrt{\frac{108}{h}}\sqrt{h^2 + \frac{108}{h}}$
$A(r) = \frac{\pi}{r}\sqrt{r^6 + 108^2}$	1	$A(h) = \pi\sqrt{108h + \frac{108^2}{h^2}}$
$\frac{dA}{dr} = \pi\left[-\frac{1}{r^2}\sqrt{r^6 + 108^2} + \frac{1}{r}\frac{6r^5}{2\sqrt{r^6 + 108^2}}\right]$	1	$A'(h) = \frac{\pi}{2\sqrt{\cdots}}\left(108 - \frac{2 \times 108^2}{h^3}\right)$
$= \pi\left[\frac{3r^4}{\sqrt{r^6 + 108^2}} - \frac{\sqrt{r^6 + 108^2}}{r^2}\right]$		$= \frac{\pi}{\sqrt{\cdots}}\left(54 - \frac{108^2}{h^3}\right)$
For possible max/min value of A to occur $\frac{dA}{dr} = 0$		
$\therefore \frac{3r^4}{\sqrt{r^6 + 108^2}} = \frac{\sqrt{r^6 + 108^2}}{r^2}$		$54 = \frac{108^2}{h^3}$
$3r^6 = r^6 + 108^2$		$h^3 = \frac{108^2}{54} = 216$
$2r^6 = 108^2 = 11\,664$		
$r^6 = 5832 = 54 \times 108 = 2^3 \times 3^6$	$\frac{1}{2}$	$h = 6, (h > 0)$
$r = \sqrt[6]{5832} = 3\sqrt{2} = 4.24\ldots$		
as $r > 0$		

\\\\CALLISTO\\StaffHome$\\WOH\\JRAH M Fac Admin\\Assessment info\\Suggested Mk solns template_V4_half Ls.doc

$r = 4.24 \quad A' = -0.06$

MATHEMATICS Extension 1 : Question 12

Suggested Solutions	Marks	Marker's Comments

CONT:

(e) $r = 3\sqrt{2} = \sqrt{18} = 4.24\ldots$

$\therefore h = \dfrac{108}{r^2} = \dfrac{108}{18} = 6 \quad ; \quad [A = 18\pi\sqrt{3}]$

$\boxed{5}$

TEST NATURE at $r = 3\sqrt{2} = 4.24\ldots$

r	4	4.1	4.2	4.24...	4.25	4.3	4.5	5
$\dfrac{dA}{dr}$	−5.43 ⊖ \	−3.16 ⊖ \	−0.93 ⊖ \	0 —	0.16... ⊕ /	1.24 ⊕ /	5.43 ⊕ /	14.90 ⊕ /

$\frac{1}{2}$ each

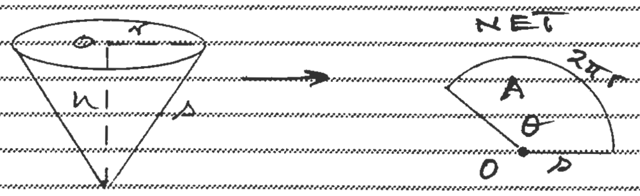
$\therefore$ a Relative min T.P at $r = 3\sqrt{2}$

And since there exist only 1TP for $r > 0$

$\therefore$ absolute min Area occurs when dimensions $3\sqrt{2}$ cm for radius and 6 cm for height

$r = 3\sqrt{2}$
$h = 6$ $\quad \frac{1}{2}$

NOTE: Background Y10



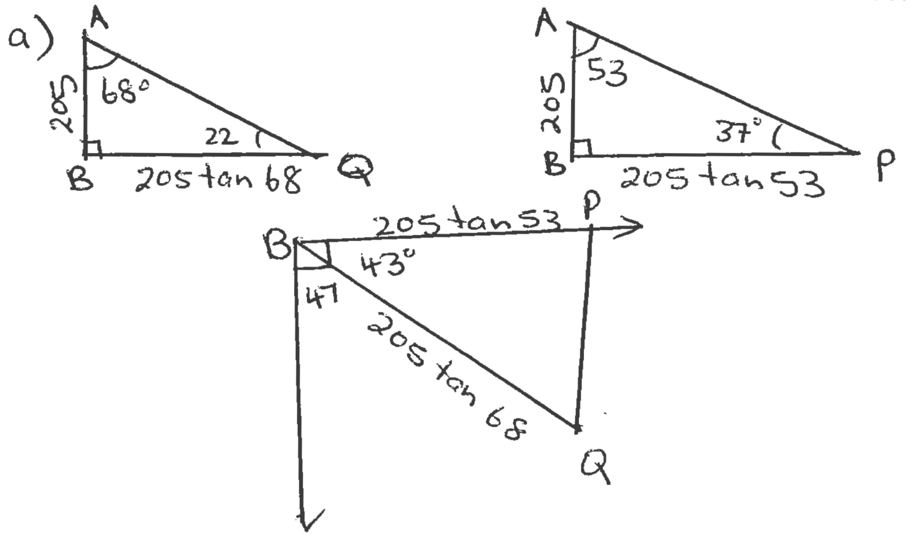
$\dfrac{\theta}{2\pi} = \dfrac{2\pi r}{2\pi s} = \dfrac{A}{\pi s^2}$

$\therefore \theta = \dfrac{2\pi r}{s}$

and $A = \frac{1}{2}s^2\theta = \frac{1}{2}s^2 \times \dfrac{2\pi r}{s}$

$A = \pi r s$ [Y10 recall formula]

MATHEMATICS Extension 1 : Question 13 Pg 1

Suggested Solutions	Marks	Marker's Comments
a) 		Alternatively BP could also be. $BQ = \dfrac{205}{\tan 22}$ $BP = \dfrac{205}{\tan 37}$
$PQ^2 = (205\tan 53)^2 + (205\tan 68)^2 - 2(205^2 \tan 53 \tan 68)\cos 43°$ $= 205^2(\tan^2 53 + \tan^2 68 - 2\tan 53 \tan 68 \cos 43$ $PQ = \sqrt{205^2(\tan^2 53 + \tan^2 68 - 2\tan 53 \tan 68 \cos 43}$ $= 359.9350248 \ldots$ $= 360\text{m}$ (nearest metre).		✓ 1 mark for getting Lengths of BP and BQ ✓ 1 mark calculating angle PBQ as 43° ✓ 1 mark for correct formula and working
b) $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right) \cdots\cdots \left(1-\frac{1}{n^2}\right) = \frac{n+1}{2n}$ Step 1 Prove true for $n=2$: $P(2)$ LHS $1-\frac{1}{2^2} = 1-\frac{1}{4} = \frac{3}{4}$ RHS $\frac{2+1}{2\times 2} = \frac{3}{4}$ = LHS $\therefore$ Statement is true for $n=2$.		✓ 1 mark step 1 $\frac{1}{2}$ mark for steps 2 ~~+ 3~~ 2 step 3 Prove LHS = RHS $\frac{1}{2}$ mark for conclusion statement must mention true for all integers $n \geq 2$ or PMI.

J:\Maths\Suggested Mk solns template_V2_no Ls.doc

MATHEMATICS Extension 1 : Question 13

Suggested Solutions	Marks	Marker's Comments

Step 2 : Assume true for $n=k$.

$$\text{ie } \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\text{------}\left(1-\frac{1}{k^2}\right)=\frac{k+1}{2k}.$$

Marks: $\frac{1}{2}$.

Step 3 prove true for $n=k+1$

$$\text{ie } \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\text{------}\left(1-\frac{1}{k^2}\right)\left(1-\frac{1}{(k+1)^2}\right)=\frac{k+2}{2k+1}$$

Marker's Comments: Some students had this step missing

LHS $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\text{-----}\left(1-\frac{1}{k^2}\right)\left(1-\frac{1}{(k+1)^2}\right)$

$$=\left(\frac{k+1}{2k}\right)\left(1-\frac{1}{(k+1)^2}\right)\text{(by assumption)}$$

Marker's Comments: $-\frac{1}{2}$ mark if assumption is not mentioned

$$=\left(\frac{k+1}{2k}\right)\left(\frac{(k+1)^2-1}{(k+1)^2}\right)$$

$$=\frac{k^2+2k+1-1}{2k(k+1)}$$

$$=\frac{k^2+2k}{2k(k+1)}$$

$$=\frac{k^2+2k}{2k(k+1)}$$

$$=\frac{k+2}{2(k+1)} \quad = RHS$$

Step 4 statement is true for $n=k+1$ if assumed true for $n=k$. Since the statement has been proven true for $n=2$, it is true for all integers $n\geq 2$.

or $\therefore$ by the PMI $P(n)$ is ~~true~~ for $n=2,3,4\ldots$

MATHEMATICS Extension 1 : Question 13... Pg 3

Suggested Solutions	Marks	Marker's Comments

c (i) $\dfrac{d^2x}{dt^2} = 36 - 9x$

compare $\ddot{x} = -n^2(x - x_0)$ ↑centre.

$\ddot{x} = -9(x - 4)$

$\therefore$ centre is $x = 4$ — ✓ 1 mark

ii) $\dfrac{d}{dx}\left(\frac{1}{2}v^2\right) = 36 - 9x$

$\frac{1}{2}v^2 = \int (36 - 9x)\,dx$

$\dfrac{v^2}{2} = 36x - \dfrac{9x^2}{x} + c.$

when $x = 6$, $v = 0$

$0 = 216 - 162 + c$

$\therefore c = -54$

Equation is $\dfrac{v^2}{2} = 36x - \dfrac{9x^2}{2} - 54$

$v^2 = 72x - 9x^2 - 108$

$= 9(8x - x^2 - 12)$

$= -9(x^2 - 8x + 12)$

$v^2 = -9((x-4)^2 - 4)$

$= -9(4 - (x-4)^2)$

Compare $v^2 = n^2(a^2 - (x - x_0)^2)$

$a^2 = 4$

$a = 2.\ (a > 0).$

OR.

write it in the form

$x = b + a\cos nt + c$

↖ centre

so $x = 4 + a\cos 3t$ (from part (i) $n^2 = 9$ so $n = 3$)

when $t = 0$, $x = 6$

$6 = 4 + a\cos 3t$

$a\cos 3t = 2.$

$a.(1) = 2$

$a = 2$

Marks: 1 mark

Marker's comments:

For 1 mark be careful.

If students got $6 - 4 = 2$ only they must justify and mention particle is at end point

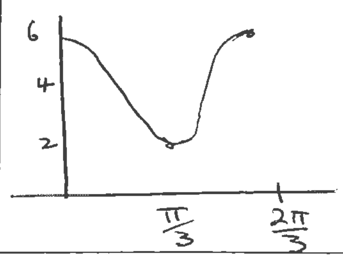
$v^2 = n^2[a^2 - (x - x_0)^2]$

$v^2 = 9[a^2 - (x-4)^2]$

$x = 6$ $v = 0$

Leads to $a = 2$

Because it is initially at rest and at end of motion $c = 0$

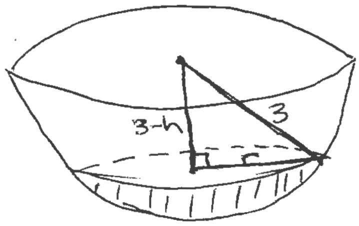


MATHEMATICS: Question 13

Pg 4

Suggested Solutions	Marks Awarded	Marker's Comments

d) i)



Using pythagoras

$$3^2 = r^2 + (3-h)^2$$

$$r^2 = 9 - (3-h)^2$$

$$= 9 - (9 - 6h + h^2)$$

$$= 6h - h^2$$

$$\text{Volume} = \pi \int_0^h (6h - h^2)\, dh \quad ✓$$

$$= \pi \left[3h^2 - \frac{h^3}{3} \right]_0^h$$

$$= \pi \left(3h^2 - \frac{h^3}{3} \right)$$

$$✓ = \frac{\pi}{3}\left(9h^2 - h^3\right) \text{ as required } ✓$$

$\therefore$ volume is $\frac{\pi}{3}(9h^2 - h^3)\ m^3$

Alternatively

Rotated $x^2 + y^2 = 9$ around x axis between $x = 3$ and $x = 3 - h$.

$$\therefore y^2 = 9 - x^2$$

$$\text{Volume} = \pi \int_{3-h}^{3} (y^2)\, dx$$

$$= \pi \int_{3-h}^{3} (9 - x^2)\, dx \quad ✓$$

$$= \left[9x - \frac{x^3}{3} \right]_{3-h}^{3}$$

$$= \pi \left[(27 - 9) - \left\{ 9(3-h) - \frac{1}{3}(3-h)^3 \right\} \right]$$

$$= \pi \left(3h^2 - \frac{h^3}{3} \right)$$

$$= \frac{\pi}{3}\left(9h^2 - h^3\right)\ m^3 \quad ✓$$

Marks Awarded: 2 marks

Marker's Comments:

$x^2 = 9 - y^2$

$\therefore V = \pi \int_0^h (9 - y)^2$

is incorrect. limits should be 3 and 3-h.

If recognised circle is rotated with correct limits

$\pi \int_{3-h}^{3}$ —— dx

is 1 mark.

and 1 mark for correct subsequent working without making the question simpler.

$$\pi\left[18 - 9\left\{\frac{(3-h)}{3}\left(27 - (3-h)^2\right)\right\}\right]$$

$= \pi($

$=$

$=$

MATHEMATICS: Question 13

Suggested Solutions	Marks Awarded	Marker's Comments
d(ii) After 8 minutes, $V = \frac{8\pi}{3}$ $\therefore \frac{8\pi}{3} = \frac{\pi}{3}(9h^2 - h^3)$ $8 = 9h^2 - h^3$ $h^3 - 9h^2 + 8 = 0$ $(h-1)(h^2-8h-8) = 0$	→ 1 mark.	$h-1 \overline{)h^3 - 9h^2 + 0h + 8}$ quotient $h^2 - 8h - 8$ $h^3 - h^2$ $-8h^2 + 0h$ $-8h^2 + 8h$ $-8h + 8$ $-8h + 8$ 0
$h - 1 = 0$ or $h^2 - 8h - 8 = 0$ $\therefore h = 1$ $\frac{8 \pm \sqrt{64-(-32)}}{2}$ $= \frac{8 \pm \sqrt{96}}{2}$ $= 4 \pm 2\sqrt{6}$ as $0 \le h \le 3$ $h = 1$ is only ~~th~~ solution	→ $\frac{1}{2}$ mark → ½ mark	Some ended up with $h^2 - 8$ instead of $h^2 - 8h - 8$ Some tested $h = 1$ on $h^3 - 9h^2 + 8$ only. Did not mention other solutions, not working
d(iii) $\frac{dV}{dt} = \frac{\pi}{3}$ (given) $\frac{dV}{dh} = \frac{\pi}{3}(18h - 3h^2)$ $= \frac{\pi}{3}(6h - h^2)$ $\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$ $= \frac{1}{\pi(6h-h^2)} \times \frac{\pi}{3}$ ✓ 1 mark. $\frac{dh}{dt} = \frac{1}{3(6h-h^2)}$ when $h = 1$ $\frac{dh}{dt} = \frac{1}{3(6-1)} = \frac{1}{15}$ ✓ 1 mark. $\therefore$ when oil is 1 m deep, h is increasing at a rate of $\frac{1}{15}$ metres/min.		$\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}$ $\frac{\pi}{3} = \frac{\pi}{3}(6h - h^2)\frac{dh}{dt}$ $\Rightarrow \frac{dh}{dt} = \frac{1}{3(6h-h^2)}$ Some forgot to invert $\frac{dV}{dh} = \pi(6h - h^2)$ $\therefore \frac{dh}{dV} = \frac{1}{\pi(6h-h^2)}$ Some calculated the first few steps in part (ii).

MATHEMATICS Extension 1 : Question 14 V2

Suggested Solutions	Marks	Marker's Comments
a) $x = 3\cos 2t + 4\sin 2t + 2$		
(i) $3\cos 2t + 4\sin 2t = R\cos(2t-\alpha)$ $R>0$, $0<\alpha<\frac{\pi}{2}$ $3\cos 2t + 4\sin 2t = R\cos 2t\cos\alpha + R\sin 2t\sin\alpha$ $\therefore R\cos\alpha = 3$ $R\sin\alpha = 4$ $\therefore R^2 = 3^2+4^2$ $R = 5$ $R>0$. $\cos\alpha = 3/5$ $\sin\alpha = 4/5$ $\therefore 0<\alpha<\pi/2$ $\therefore \alpha = \tan^{-1} 4/3$ or $\cos^{-1} 3/5$ or $\sin^{-1} 4/5$ $\therefore 3\cos 2t + 4\sin 2t = 5\cos(2t - \tan^{-1} 4/3)$	2	½ for $R=5$ ½ Both sin + cos equations ½ α value. ½ answer. last answer only accepted.
(ii) $x = 5\cos(2t-\tan^{-1}4/3)+2$. $\dot{x} = -10\sin(2t-\tan^{-1}4/5)$ $\ddot{x} = -20\cos(2t-\tan^{-1}4/3)$ $\ddot{x} = -20\dfrac{(x-2)}{5}$ $= -4(x-2)$ $= -(2)^2(x-2)$ $\therefore \ddot{x}$ is in the form $\ddot{x} = -n^2(x-b)$ where $n=2$ $b=2$. $\therefore$ motion is SHM. amplitude is 5 metres	2	½ $\dot{x}$ ½ $\ddot{x}$ ½ reason. ½ amplitude
(iii) $\dot{x} = -10\sin(2t-\tan^{-1}4/3)$ $\therefore$ max speed is 10 m/s. $\therefore \lvert\sin(2t-\tan^{-1}4/3)\rvert = 1$ $2t-\tan^{-1}(4/5) = \frac{(2n+1)\pi}{2}$ $n\in Z$ $\therefore$ first time $2t = \frac{\pi}{2} + \tan^{-1}(\frac{4}{3})$ $t = \frac{1}{2}\left[\frac{\pi}{2}+\tan^{-1}(4/3)\right]$ $= 1.249045772$. $= 1.25$ (2dP)	2	Several methods. eg $\lvert\dot{x}\rvert = 10$, $\ddot{x} = 0$. ½ trig equation 1 solution + 1dP. ½ max speed.
b) (i) $b_k = a_{k-1} + a_k$ from Pascal's triangle (alternatively $b_k = \left[\frac{n+1}{n+1-k}\right]a_k$	1	1 correct relationship
(ii) $\dfrac{1}{a_0a_1\cdots a_n}\times(a_0+a_1)(a_1+a_2)\cdots(a_{n-1}+a_n)$ $= \dfrac{1}{a_0}\times\dfrac{1}{a_1a_2\cdots a_n}\times(b_1b_2\cdots b_n)$ $= 1\times\dfrac{b_1b_2\cdots b_n}{a_1a_2\cdots a_n}$ $a_0=1$ now $\dfrac{b_k}{a_k} = \dfrac{{}^{n+1}C_k}{{}^nC_k} = \dfrac{(n+1)!}{(n-k+1)!\,k!}\times\dfrac{(n-k)!\,k!}{n!}$ $= \dfrac{n+1}{n-k+1}$	3	½ product of b's ½ $a_0=1$ 1 coefficients expansions 1 answer with working

MATHEMATICS Extension 1 : Question 14 V2

Suggested Solutions	Marks	Marker's Comments
(iii) continued $\dfrac{b_1 b_2 \cdots b_n}{a_1 a_2 \cdots a_n} = \dfrac{(n+1)(n+1)\cdots(n+1)}{(n)(n-1)\cdots(1)} = \dfrac{(n+1)^n}{n!}$		

c) (i) To show $\tan^{-1}(n+1) - \tan^{-1}(n-1) = \tan^{-1}(2/n^2)$ $n \geq 1$

consider $\tan(LHS)$

Let $\tan^{-1}(n+1) = \alpha$ $\therefore \tan\alpha = n+1$ $\pi/4 < \alpha < \pi/2$

$\tan^{-1}(n-1) = \beta$ $\tan\beta = n-1$ $0 < \beta < \pi/2$

$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$$ $\frac{1}{2}$ tan expansion

$$= \frac{(n+1)-(n-1)}{1+(n+1)(n-1)}$$ $\frac{1}{2}$ simplifying

$$= \frac{2}{1+n^2-1} = \frac{2}{n^2}$$ $\frac{1}{2}$ solution

$\therefore \alpha - \beta = k\pi + \tan^{-1}(2/n^2)$ $k \in \mathbb{Z}$ $\frac{1}{2}$ restrictions

But $-\pi/4 < (\alpha-\beta) < \pi/2$ $\therefore k = 0$

$\tan^{-1}(n+1) - \tan^{-1}(n-1) = \tan^{-1}\left(\frac{2}{n^2}\right)$

(ii) To show $\displaystyle\sum_{r=1}^{n} \tan^{-1}\left(\frac{2}{r^2}\right) = \tan^{-1}\left(\frac{2n+1}{1-n-n^2}\right) + \frac{3\pi}{4}$

$$LHS = \sum_{r=1}^{n}\left[\tan^{-1}(r+1) - \tan^{-1}(r-1)\right]$$

$= \tan^{-1}2 - \tan^{-1}0 + \tan^{-1}3 - \tan^{-1}1 \cdots$ $\frac{1}{2}$ substitution

$\cdots \tan^{-1}n + \tan^{-1}(n-2) + \tan^{-1}(n+1) + \tan^{-1}(n-1)$

$= \tan^{-1}(n) + \tan^{-1}(n+1) - \tan^{-1}1$

consider

$\tan\left(\tan^{-1}n + \tan^{-1}(n+1)\right)$

$$= \frac{\tan(\tan^{-1}n) + \tan(\tan^{-1}(n+1))}{1 - \tan(\tan^{-1}n) \times \tan(\tan^{-1}(n+1))}$$ $\frac{1}{2}$ expansion + cancelling

$$= \frac{n+n+1}{1-n(n+1)}$$ $\frac{1}{2}$ tan expression and simplifying

$$= \frac{2n+1}{1-n-n^2}$$

$\tan^{-1}n + \tan^{-1}(n+1) = k\pi + \tan^{-1}\left(\frac{2n+1}{1-n-n^2}\right)$ $k \in \mathbb{Z}$ $\frac{1}{2}$ general solution

But $\frac{\pi}{4} \leq \tan^{-1}n < \frac{\pi}{2}$ $n \geq 1$

and $\pi/4 < \tan^{-1}(n+1) < \pi/2$ $\frac{1}{2}$ correct restriction

$\therefore \frac{\pi}{2} < \tan^{-1}n + \tan^{-1}(n+1) < \pi$

$k = 1$

$\therefore \tan^{-1}n + \tan^{-1}(n+1) = \pi + \tan^{-1}\left(\frac{2n+1}{1-n-n^2}\right)$ $\frac{1}{2}$ answer with working

$$\therefore \sum_{r=1}^{n}\tan^{-1}\left(\frac{2}{r^2}\right) = \pi + \tan^{-1}\left(\frac{2n+1}{1-n-n^2}\right) - \tan^{-1}(1)$$

$$= \pi + \tan^{-1}\left(\frac{2n+1}{1-n-n^2}\right) - \pi/4$$

$$= \tan^{-1}\left(\frac{2n+1}{1-n-n^2}\right) + \frac{3\pi}{4}$$