Name:	
Class:	

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION 2017

MATHEMATICS EXTENSION 1

General Instructions:

- Reading Time: 5 minutes.
- Working Time: 2 hours.
- Write in black or blue pen.
- Board approved calculators & templates may be used
- A Standard Integral Sheet is provided.
- In Question 11 - 14, show all relevant mathematical reasoning and/or calculations.
- Marks may not be awarded for careless or badly arranged working.

Total Marks 70

Section I: 10 marks

- Attempt Question 1 – 10.
- Answer on the Multiple Choice answer sheet provided.
- Allow about 15 minutes for this section.

Section II: 60 Marks

- Attempt Question 11 - 14
- Answer on blank paper unless otherwise instructed. Start a new page for each new question.
- Allow about 1 hours & 45 minutes for this section.

The answers to all questions are to be returned in separate *stapled* bundles clearly labelled Question 11, Question 12, etc. Each question must show your Candidate Number.

Multiple Choice Questions

Choose the best answer for each of the following questions:

1. Differentiate $\tan^2 5x$.

   A $2\tan 5x$

   B $10\tan 5x$

   C $10\tan 5x \sec 5x$

   D $10\tan 5x \sec^2 5x$

2. Evaluate $\lim_{x \to 0} \dfrac{\sin \frac{x}{5}}{3x}$.

   A $\frac{3}{5}$

   B $\frac{5}{3}$

   C 15

   D none of the above

3. Let each different arrangement of all the letters of the word " DELETED" be considered a word. How many words are possible altogether?

   A 420

   B 630

   C 840

   D None of the above

4. Suppose the rate of change of a quantity N is proportional to the excess of N over a fixed quantity P.
i.e. $\frac{dN}{dt} = -k(N - P)$, where k is a constant.
If t is the time and the value of A is $(N\text{-}P)$ when $t = 0$, then which of the following is true?

A $N = P - Ae^{-kt}$

B $N = P - Ae^{kt}$

C $N = P + Ae^{-kt}$

D $N = P + Ae^{kt}$

5. A particle is travelling in Simple Harmonic Motion. If x is the displacement at time t, v is the velocity and $\ddot{x}$ is the acceleration. Which of the following cannot be true?

A $x = -3\cos 4t$

B $\ddot{x} = 4 - 4x^2$

C $\ddot{x} = 4 - 4x$

D $v^2 = 4(1 - x^2)$

6. Which expression is equal to $\int \sin^2 2x\,dx$?

A $\frac{-\cos^3 2x}{6} + C$

B $\frac{\sin^3 2x}{6} + C$

C $\frac{1}{2}(x - \frac{1}{4}\sin 4x) + C$

D $\frac{1}{2}(x + \frac{1}{4}\sin 4x) + C$

7. Find the equation of the chord of contact of the tangents to the parabola $x^2 = 8y$ from the point (3,-2).

A $3x + 4y + 8=0$

B $3x - 4y + 8=0$

C $3x - 4y - 8=0$

D $3x + 4y - 8=0$

8. Which of the following may not be true in a circle?

i Equal arcs subtend equal chords.
ii Angles at the circumference subtended by equal chords are equal.
iii Angle at the centre is twice the angle at the circumference.
iv If a line subtends equal angles then the 4 end points are concylic.

A ii only

B i, ii, iv only

C ii, iii, iv only

D All of i, ii, iii, iv

9. The point A is (-2,1) and the point B is (b,-3). The point P(13,-9) divides the interval AB externally in the ratio of 5:3. Find the value of b.

A 4

B -4

C $\frac{6}{7}$

D $-\frac{6}{7}$

10 The quadratic equation $ax^2 + bx + c = 0$ has roots $x = \tan\alpha$ and $x = \tan\beta$.
Which of the following is/are true?

i $\tan(\alpha+\beta) = \frac{b}{c-a}$ ii $\tan^2(\alpha-\beta) = \frac{b^2 - 2ac}{(a+c)^2}$ iii $\tan^2(\alpha-\beta) = \frac{b^2 - 4ac}{(a+c)^2}$

A i only

B ii only

C i & ii only

D i & iii only

Question 11 **Marks**

a) Find $\int \frac{5}{4+8x^2} dx$. **2**

b) Find $\int \frac{x^3}{\sqrt{x^2+1}} dx$ (use substitution $u = \sqrt{x^2+1}$). **3**

c) (i) A particle is travelling in a straight line. Its displacement from the origin is x m/s at time t seconds. **2**

If $x = \sqrt{3}\cos 2t - \sin 2t$, express x in the form of $R\cos(2t+\alpha)$, where R>0 and $0 \le \alpha \le 2\pi$.

(ii) Find the maximum speed of the particle and the time when it first occurs. **2**

d) Find the area of then region bounded by the curve $y = \frac{x}{\sqrt{x^2-1}}$, the x-axis and the lines $x = \sqrt{10}$, $x = \sqrt{5}$. **3**

e) (i) Given the equation $x - 2\sin x = 0$. **1**

Show that there is at least one root between x =1.5 and x = 2.

(ii) Use Newton's method to find a second approximation to the positive root of $x - 2\sin x = 0$. **2**

Take $x = 1.7$ as the first approximation.

Question 12 (15 marks) (START A NEW PAGE)

a) Consider $f(x) = e^x - e^{-x}$.

(i) Justify that the inverse of $f(x)$ exists. **1**

(ii) Find the equation of $f^{-1}(x)$. **3**

b) Find the coefficient of x^4 in the expansion of $(2-5x)^7$. **2**

c) (i) A fair coin is tossed $2n$ times. Write down the probability (P_k) of observing k heads and $(2n-k)$ tails. **1**

(ii) Find the probability of the most likely outcome and simplify it. Give reason(s). **2**

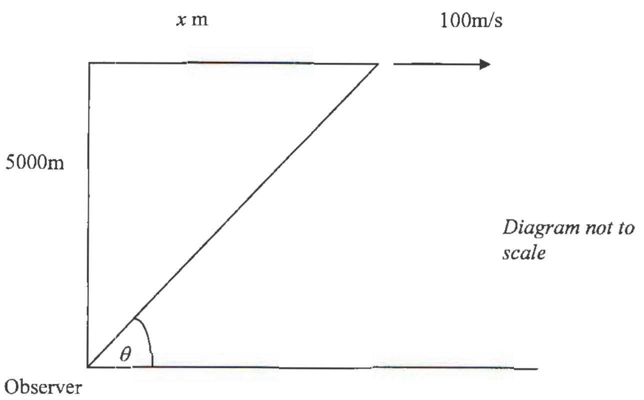
d) (i) Find the derivative of $y = \sin^{-1}[2x(1-x)]$. **1**

(ii) Hence find the maximum value of $y = \sin^{-1}[2x(1-x)]$. **2**

(iii) Sketch the graph of $y = \sin^{-1}[2x(1-x)]$. **3**

Question 13 (15 marks) (START A NEW PAGE)

a)



At a certain instant, a plane flies overhead at a constant altitude of 5000 metres and at a constant speed of 100 metres per second.

When the plane has travelled x metres from the overhead position, its angle of elevation from the observer is θ radians.

(i) Show that $\dfrac{dx}{d\theta} = -\dfrac{5000}{\sin^2\theta}$. **2**

(ii) Hence show that $\dfrac{d\theta}{dt} = -\dfrac{1}{50}\sin^2\theta$. **1**

(iii) Find the rate at which the angle of elevation is changing 50 seconds after the plane is overhead. **2**

Question 13 continues on the next page

Question 13 (continued)

b) There are four couples: Katie and Ming, David and Betty, Margaret and Danny, May and Kent. They go to a cinema and decide to sit in the last row. **2**

How many different arrangements are possible if Katie and Ming want to sit together but May and Kent do not want to sit together ?

c) (i) The acceleration ($\ddot{x}$) of a particle at a displacement from the origin (x) is given by $\ddot{x} = \dfrac{1}{x}$. **2**

Initially the particle is at rest at a point 25m from the origin, find its velocity (v) in terms of x.

(ii) Briefly describe its motion. **1**

d) (i) Prove $(x+y) \geq 2\sqrt{xy}$, $x > 0, y > 0$. **2**

(ii) Hence or otherwise, find the largest possible value for $\log_\alpha\left(\dfrac{\alpha}{\beta}\right) + \log_\beta\left(\dfrac{\beta}{\alpha}\right)$, $\alpha, \beta > 1$ **3**

Question 14 (15 marks) (START A NEW PAGE)

a) Prove by Mathematical induction that $3^n > n^3$ for $n \geq 4$. **4**

b) The diagram below shows a plane P which is flying at a constant speed of $\sqrt{8gh}$ m/s upwards at an angle of elevation 30°.

At the instant when the plane is at a height h metres vertically above a missile silo, which is located at a point O on the ground, a cruise missile from the silo is launched at an angle of elevation α to hit the plane where $0^\circ \leq \alpha \leq 90^\circ$.

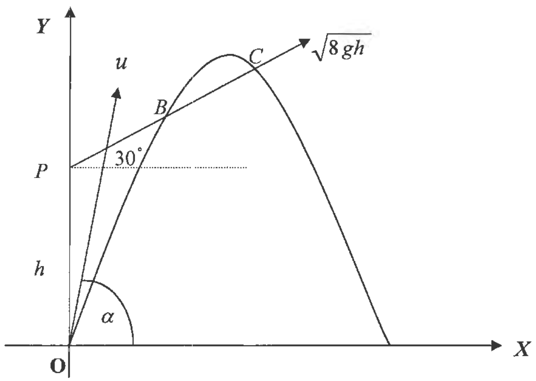


Diagram not to scale

The launching speed of the missile is u m/s, t is the time in seconds after launch and g is the acceleration due to gravity in m/s^2.

With the axes shown in the diagram above, you may assume the following equations:
(DO NOT PROVE)

Position of the missile is given by $x = ut\cos\alpha$, $y = ut\sin\alpha - \frac{1}{2}gt^2$

The trajectory of the missile is given by $y = x\tan\alpha - \frac{gx^2}{2u^2}(1+\tan^2\alpha)$

(i) Show that the trajectory of the plane is given by $y = \frac{x}{\sqrt{3}} + h$. **1**

(ii) Show that $u\cos\alpha = \sqrt{6gh}$. **2**

(iii) Assuming the missile can hit the plane, hence, from part (i), show that the x-coordinates of the plane of collision must satisfy $\frac{x^2}{12} + (\frac{1}{\sqrt{3}} - \tan\alpha)hx + h^2 = 0$. **3**

Question 14 continues on the next page

(iv) Suppose that $\tan\alpha > \frac{2}{\sqrt{3}}$.

(α) Show that there are two possible points of collision, at B and C, between the plane and the missile. **2**

(β) Show that the time T (in seconds) elapsed between the two points of collision is given by : **3**

$$T = \sqrt{\frac{8h\tan\alpha}{g}(3\tan\alpha - 2\sqrt{3})}.$$

END OF PAPER

Student ID number: | | | | |

Section I **Multiple Choice** **10 Marks**

Attempt Question 1 – 10 (1 mark each)
Allow approximately 15 minutes for this section.

Use the multiple choice answer sheet below to record your answers to Question 1 – 10.

Select the alternative: A, B, C or D that best answers the question.

Colour in the response oval completely.

Sample:

2 + 4 = ? (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, draw a cross through the incorrect answer and colour in the new answer

ie A ● B ☒ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate this by writing the word "*correct*" and draw an arrow as follows:

correct

A ● B ☒ C ○ D ○

Trial HSC Examination 2017

Multiple Choice Answer Sheet

2 Unit / Ext 1 / Ext 2

Completely colour in the response oval representing the most correct answer.

	A	B	C	D
1	○	○	○	●
2	○	○	○	●
3	●	○	○	○
4	○	○	●	○
5	○	●	○	○
6	○	○	●	○
7	○	●	○	○
8	○	○	●	○
9	●	○	○	○
10	○	○	○	●

Marks: 10/10

JRAHS TRIALS **MATHEMATICS Extension 1 : Question** 11 2017

Suggested Solutions	Marks	Marker's Comments
a) $\displaystyle\int \frac{5}{4+8x^2}\,dx = \frac{5}{8}\int \frac{1}{\frac12+x^2}\,dx$		
$= \frac{5}{8}\cdot\frac{1}{1/\sqrt2}\tan^{-1}\frac{x}{1/\sqrt2}+C$	1	
$= \frac{5\sqrt2}{8}\tan^{-1}(\sqrt2\,x)+C$	1	
b) Let $I=\displaystyle\int\frac{x^3}{\sqrt{x^2+1}}\,dx$		
$u=\sqrt{x^2+1}$ $u^2=x^2+1$ and $x^2=u^2-1$ $2u\,du=2x\,dx$ $x\,dx=u\,du$	1	changing variables into u
$\therefore\ I=\displaystyle\int\frac{(u^2-1)\,u\,du}{u}$	1	Converting integrand into u
$=\displaystyle\int(u^2-1)\,du$ $=\frac13u^3-u^2+c$		
$=\frac13(x^2+1)^{3/2}-(x^2+1)^{\frac12}+c$	1	For undoing substitution

JRAHS TRIALS **MATHEMATICS Extension 1 : Question** 11 2017

Suggested Solutions	Marks	Marker's Comments

c) (i) Method I:

$x = \sqrt{3}\cos 2t - \sin 2t$

$= 2\left[\frac{\sqrt{3}}{2}\cos 2t - \frac{1}{2}\sin 2t\right]$

$= 2\left[\cos\frac{\pi}{6}\cos 2t - \sin\frac{\pi}{6}\sin 2t\right]$

$= 2\cos\left(2t + \frac{\pi}{6}\right) \equiv R\cos(2t+\alpha)$ — 1

$\therefore R = 2$ and $\alpha = \pi/6$

Method II:

$\sqrt{3}\cos 2t - \sin 2t = R\cos(2t+\alpha)$

$= R\cos 2t\cos\alpha - R\sin 2t\sin\alpha$

Equating coefficients

$R\cos\alpha = \sqrt{3}$ and $R\sin\alpha = 1$

$\Rightarrow R^2(\cos^2\alpha + \sin^2\alpha) = 3+1$

$\therefore R = 2 \ (R>0)$ and $\sin^2\alpha + \cos^2\alpha = 1$

$\therefore \cos\alpha = \frac{\sqrt{3}}{2}$ and $\sin\alpha = \frac{1}{2}$

$\Rightarrow \alpha$ in 1st quadrant and $\alpha = \pi/6$ only — 1

Marker's comments: Mention why $7\pi/6$ excluded when $\tan\alpha = \frac{1}{\sqrt{3}}$

$\therefore x = \sqrt{3}\cos 2t - \sin 2t = 2\cos\left(2t + \frac{\pi}{6}\right)$

JRAHS TRIALS **MATHEMATICS Extension 1 : Question** 11 2017

Suggested Solutions	Marks	Marker's Comments

c) (ii) From $x = 2\cos\left(2t + \frac{\pi}{6}\right)$

$\dot{x} = -4\sin\left(2t + \frac{\pi}{6}\right)$

maximum speed $= |\dot{x}| = 4$ m/s — 1 — For maximum speed

Maxima/minima occur when

$-4 = -4\sin\left(2t + \frac{\pi}{6}\right)$

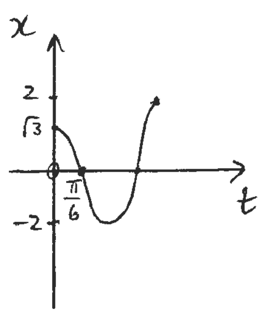
or $\sin\left(2t + \frac{\pi}{6}\right) = 1$

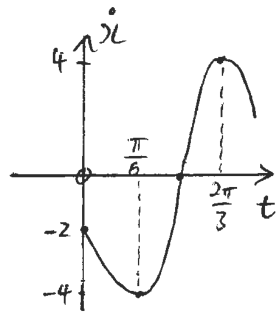
$\Rightarrow 2t + \frac{\pi}{6} = \frac{\pi}{2}$

$2t = \frac{3\pi}{6} - \frac{\pi}{6}$

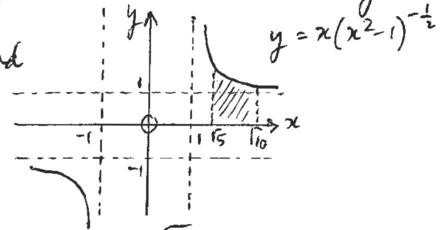
$\therefore t = \frac{\pi}{6}$

$\therefore$ maximum speed first occurs when $t = \pi/6$ — 1 — For 1st time max occurs





Suggested Solutions	Marks	Marker's Comments
c) (ii) Method I:		
$x = 2\cos\left(2t+\frac{\pi}{6}\right)$		
$\dot{x} = -4\sin\left(2t+\frac{\pi}{6}\right)$		
$\ddot{x} = -8\cos\left(2t+\frac{\pi}{6}\right)$		
maximum speed occurs when $\ddot{x}=0$		
i.e $\cos\left(2t+\frac{\pi}{6}\right)=0$		
$\Rightarrow 2t+\frac{\pi}{6} = \frac{\pi}{2}$ or $\frac{3\pi}{2}$ etc		
$2t = \frac{2\pi}{6}$ or		
$\therefore t = \frac{\pi}{6}$	1	
$\therefore$ maximum speed $= \left\lvert\dot{x}\left(\frac{\pi}{6}\right)\right\rvert$		
$= 4\sin\left(\frac{\pi}{3}+\frac{\pi}{6}\right)$		
$= 4\sin\frac{\pi}{2}$		
$= 4$ m/s	1	
Method II: $\ddot{x} = -4\left(2\cos\left(2t+\frac{\pi}{6}\right)\right)$		
$= -4x = -2^2x$		
$\therefore$ motion is SHM		
$\therefore$ max speed at centre of motion i.e at $x=0 \Rightarrow t=\pi/6$ and $\lvert\dot{x}\rvert=4$	1+1	

Suggested Solutions	Marks	Marker's Comments
d) $y=\frac{x}{\sqrt{x^2-1}}$ is undefined for $-1<x<1$		
For $x>1$, the function is decreasing		$y=\frac{1}{\sqrt{1-\frac{1}{x^2}}}$
Between $\sqrt{5}$ and $\sqrt{10}$, $y>0$ 		as $x\to\infty$, $y\to 1$
$\Rightarrow$ Area $=\int$		$\therefore \left.\begin{array}{l} y=1 \\ y=-1\end{array}\right\}$ H.A.
$\therefore$ Area, $A = \int_{\sqrt{5}}^{\sqrt{10}} y\,dx$		
$= \int_{\sqrt{5}}^{\sqrt{10}} x(x^2-1)^{-\frac{1}{2}}\,dx$	1	For setting up integral
$= \frac{1}{2}\int_{\sqrt{5}}^{\sqrt{10}} 2x(x^2-1)^{-\frac{1}{2}}\,dx$		$\int f'(x)[f(x)]^n\,dx = \frac{[f(x)]^{n+1}}{n+1}$
$= \frac{1}{2}\left.\frac{(x^2-1)^{\frac{1}{2}}}{\frac{1}{2}}\right\rvert_{\sqrt{5}}^{\sqrt{10}}$		
$= \left.\sqrt{x^2-1}\,\right\rvert_{\sqrt{5}}^{\sqrt{10}}$	1	For integrating
$= \sqrt{9}-\sqrt{4}$		
$= 1\ u^2$		
$\therefore$ Area is 1 unit^2	1	For correct Area

JRAHS TRIALS **MATHEMATICS Extension 1 : Question** 11 2017

Suggested Solutions	Marks	Marker's Comments
e) (i) Let $f(x) = x - 2\sin x$		
$f(1.5) = 1.5 - 2\sin 1.5 = -0.495$ (3dp)	<0	
$f(2) = 2 - 2\sin 2 = 0.181$ (3dp)	>0	
Since $f(x)$ is continuous between 1.5 and 2 AND $f(x)$ changes sign, a root exists between 1.5 and 2	1	
(ii) $f(x) = x - 2\sin x$ $f'(x) = 1 - 2\cos x$		
If $x_1 = 1.7$, then by Newton's Method x_2 is given by $x_2 = x_1 - \dfrac{f(x)}{f'(x)}$		
$= 1.7 - \dfrac{(1.7 - 2.\sin 1.7)}{(1 - 2\cos 1.7)}$	1	For $f'(1.7)$ in the expression
$= 1.9257\ldots$		
$= 1.9$ (1dp) $\therefore$ 2nd approximation is 1.9	1	For final correct answer!

Mathematics Ext 1 Question 12

Suggested Solutions	Marks	Marker's Comments
a) i $f(x) = e^x - e^{-x}$ $f'(x) = e^x + e^{-x}$ >0 (Since $e^x > 0$ and $e^{-x} > 0$ for $x \in \mathbb{R}$)		
$\therefore f(x)$ is monotonically increasing (passes horizontal line test)	1	
$\therefore f^{-1}(x)$ exists.		
ii let $y = f(x)$ $\therefore y = e^x - e^{-x}$ for $f^{-1}(x)$:		
$x = e^y - e^{-y}$ (swap x and y)	1	
$= e^y - \dfrac{1}{e^y}$ $xe^y = e^{2y} - 1$		
$e^{2y} - xe^y - 1 = 0$	1	
$\therefore e^y = \dfrac{x \pm \sqrt{x^2+4}}{2}$ Since $e^y > 0$, then $e^y = \dfrac{x + \sqrt{x^2+4}}{2}$ only		
$\therefore y = \ln\left[\dfrac{x + \sqrt{x^2+4}}{2}\right]$	1	

Mathematics _______ Question _____

Suggested Solutions	Marks	Marker's Comments
b) $(2-5x)^7$		
General term for expansion ${}^7C_k(2)^{7-k}(-5x)^k$		
Coefficient of $x^4 \rightarrow k=4$	1	
$\therefore$ coefficient is ${}^7C_4(2)^3(-5)^4$		
$= 175000$	1	
c) i ${}^{2n}C_k\left(\frac{1}{2}\right)^k\left(\frac{1}{2}\right)^{2n-k}$		
ii $(P_k) = {}^{2n}C_k\left(\frac{1}{2}\right)^{2n}$		
Since $\left(\frac{1}{2}\right)^{2n}$ is constant for all k,		Must state that fact
then maximum probability happens when ${}^{2n}C_k$ is maximum.	1	
Max value when $k=\frac{1}{2}(2n)=n$		
(By Pascal's triangle)		
$\therefore$ Max $P_k = {}^{2n}C_n\left(\frac{1}{2}\right)^{2n}$	1	
Or		

Mathematics _______ Question _____

Suggested Solutions	Marks	Marker's Comments
ii) Since the coin is fair, the most likely outcome is half the number of tosses.		
i.e. n heads and n tails	1	
$\therefore$ Max $P_k = {}^{2n}C_n\left(\frac{1}{2}\right)^n\left(\frac{1}{2}\right)^n$		
$= {}^{2n}C_n\left(\frac{1}{2}\right)^{2n}$	1	
Or		
ii) $P_k = {}^{2n}C_k\left(\frac{1}{2}\right)^k\left(\frac{1}{2}\right)^{2n-k}$		
$= {}^{2n}C_k\left(\frac{1}{2}\right)^{2n}$		
Max Probability $\rightarrow \dfrac{P_{k+1}}{P_k} = \dfrac{{}^{2n}C_{k+1}\left(\frac{1}{2}\right)^{2n}}{{}^{2n}C_k\left(\frac{1}{2}\right)^{2n}} \geq 1$		
$\therefore {}^{2n}C_{k+1} \geq {}^{2n}C_k$		
$\dfrac{(2n)!}{(k+1)!(2n-k-1)!} \geq \dfrac{(2n)!}{k!(2n-k)!}$		
(Both are positive) $\dfrac{(k+1)!(2n-k-1)!}{(2n)!} \leq \dfrac{k!(2n-k)!}{(2n)!}$		Note: If you used $\dfrac{P_k}{P_{k-1}} \geq 1$, you must end up with $k \leq n+\frac{1}{2}$, $k=n$.
$k+1 \leq 2n-k$	1	
$2k \leq 2n-1$		
$k \leq n-\frac{1}{2}$		
$k = n-1 \quad (k \in \mathbb{Z})$		
$\therefore$ Max Probability $= P_n = {}^{2n}C_n\left(\frac{1}{2}\right)^{2n}$	1	

Mathematics _______ Question _____

Suggested Solutions	Marks	Marker's Comments

d) i $y = \sin^{-1}[2x(1-x)] = \sin^{-1}(2x - 2x^2)$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(2x-2x^2)^2}} \times (2-4x)$$

$$= \frac{2-4x}{\sqrt{1-4x^2(1-x)^2}}$$ (1)

Must be under one fraction!

ii) $\frac{dy}{dx} = 0 \longrightarrow \frac{2-4x}{\sqrt{1-4x^2(1-x)^2}} = 0$

$\therefore 2 - 4x = 0$

$x = \frac{1}{2}$

x	0	$\frac{1}{2}$	1
$\frac{dy}{dx}$	2	0	-2

(1)

$\therefore$ maximum at $x = \frac{1}{2}$ as $\sin^{-1}[2x(1-x)]$ is continuous

when $x = \frac{1}{2}$, $y = \sin^{-1}(1 - \frac{1}{2})$

$= \sin^{-1}(\frac{1}{2})$

$= \frac{\pi}{6}$ (1)

Note:

x	0	$\frac{1}{4}$	$\frac{1}{3}$	0.4	0.49	$\frac{1}{2}$	0.51	0.6	$\frac{2}{3}$	$\frac{3}{4}$	1
$\frac{dy}{dx}$	2	1.08	0.46	0.05	0.74	0	-0.74	-0.05	-0.46	-1.08	-2

symmetrical values!

Mathematics _______ Question _____

Suggested Solutions	Marks	Marker's Comments

iii) Domain:

$-1 \le 2x - 2x^2 \le 1$

$-\frac{1}{2} \le x - x^2 \le \frac{1}{2}$

$-\frac{1}{2} \le x^2 - x \le \frac{1}{2}$

$-\frac{1}{4} \le x^2 - x + \frac{1}{4} \le \frac{3}{4}$

$-\frac{1}{4} \le (x - \frac{1}{2})^2 \le \frac{3}{4}$

Since $(x - \frac{1}{2})^2 \ge -\frac{1}{4}$ for all x,

check $(x - \frac{1}{2})^2 \le \frac{3}{4}$

$x - \frac{1}{2} \le \frac{\pm\sqrt{3}}{2}$

$x \le \frac{1 \pm \sqrt{3}}{2}$

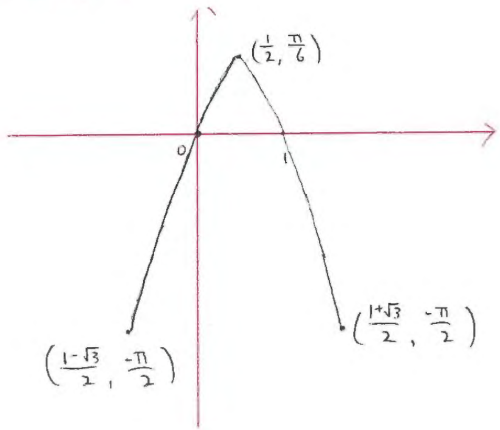
$\frac{1-\sqrt{3}}{2} \le x \le \frac{1+\sqrt{3}}{2}$

when $x = \frac{1 \pm \sqrt{3}}{2}$, $y = -\frac{\pi}{2}$ (By calculator)

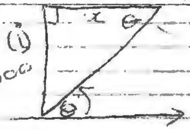
x-intercepts: $y = 0 \longrightarrow \sin^{-1}[2x(1-x)] = 0$

$2x(1-x) = 0$

$\therefore x = 0$ or 1

Suggested Solutions (Mathematics ____ Question ____)	Marks	Marker's Comments
iii) Continued		
		1 Domain 1 Shape + intercepts 1 correct scale + Max turning point
Note: If you did not find the domain of curve you receive 1/3 maximum.		

MATHEMATICS Extension 1 : Question 13. 1/3

Suggested Solutions	Marks	Marker's Comments
(a) (i)  → alternate angles on ‖ lines are equal.		
$\tan\theta = \frac{5000}{x}$		
$x = \frac{5000}{\tan\theta}$		
$x = 5000\cot\theta$		1
$\frac{dx}{d\theta} = 5000(-\mathrm{cosec}^2\theta)$ ↔		needed a line or two of working to get the second mark.
$= \frac{-5000}{\sin^2\theta}$		1
(ii) $\frac{dx}{dt} = 100$ $\quad \frac{d\theta}{dt} = \frac{d\theta}{dx} \times \frac{dx}{dt}$		
$= \frac{-\sin^2\theta}{5000} \times 100$	1	
$= \frac{-\sin^2\theta}{50}$		
(iii) $t = 50$ $\quad x = 100t$ $\quad \tan\theta = \frac{5000}{5000}$		
$x = 5000$ $\quad \tan\theta = 1$		
$\theta = \pi/4$		1
$\frac{d\theta}{dt} = \frac{-\sin^2\theta}{50} = \frac{-\sin^2(\pi/4)}{50} = \frac{-1/2}{50}$		
$\frac{d\theta}{dt} = -\frac{1}{100}$ rad/sec		1
(b) 5! ways of arranging Katie, Ming, D, B, M, D		
2! internal arrangements of Katie + Ming		
(KM) (D) (B) (M) (D)		1
6 places for Mary		
5 places for Kat.		1
Total $= 5! \times 2 \times 6 \times 5 = 7200$		
or All arrangements with Ming + Katie together − all arrangements with Ming + Katie, Mary + Kat together		1
$7! \times 2! - 6! 2! 2! = 10\,080 - 2880$		1
$= 7200$		

Suggested Solutions	Marks	Marker's Comments
(c)(i) $v\frac{dv}{dx} = \frac{1}{x}$		
$\int v\,dv = \int \frac{dx}{x}$		
$\frac{1}{2}v^2 = \ln x + c$		
when $t=0, v=0, x=25$		
$0 = \ln 25 + c$		
$c = -\ln 25$		
$\therefore \frac{1}{2}v^2 = \ln x - \ln 25$		
$\frac{1}{2}v^2 = \ln\frac{x}{25}$		
$v^2 = 2\ln\frac{x}{25}$		
$v = \pm\sqrt{2\ln\left(\frac{x}{25}\right)}$	1	
now initially $\ddot{x} = \frac{1}{25}$, so particle moves to the right & velocity is only zero when $x=25$, so particle won't change direction ie keeps moving right		← needed an explanation to get the second mark.
$\therefore\ v = \sqrt{2\ln{x/25}}$ only	1	
(ii) Initially the particle is at rest at 25m to the right of the origin the acceleration is positive so it moves to the right. It continues to move to the right with increasing speed at a decreasing acceleration, never stops and never changes direction.	1	MUST USE WORDS
(d) (i) consider $(x-y)^2 \geq 0$		
$x^2+y^2-2xy \geq 0 \quad x>0, y>0$		
$x^2+y^2 \geq 2xy$	1	
$x^2+y^2+2xy \geq 4xy$		
$(x+y)^2 \geq 4xy$	1	
$x+y \geq 2\sqrt{xy}$		

Suggested Solutions	Marks	Marker's Comments
or $(\sqrt{x}-\sqrt{y})^2 \geq 0$ $\quad x>0, y>0$	1	
$x - 2\sqrt{xy} + y \geq 0$		
$x+y \geq 2\sqrt{xy}$	1	
(ii) $\log_\alpha\left(\frac{\alpha}{\beta}\right) + \log_\beta\left(\frac{\beta}{\alpha}\right) = \log_\alpha\alpha - \log_\alpha\beta + \log_\beta\beta - \log_\beta\alpha$		
$= 1 - \log_\alpha\beta + 1 - \log_\beta\alpha$		
$= 2 - \log_\alpha\beta - \log_\beta\alpha$	1	
now $\log_\beta\alpha + \log_\alpha\beta \geq 2\sqrt{\log_\alpha\beta\log_\beta\alpha}$		
$\geq 2\sqrt{\frac{\log_\alpha\beta}{\log_\beta\alpha}\cdot\log_\beta\alpha}$		
$\geq 2\sqrt{1}$		
≥ 2	1	
so $-\log_\beta\alpha - \log_\alpha\beta \leq -2$		
$\therefore \log_\alpha\left(\frac{\alpha}{\beta}\right) + \log_\beta\left(\frac{\beta}{\alpha}\right) \leq 2-2$		
$\log_\alpha\left(\frac{\alpha}{\beta}\right) + \log_\beta\left(\frac{\beta}{\alpha}\right) \leq 0$	1	
$\therefore$ max. value is zero		

* If you let $x=\log_\alpha\left(\frac{\alpha}{\beta}\right)$ and $y=\log_\beta\left(\frac{\beta}{\alpha}\right)$ ⇒ maximum of one mark if the algebra is correct

* Some students proved it was a minimum but wrote a max value of 0 without an explanation ⇒ 2 marks only :)

MATHEMATICS Extension 1 : Question 14 ①

Suggested Solutions	Marks	Marker's Comments

a) $3^n > n^3$, $n \geq 4$

For $n = 4$,

$3^4 = 81$

$4^3 = 64$

$\therefore 3^4 > 4^3$

$\therefore$ true for $n = 4$ — 1

Assume true for $n = k$,

ie $3^k > k^3$ for $k \geq 4$

Prove true for $n = k+1$

ie. $3^{k+1} > (k+1)^3$

$LHS = 3^{k+1}$

$= 3 \times 3^k$

$= 3^k + 3^k + 3^k$

$> k^3 + k^3 + k^3$ (by assumption) — 1

$\geq k^3 + 4k^2 + 4k^2$ $(k \geq 4)$

$= k^3 + 3k^2 + 3k^2 + 2k^2$

$> k^3 + 3k^2 + 3k + 2$ (since $k \geq 4$)

$> k^3 + 3k^2 + 3k + 1$

$= (k+1)^3$

$\therefore 3^{k+1} > (k+1)^3$

$\therefore$ the statement is true by the principle of mathematical induction

Marker's comment: 2 for working out and conclusion.

OR $3^k - k^3 > 0$

RTP: $3^{k+1} - (k-1)^3 > 0$

$LHS = 3^k \times 3 - k^3 - 3k^2 - 3k - 1$

$= 3(3^k - k^3) + 2k^3 - 3k^2 - 3k - 1$

from assumption: $3^k - k^3 > 0$

$\therefore 3(3^k - k^3) > 0$ — ⇐ 2 marks up to this step.

$\therefore$ RTP : $2k^3 - 3k^2 - 3k - 1 > 0$ for $k \geq 4$

MATHEMATICS Extension 1 : Question 14 ②

Suggested Solutions	Marks	Marker's Comments

let $f(k) = 2k^3 - 3k^2 - 3k - 1$

$f'(k) = 6k^2 - 6k - 3$

when $f'(x) = 0$

$k = \dfrac{6 \pm \sqrt{36 + 4(6)(3)}}{2(6)}$

$= \dfrac{6 \pm \sqrt{108}}{12}$

$= \dfrac{1 \pm \sqrt{3}}{2}$

for $k = \dfrac{1+\sqrt{3}}{2}$

k	1	$\frac{1+\sqrt{3}}{2}$	2
$f'(k)$	-3	0	9

$\searrow \quad - \quad \nearrow$

$4 > \dfrac{1+\sqrt{3}}{2}$

$\therefore f'(k) > 0$ for $k \geq 4$ — 1

$f(4) = 2(4)^3 - 3(4)^2 - 3(k) - 1$

$= 67 > 0$

since $f(4) > 0$ and $f'(k) > 0$ for $k \geq 4$,

$2k^3 - 3k^2 - 3k - 1 > 0$ for $k \geq 4$ — 1

$\therefore 3^{k+1} - (k-1)^3 > 0$

$\therefore$ statement is true by the process of mathematical induction

b) i) $y = mx + b$

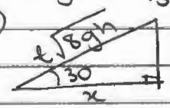
where $b = h$

and $m = \tan 30$

$= \dfrac{1}{\sqrt{3}}$ — 1

$\therefore y = \dfrac{x}{\sqrt{3}} + h$

ii)



$\cos 30 = \dfrac{x}{t\sqrt{8gh}}$

$\therefore x = t\sqrt{8gh} \cos 30$ — 1

Suggested Solutions	Marks	Marker's Comments
the missile will hit the plane at time t when:		
$x_{missile} = x_{plane}$		
$ut\cos\alpha = t\sqrt{8gh}\cos 30$		
$u\cos\alpha = \sqrt{8gh} \times \frac{\sqrt{3}}{2}$		
$u\cos\alpha = \sqrt{\frac{8\times 3}{4}gh}$	1	
$\therefore u\cos\alpha = \sqrt{6gh}$		
iii) $y = \frac{x}{\sqrt{3}} + h$ from (i)		
$y = x\tan\alpha - \frac{gx^2}{2u^2}(1+\tan^2\alpha)$ (given)		
when they collide:		
$\frac{x}{\sqrt{3}} + h = x\tan\alpha - \frac{gx^2}{2u^2}(1+\tan^2\alpha)$	1	
$\frac{x}{\sqrt{3}} + h - x\tan\alpha + \frac{gx^2}{2u^2}(\sec^2\alpha) = 0$		
$\frac{x}{\sqrt{3}} + h - x\tan\alpha + \frac{gx^2}{2u^2\cos^2\alpha} = 0$		
$\frac{gx^2}{2(6gh)} + x\left(\frac{1}{\sqrt{3}} - \tan\alpha\right) + h = 0$	1	
since $u^2\cos^2\alpha = (\sqrt{6gh})^2$ from (ii)		
$\frac{x^2}{12h} + x\left(\frac{1}{\sqrt{3}} - \tan\alpha\right) + h = 0$		
$\therefore \frac{x^2}{12} + xh\left(\frac{1}{\sqrt{3}} - \tan\alpha\right) + h^2 = 0$	1 $\Rightarrow$ working out.	
iv) α) from equation of (iii)		
$\Delta = h^2\left(\frac{1}{\sqrt{3}} - \tan\alpha\right)^2 - 4\left(\frac{1}{12}\right)h^2$		
$= h^2\left(\frac{1}{3} - \frac{2}{\sqrt{3}}\tan\alpha + \tan^2\alpha\right) - \frac{h^2}{3}$		
$= h^2\left(\tan^2\alpha - \frac{2}{\sqrt{3}}\tan\alpha\right)$		
$= h^2\tan\alpha\left(\tan\alpha - \frac{2}{\sqrt{3}}\right)$	1	

Suggested Solutions	Marks	Marker's Comments
since $\tan\alpha > \frac{2}{\sqrt{3}}$		must have clear explanation of why $\Delta>0$ cannot just substitute $\tan\alpha = \frac{2}{\sqrt{3}}$ and $\therefore \Delta > 0$
$\tan\alpha - \frac{2}{\sqrt{3}} > 0$ and $h^2 > 0$		
$\therefore h^2\tan\alpha\left(\tan\alpha - \frac{2}{\sqrt{3}}\right) > 0$	1	
$\therefore \Delta > 0$		
$\therefore$ the equation has 2 distinct real roots at B and C.		
β) from equation in (iii)		
$x = \dfrac{-h\left(\frac{1}{\sqrt{3}} - \tan\alpha\right) \pm \sqrt{h^2\tan\alpha\left(\tan\alpha - \frac{2}{\sqrt{3}}\right)}}{2 \times \frac{1}{12}}$		
$x = -6h\left(\frac{1}{\sqrt{3}} - \tan\alpha\right) \pm 6\sqrt{h^2\tan\alpha\left(\tan\alpha - \frac{2}{\sqrt{3}}\right)}$		
$x_1 - x_2 = -6h\left(\frac{1}{\sqrt{3}} - \tan\alpha\right) + 6\sqrt{h^2\tan\alpha\left(\tan\alpha - \frac{2}{\sqrt{3}}\right)}$		
$-\left(-6h\left(\frac{1}{\sqrt{3}} - \tan\alpha\right) - 6\sqrt{h^2\tan\alpha\left(\tan\alpha - \frac{2}{\sqrt{3}}\right)}\right)$		
$= 12\sqrt{h^2\tan\alpha\left(\tan\alpha - \frac{2}{\sqrt{3}}\right)}$	1	
time $= \frac{\text{distance}}{\text{speed}}$		
$T = \dfrac{12\sqrt{h^2\tan\alpha\left(\tan\alpha - \frac{2}{\sqrt{3}}\right)}}{u\cos\alpha}$	1	
$= \dfrac{\sqrt{144h^2\tan\alpha\left(\tan\alpha - \frac{2}{\sqrt{3}}\right)}}{\sqrt{6gh}}$		
$= \sqrt{\dfrac{24h\tan\alpha\left(\tan\alpha - \frac{2\sqrt{3}}{3}\right)}{g}}$		
$= \sqrt{\dfrac{\frac{24}{3}\times h\tan\alpha\,(3\tan\alpha - 2\sqrt{3})}{g}}$		
$\therefore T = \sqrt{\dfrac{8h\tan\alpha\,(3\tan\alpha - 2\sqrt{3})}{g}}$	1 $\Rightarrow$ working out.	