Name: ______________________________________

Class: _________________

2019

Higher School Certificate

Trial Examination

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided
- In Questions 11–14, show relevant mathematical reasoning and/or calculations

Total marks – 70

Section I – 10 marks (pages 2–6)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 7–10)

- Attempt Questions 11–14
- Allow about 1 hour and 45 minutes for this section

Section I

Write your answers on the multiple choice answer sheet provided.

1. The solution to the equation $2^x + 2^{x+1} = 8$ is:

   (A) $x = 1$

   (B) $x = 3 - \log_2 3$

   (C) $x = \log_2 5$

   (D) $x = \frac{1}{4}$

2. A polynomial equation has roots α , β and γ , where

   $\alpha + \beta + \gamma = 0; \ \ \alpha\beta\gamma = 1 ; \ \ \alpha\beta + \alpha\gamma + \beta\gamma = 1$

   Which polynomial equation has roots α , β and γ

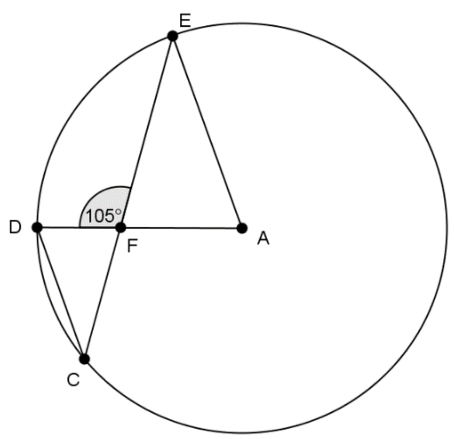
   (A) $x^3 + x - 1 = 0$

   (B) $x^3 + x + 1 = 0$

   (C) $x^3 - x + 1 = 0$

   (D) $x^3 - x - 1 = 0$

3. *D, C,* and *E* are points on a circle with centre *A.*
   DC is parallel to *AE. AD* intersect *CE* at *F.* ∠DFE = 105°



Not to scale

The value of $\angle FAE$ is

(A) 75°

(B) 52.5°

(C) 70°

(D) 105°

4. Consider the function $f(x) = e^{x+2}$ and its inverse function $f^{-1}(x)$. What is the value of $f^{-1}(e^2)$?

(A) $f^{-1}(x) = e^{-(e^2)-2}$

(B) $f^{-1}(x) = e^{(e^2)+2}$

(C) $f^{-1}(x) = 0$

(D) $f^{-1}(x) = 4$

5. The value of $\lim_{x \to 3} \frac{x-3}{\sqrt{x+1}-2}$ is:

(A) 4

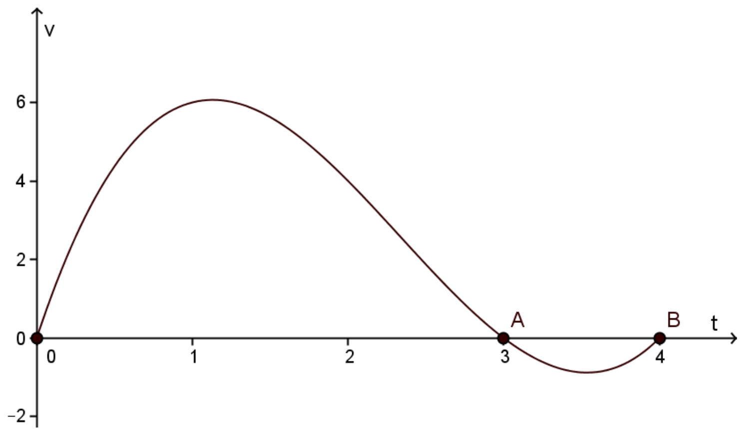
(B) 0

(C) Not defined

(D) 1

6. The graph describes the velocity *v* of a particle at time *t*.

Which of the following statements is true?



(A) The particle is at the origin at A and B.

(B) The particle moves in a positive direction and changes direction after 3 seconds.

(C) The particle has a negative acceleration throughout the last second of its trajectory.

(D) The particle returns to the origin after 4 seconds and comes to rest.

7. A particle is moving along a straight line so that initially its displacement is $x = 1$, its velocity is $v = 2$, and its acceleration is $a = 4$.

Which is a possible equation describing the motion of the particle?

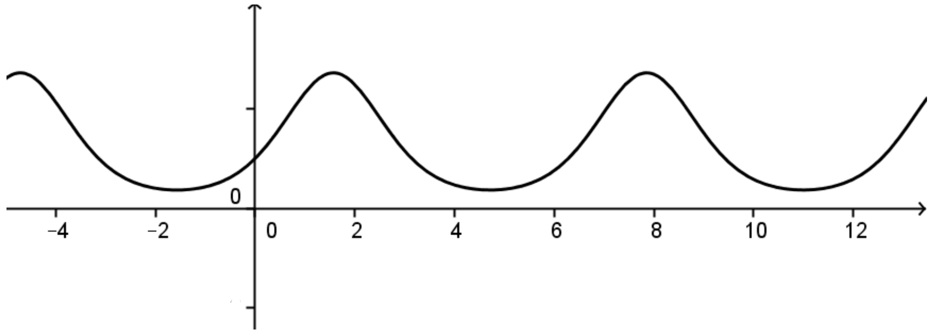
(A) $v = x^2 + 2x + 4$

(B) $v^2 = 4(x^2 - 2)$

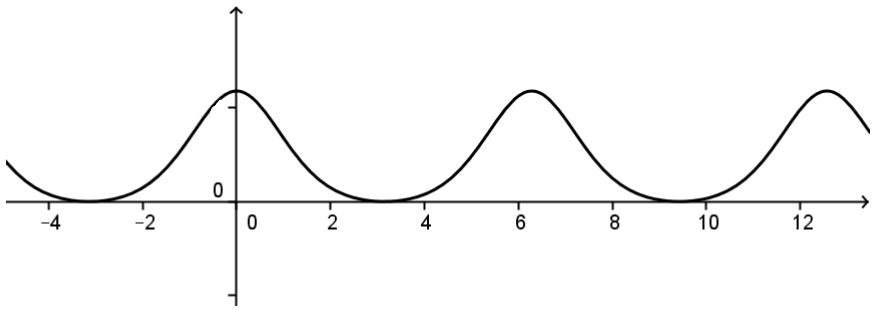
(C) $v = 2 + 4\ln x$

(D) $v = 2\sin(x - 1) + 2$

8. The following curve is the graph of the function $y = e^{\sin x}$



The graph is translated vertically and horizontally so that the resulting graph describes an even function which touches the x-axis, as shown:



A possible equation for the resulting graph is:

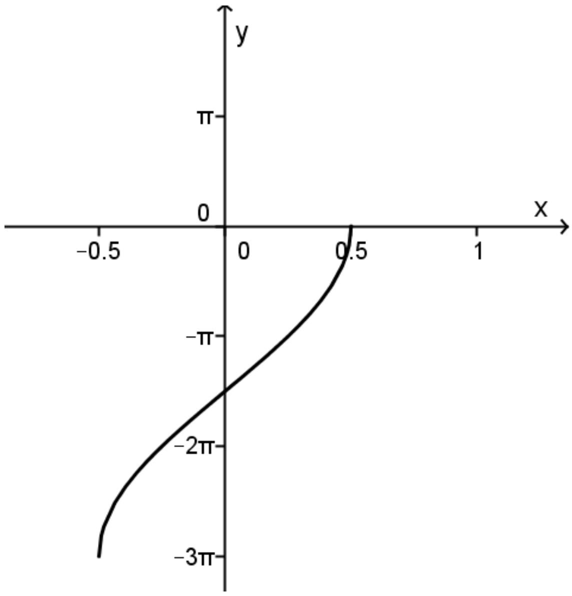
(A) $y = e^{\sin\left(x - \frac{\pi}{2}\right)} - e$

(B) $y = e^{\sin\left(x + \frac{3\pi}{2}\right)} - \frac{1}{e}$

(C) $y = e^{\sin\left(x + \frac{\pi}{2}\right)} - \frac{1}{e}$

(D) $y = e^{\sin\left(x + \frac{3\pi}{2}\right)} - e$

9. The following curve is the graph of which of the following function?



(A) $y = -3\sin^{-1}\frac{x}{2}$

(B) $y = -3\cos^{-1}2x$

(C) $y = 3\cos^{-1}2x$

(D) $y = \frac{1}{2}\cos^{-1}3x$

10. Let $f(x) = ax^m$ and $g(x) = bx^n$, where *a*, *b*, *m* and *n* are positive integers. For both *f* and *g*, the domain is all real *x*.

If $f'(x)$ is a primitive of $g(x)$, then which one of the following must be true?

(A) $\frac{m}{n}$ is an integer

(B) $\frac{n}{m}$ is an integer

(C) $\frac{a}{b}$ is an integer

(D) $\frac{b}{a}$ is an integer

Section II begins on the next page

Section II

60 marks

Attempt Questions 11–14

Allow about 1 hour and 45 minutes for this section

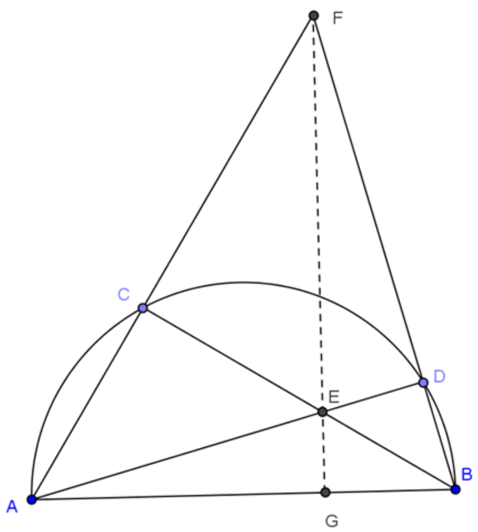
Start each question on a new page. Extra paper is available.

In Questions 11–14, your responses should include relevant mathematical reasoning and/ or calculations.

Question 11 (15 marks) **Start a new page**.

(a) Find the exact value of $\int_0^{\frac{\pi}{2}} \cos x \sin^2 x dx$ **2**

(b)



The points D, C lie on a semicircle with AB as diameter.
AC, BD produced intersect at F; AD, BC intersect at E.

(i) Prove that $CEDF$ is a cyclic quadrilateral. **1**

(ii) Prove that FE is perpendicular to AB. **2**

(c) Let α, β and γ, be the three angles of a given triangle.

(i) Show that $\tan(\alpha+\beta) = -\tan\gamma$ **1**

(ii) Hence, or otherwise, prove that: $\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha \times \tan\beta \times \tan\gamma$ **2**

(d) (i) Show that the equation $\ln(x) = \cos x$ has a solution between $x = 1$ and $x = 2$ **1**

(ii) Using one application of Newton's Method, with $x = 1.5$ as your initial value, find a better estimate for this solution. Present your answer accurate to three decimal places. **2**

(e) The velocity, *v*, measured in centimetres per second, of a particle moving in simple harmonic motion along the *x* axis, is given by $v^2 = 16x - 4x^2 + 20$

(i) Show that $\ddot{x} = -4(x-2)$ **2**

(ii) Find the maximum speed of the particle. **2**

Question 12 (15 marks) **Start a new page**.

(a) Find a general solution to the equation $\sin 3x = \cos 2x$ where *x* is measured in radians. **2**

(b) Air is escaping from a spherical balloon at the rate of 2 cm^3 per second. How fast is the surface area of the balloon shrinking when the radius is 2cm? **3**

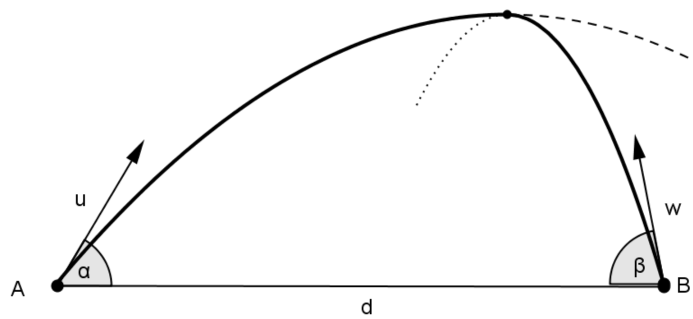
(c) If there exists a non-zero constant term in the expansion of $\left(x^2 - \frac{1}{\sqrt{x}}\right)^n$, show that *n* is a multiple of 5. **2**

(d) From a group of 6 men and 7 women, a committee of 5 people is to be formed.

(i) What is the probability that in the committee there is at least one man and at least one woman? **2**

(ii) Three particular women and two particular men are chosen to be on the committee and the committee members are seated at random around a circular table. What is the probability that the two men are not seated next to each other? **2**

(e) Points *A* and *B* are located *d* metres apart on a horizontal plane. A projectile is fired from A towards B with initial velocity *u* m/s at angle α to the horizontal. At the same time, another projectile is fired from *B* towards *A* with initial velocity w ms^{-1} d at angle β to the horizontal, as shown in the diagram. The projectiles collide when they both reach their maximum height.



The equations of motion of a projectile fired from the origin with initial velocity *V* m/s at angle θ to the horizontal are: $x = Vt\cos\theta$ and $y = Vt\sin\theta - \frac{1}{2}gt^2$ (Do NOT prove this)

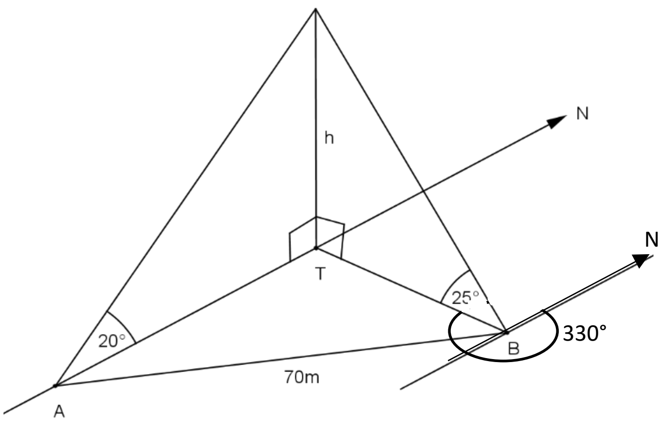
(i) Show that the projectile fired from *A* reaches its maximum height at time $t = \frac{u\sin\alpha}{g}$ **1**

(ii) Show that $u\sin\alpha = w\sin\beta$ **1**

(iii) The distance between *A* and *B*, is given by: $d = \frac{uw}{g}\sin(\alpha+\beta)$ **2**

Question 13 (15 marks) **Start a new page**.

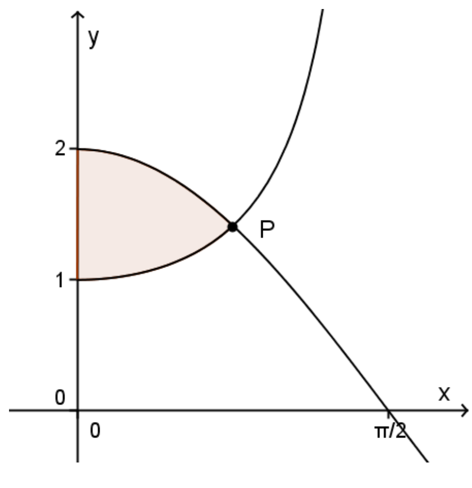
(a) John stands at point *A* and sees a tower due north. The angle of elevation from *A* to the top of the tower is 20°. He then walks in a straight line 70 m to point *B* and notices the tower is now positioned at a bearing of 330° from him. The angle of elevation from point *B* to the top of the tower is now 25°. **4**



Copy the diagram on to your answer sheet showing points *A, B* and *T* and relevant lengths and bearings.

Calculate the height of the tower, correct to 1 decimal place.

(b) *P* is the point of intersection between $x = 0$ and $x = \frac{\pi}{2}$ of the graphs of $y = \sec x$ and $y = 2\cos x$, as shown.



(i) Verify that the x-coordinate of *P* is $\frac{\pi}{4}$ **1**

(ii) The shaded region makes a revolution about the x-axis. Show that the volume of the resulting solid is $\frac{\pi^2}{2}$ cubic units. **3**

(c) Use mathematical induction to prove that for all positive integers $n \geq 1$: **3**

$$1 \times 2^0 + 2 \times 2^1 + 3 \times 2^2 + \ldots + n \times 2^{n-1} = 1 + (n-1) \times 2^n$$

(d) Four fair dice are rolled. Any die showing 6 is left alone, while the remaining dice are rolled again.

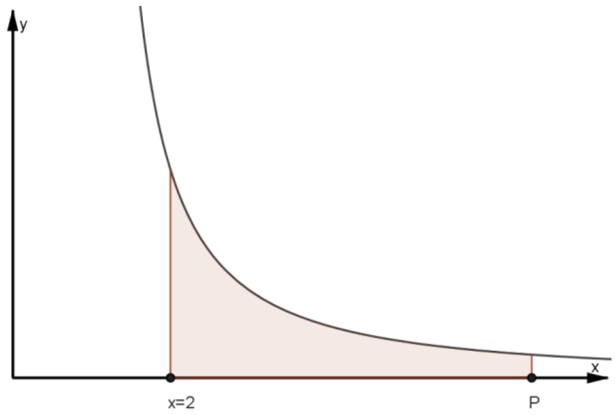
(i) Find the probability (correct to 2 decimal places) that after the first roll of the dice, exactly one of the four dice is showing 6. **1**

(ii) Find the probability (correct to 2 decimal places) that after the second roll of the dice exactly two of the four dice are showing 6. **3**

Question 14 continues over the page

Question 14 (15 marks) **Start a new page**.

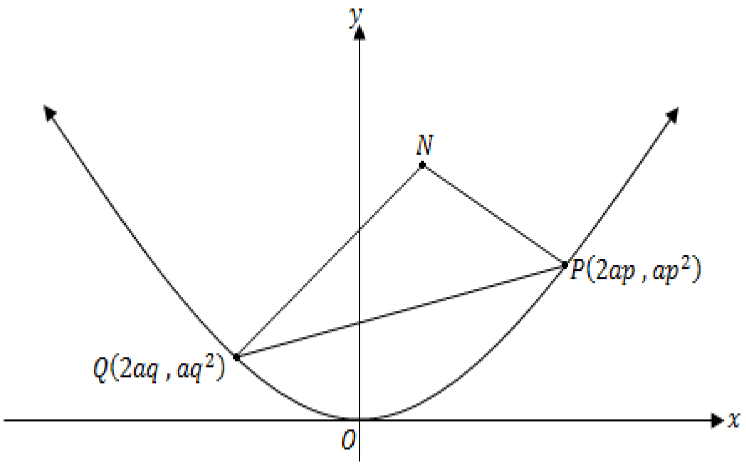
(a) (i) Find $\int \frac{1}{x \ln x} dx$ using the substitution $u = \ln x$ **1**



The curve above is the graph of the function $y = \frac{1}{x \ln x}$

(ii) The area shaded is bounded by the curve, the x axis and the lines $x = 2$ and $x = p$ is 1 square unit. Using part (i), or otherwise, show that $p = 2^e$. **3**

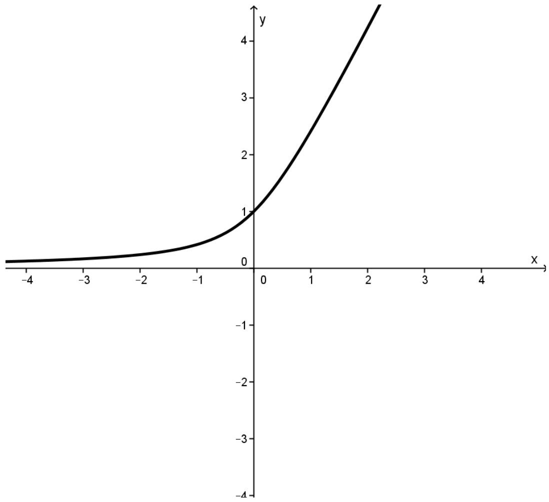
(b) $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are two variable points on a parabola $x^2 = 4ay$.



(i) If the variable chord PQ is always parallel to the line $y = x$, show that $p + q = 2$. **2**

(ii) The normals at P and Q meet at N. Prove that the locus of *N* is a straight line. **3**
[You may assume that the gradient of the tangent at P is p.]

(c) Consider the function $f(x) = x + \sqrt{x^2 + 1}$



(i) Copy the graph of *f*(*x*) and sketch the graph of $f^{-1}(x)$, the inverse function of $f(x)$, on the same axes. **1**

(ii) Show that $f^{-1}(x) = \frac{1}{2}\left(x - \frac{1}{x}\right)$ **2**

(iii) By comparing the graphs of $f(x)$ and $f^{-1}(x)$, or otherwise, show that: **3**

$$\int_0^1 \left(x + \sqrt{x^2 + 1}\right)dx = \frac{1}{2}\left(1 + \sqrt{2} + \ln(1 + \sqrt{2})\right)$$

END OF EXAMINATION

Name: ______________________________________

Class: _________________

2019

Higher School Certificate

Trial Examination

Mathematics Extension 1

SOLUTIONS

Section I B, A, C, C, A, B, C, C, B

Write your answers on the multiple choice answer sheet provided.

1.

$$2^x + 2^x \times 2 = 8$$
$$2^x \times 3 = 8$$
$$2^x = \frac{8}{3}$$
$$x = \log_2\left(\frac{8}{3}\right)$$
$$x = \log_2 8 - \log_2 3$$
$$\underline{x = 3 - \log_2 3}$$

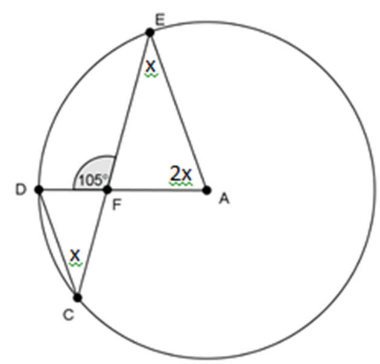
2. A polynomial equation has roots α, β and γ, where

$\alpha + \beta + \gamma =$ - B $\quad\quad$ $\alpha\beta + \alpha\gamma + \beta\gamma =$ C; $\alpha\beta\gamma =$-D

$$B = 0$$
$$C = 1$$
$$D = -1$$
$$\underline{x^3 + x - 1 = 0}$$

3.

$$3x = 105$$
$$x = 35$$
$$\angle FAE = 2x = 70$$



Not to scale

4. $y = e^{x+2}$

$$x = e^{y+2}$$
$$y = \ln(x) - 2$$
$$f^{-1}(x) = \ln(x) - 2$$
$$f^{-1}(e^2) = \ln(e^2) - 2 = 0$$

5.

$$\lim_{x\to 3}\frac{x-3}{\sqrt{x+1}-2}$$

$$\lim_{x\to 3}\frac{x-3}{\sqrt{x+1}-2}\times\frac{\sqrt{x+1}+2}{\sqrt{x+1}+2}$$

$$\lim_{x\to 3}\frac{(x-3)\left(\sqrt{x+1}+2\right)}{x+1-4}$$

$$\lim_{x\to 3}\frac{(x-3)\left(\sqrt{x+1}+2\right)}{x-3}$$

$$\lim_{x\to 3}\frac{(x-3)\left(\sqrt{x+1}+2\right)}{x-3}$$

$$\lim_{x\to 3}\left(\sqrt{x+1}+2\right)$$

$$\lim_{x\to 3}\left(\sqrt{4}+2\right)=4$$

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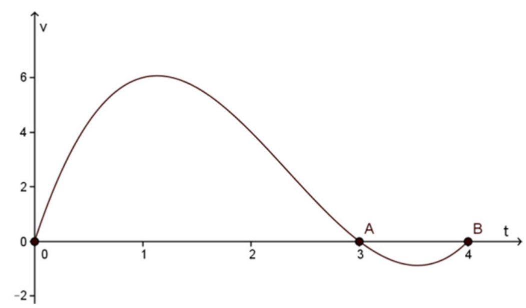
6. The particle moves in a positive direction and changes direction after 3 seconds.



7. $v=2\sin(x-1)+2\quad v^2=4\,[\sin(x-1)+1]^2$

At x=1 V=2

Use $a=\frac{d}{dx}\frac{v^2}{2}=\frac{d}{dx}2\,[\sin(x-1)+1]^2$

$a=4[\sin(x-1)+1]\cos(x-1)$

At x=1 a=4

So, D

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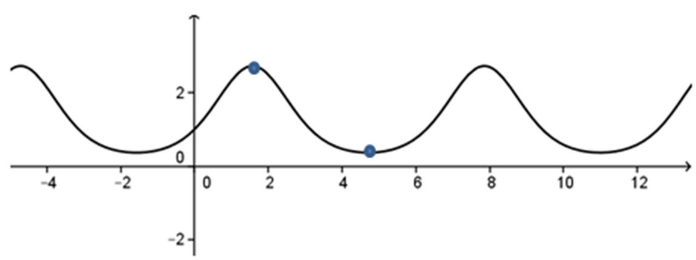
8. The following curve is the graph of the function $y=e^{\sin x}$

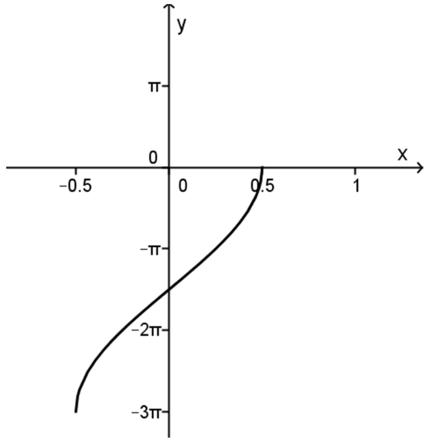


$-1\le\sin x\le 1$

Maximum point is at $y=e$ $\left(\frac{\pi}{2},e\right)$ Minimum point is at $y=e^{-1}$ $\left(\frac{3\pi}{2},\frac{1}{e}\right)$

The graph is translated vertically $\frac{1}{e}$ and horizontally $\frac{\pi}{2}$ So C

9. The following curve is the graph of the function:



At $x = 0$ $y = -\frac{3\pi}{2}$ So $y = -3\cos^{-1} 2x$ So, C

10.

Given $f'(x) = \int g(x)\, dx$,

$$amx^{m-1} = \frac{b}{n+1} x^{n+1}$$

$m - 1 = n + 1 \ldots (1)$ (equate powers)

$am = \frac{b}{n+1} \ldots (2)$ (equate coefficients)

Solving simultaneous equations:

$$m(n+1) = \frac{b}{a}$$

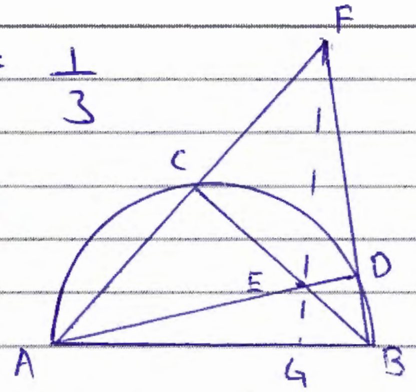
Since $m, n+1$ are both integers (i.e. $\in Z^+$),

$\Rightarrow m(n+1) \in Z^+$

$\therefore \frac{b}{a} = m(n+1) \in Z^+$

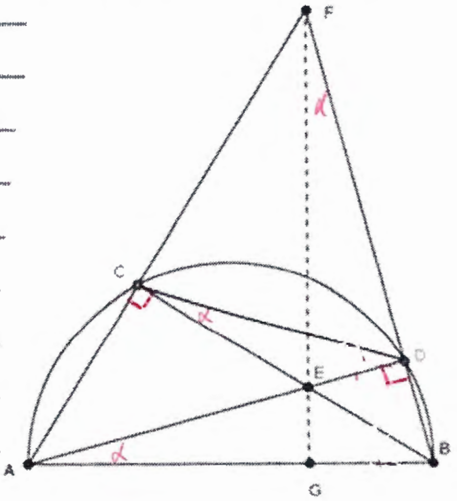
$\Rightarrow D$

MATHEMATICS Extension 1 : Question 11

Suggested Solutions	Marks	Marker's Comments
a) $\int_0^{\frac{\pi}{2}} \cos x \sin^2 x \, dx$		① for Integrating
$= \left[\frac{\sin^3 x}{3} \right]_0^{\frac{\pi}{2}}$		① for correct substitution and evaluation provided the expression wasn't made simpler.
$= \frac{1}{3}$		
b) 		
i $\angle ACB = \angle ADB = 90°$ (angles in a semi-circle)		
$\angle FCB + \angle ACB = 180°$ (angle sum of a straight angle)		
$\therefore \angle FCB = 180 - 90$ $= 90°$		
$\angle FDA + \angle BDA = 180$ (angle sum of a straight angle) $\therefore \angle FDA = 180 - 90$ $= 90°$		
$\therefore \angle FCB + \angle FDA = 180°$ $\therefore$ CEDF is a cyclic quadrilateral.		①

MATHEMATICS Extension 1 : Question.......

Suggested Solutions	Marks	Marker's Comments
ii		Note: Anyone who just states cyclic quadrilaterals without proving them will get 0.



Construct CD

let $\angle DAB = \alpha$

$\angle DAB = \angle DCB$ (angles at the circumference standing on the same arc DB)
$= \alpha$

$\angle DCB = \angle DFE$ (angles at the circumference standing on the same arc, cyclic quad $\angle$FDE)
$= \alpha$

$\angle FGB = 180 - \angle FBG - \alpha$ (angle sum of ΔFGB)

But $\angle FBG = 180 - \alpha - 90$ (angle sum of ΔDAB)

$\therefore \angle FGB = 180 - (180 - \alpha - 90) - \alpha$
$= 90$

$\therefore FE \perp AB$

Mark Allocations:

① if students were able to use 'angles at the circumference standing on the same arc' within the cyclic quadrilateral $\angle$EDF proven in part i)

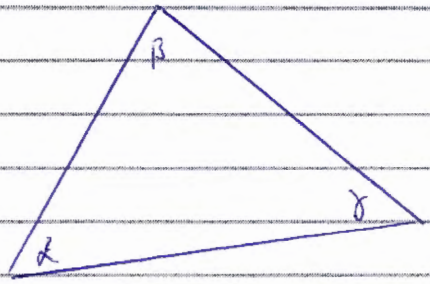
Other methods include:

* Similar Triangles
* Cyclic quadrilaterals
* Altitudes of a triangle are concurrent.

MATHEMATICS Extension 1 : Question……..

Suggested Solutions	Marks	Marker's Comments

c)



i $\alpha + \beta + \gamma = 180$ (angle sum of a triangle)

Marker's comment: Must have all or part of this line in your proof.

$\therefore \alpha + \beta = 180 - \gamma$

$\therefore \tan(\alpha + \beta) = \tan(180 - \gamma)$

$= -\tan\gamma$ (tan is negative in 2nd quadrant)

ii $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = -\tan\gamma$

$\therefore \tan\alpha + \tan\beta = \tan\alpha\tan\beta\tan\gamma - \tan\gamma$

$\therefore \tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\tan\beta\tan\gamma$

Marker's comments: ① for expanding $\tan(\alpha+\beta)$. ① finishing the proof off.

d) let $f(x) = \ln x - \cos x$

$f(1) = 0 - \cos 1$

$= -0.54$

< 0

$f(2) = \ln 2 - \cos 2$

$= 1.11$

> 0

$\therefore$ There lies a solution between $x = 1$ and $x = 2$

Marker's comment: Must show the value for both $f(1)$ and $f(2)$!

MATHEMATICS Extension 1 : Question……..

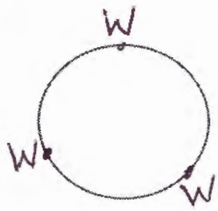
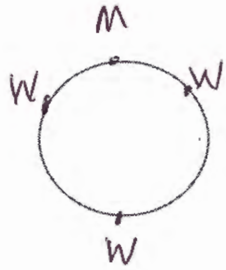
Suggested Solutions	Marks	Marker's Comments
ii) $f'(x) = \frac{1}{x} + \sin x$		①
$\therefore x_2 = 1.5 - \frac{f(1.5)}{f'(1.5)}$		
$= 1.299$		①
e) i $v^2 = 16x - 4x^2 + 20$		
$\frac{1}{2}v^2 = 8x - 2x^2 + 10$		① for multiplying half
$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 8 - 4x$		① for differentiating
$= -4(x-2)$		
$\therefore \ddot{x} = -4(x-2)$		
ii) $\ddot{x} = 0 \rightarrow x = 2$.		① $x=2$
$\therefore v^2 = 16(2) - 4(2)^2 + 20$		
$= 32 - 16 + 20$		
$= 36$		
$\therefore v = \pm 6$ cm/s		
$\therefore$ Max Speed $= 6$ cm/s.		①

Extension 1 Question 12	TRIAL	Term 3 2019	JRAHS

(a) $$\sin 3x = \cos 2x$$ $$\cos\left(\frac{\pi}{2} - 3x\right) = \cos 2x$$ $$\frac{\pi}{2} - 3x = 2\pi n \pm 2x, n \in Z$$ $$3x \pm 2x = \frac{\pi}{2} - 2\pi n$$	1	For expressing the given equation in a form that will enable one to obtain the general solution.
$$x = \frac{\pi}{10} - \frac{2\pi n}{5} \quad or \quad x = \frac{\pi}{2} - 2\pi n$$	1	For both correct solutions
(i) $$\cos 2x = \cos\left(\frac{\pi}{2} - 3x\right) \Rightarrow x = \frac{2\pi n}{5} + \frac{\pi}{10} \quad or \ x = \frac{\pi}{2} - 2\pi n$$		
(ii) $$\sin 3x = \sin\left(\frac{\pi}{2} - 2x\right) \qquad \Rightarrow x = \frac{n\pi + (-1)^n \frac{\pi}{2}}{3 + (-1)^n 2}$$		
(iii) $$\sin\left(\frac{\pi}{2} - 2x\right) = \sin 3x \qquad \Rightarrow x = \frac{\frac{\pi}{2} - n\pi}{2 + (-1)^n 3}$$		
(iv) $$\sin(2x + x) = \cos 2x$$ $$\Rightarrow \quad 4 \sin^3 x - 2\sin^2 x - 3\sin x + 1 = 0$$		
$$\Rightarrow \quad \sin x = 1 \quad or \quad \sin x = \frac{-1 \pm \sqrt{5}}{4}$$		This step is needed for the 1st mark
$\Rightarrow \quad x = \frac{\pi}{2}$ or $x = \frac{\pi}{10}$ $(or\ 18^0)$ or $\frac{3\pi}{10}$ $(or\ 54^0)$ [acute angles]		
So formulate the general solutions!!!!!		

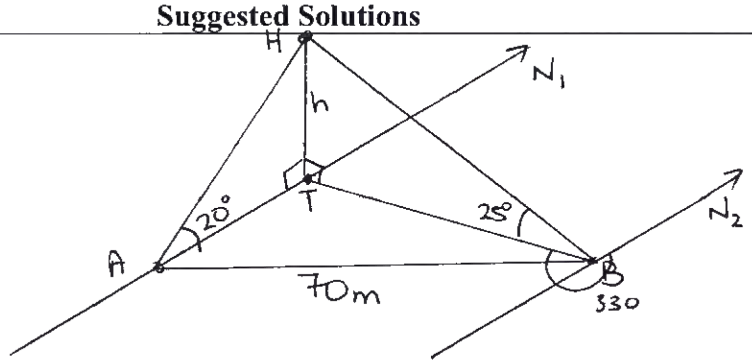
(b)		
$$V = \frac{4}{3}\pi r^3$$ $$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$$ $$\Rightarrow -2 = 4\pi r^2 \times \frac{dr}{dt}$$ $$\therefore \frac{dr}{dt} = -\frac{1}{8\pi}$$ But $$A = 4\pi r^2$$ $$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$$ $$= 8\pi r \times \left(-\frac{1}{8\pi}\right)$$ $$= -r$$ When $r = 2$, $\frac{dA}{dt} = -2$	1 1	For recognising that the rate of volume change is negative AND showing use of the related rates concept. Find the rate of change of the area.
Therefore the balloon is shrinking at 2 cm^2 /s	1	For answering the question i.e. rate at which the balloon shrinks
Note: $$V = \frac{4}{3}\pi r^3 = \frac{1}{3}A\,r$$ $$\frac{dV}{dr} \neq \frac{1}{3}A$$		Maximum 2 marks if negative rate AND/OR concluding statement omitted.

(c) $$\left(x^2-\frac{1}{\sqrt{x}}\right)^n=\sum_{k=0}^{n}\binom{n}{k}(x^2)^k\left(-\frac{1}{\sqrt{x}}\right)^{n-k}$$ $$T_{k+1}=\binom{n}{k}(x^2)^k\left(-\frac{1}{\sqrt{x}}\right)^{n-k}$$ $$=\binom{n}{k}x^{2k}(-1)^{n-k}x^{-\frac{1}{2}(n-k)}$$ For non-zero constant $\frac{5}{2}k-\frac{1}{2}n=0$ Which gives $n=5k$ But $k\in Z$, and hence n is a multiple of 5 Note: If $T_{k+1}=\binom{n}{k}(x^2)^{n-k}\left(-\frac{1}{\sqrt{x}}\right)^k$, then $n=\frac{5k}{4}=5\left(\frac{k}{4}\right)$, which still implies that n is a multiple of 5	 1 1	 For the compact form of the binomial expansion or the general term, **including** the consideration for the negative in the 2nd term of the binomial. For correctly equating the power of the term in x to zero.
(d) (i) Sample space with no restrictions: $\binom{13}{5}=1287$ $P(1M,1W)=1-(P(all\ M)+P(all\ W))$ $$=1-\frac{\binom{6}{5}+\binom{7}{5}}{\binom{13}{5}}$$ $$=\frac{140}{143}$$ OR (4M, 1W), (3M, 2W), (2M, 3W), (1M, 4W) $$P=\frac{\binom{6}{4}\times\binom{7}{1}+\binom{6}{3}\times\binom{7}{2}+\binom{6}{2}\times\binom{7}{3}+\binom{6}{1}\times\binom{7}{4}}{\binom{13}{5}}$$ $$=\frac{1260}{1287}$$ $$=\frac{140}{143}$$	1 1	Sample space - no restrictions Correct probability

(ii) With no restrictions, 5 people in a circle can Be arranged in 4! Ways. Men can be together in 2! Ways Women can be arranged in 3! Ways  $\therefore P(men\ NOT\ together) = 1 - P(men\ together)$ $= 1 - \frac{2! \times 3!}{4!}$ $= 1 - \frac{1}{2}$ $= \frac{1}{2}$ OR  Fix Man: So 2 spots for Him. Woman in 3! Ways So $P(not\ together) = \frac{3! \times 2}{4!} = \frac{1}{2}$	1 1	For sample space Correct probability
(e) (i) $y = ut \sin\alpha - \frac{1}{2}gt^2$ $\dot{y} = u \sin\alpha - gt$ Bu at maximum height $\dot{y} = 0$ Hence $u \sin\alpha - gt = 0$, from which $t = \frac{u \sin\alpha}{g}$	1	For using y and $\dot{y}$ to get time of flight
(ii) By similar arguments to (i), the time for the second particle to reach maximum height is $t = \frac{w \sin\beta}{g}$ As both particles are fired simultaneously, their flight times to collide are equal. Hence $\frac{u \sin\alpha}{g} = \frac{w \sin\beta}{g}$ From which $u \sin\alpha = w \sin\beta$	1	**NOTE**: reaching the same maximum height does not necessarily mean that their flight times are equal. Equating their vertical heights is another way to get the result i.e. $y_A = y_B$

(iii)	As distances are involved, there is no need to consider direction Now $d = x_A + x_B$ $= ut_A \cos\ \alpha + wt_B \cos\ \beta$ $= u\left(\frac{w\sin\beta}{g}\right)\cos\alpha + w\left(\frac{u\sin\alpha}{g}\right)\cos\beta$	1	This step must be shown to get the 1st mark
	since $t_A = t_B$, $\frac{u\sin\alpha}{g} = \frac{w\sin\beta}{g}$	1	The reason for the transformation of the distance equation to include sines and cosines
	$\therefore d = \frac{uw}{g}(\sin\beta\cos\alpha + \sin\alpha\cos\beta)$ $= \frac{uw}{g}\sin(\alpha+\beta)$		**NOTE**: This is a SHOW question and as the algebra is trivial, all working/reasons MUST be included

MATHEMATICS Extension 1 : Question 13

Suggested Solutions	Marks	Marker's Comments
a) 		
$\angle N_2BT = 360° - 330°$ (angles about a point) $= 30°$		
$\angle ATB = 30°$ (alternate angles, parallel lines)		
In ΔATH, $\tan 20 = \frac{h}{AT}$, $AT = \frac{h}{\tan 20}$; In ΔBTH, $\tan 25 = \frac{h}{BT}$, $BT = \frac{h}{\tan 25}$	1	
$70^2 = \left(\frac{h}{\tan 20}\right)^2 + \left(\frac{h}{\tan 25}\right)^2 - 2\left(\frac{h}{\tan 20}\right)\left(\frac{h}{\tan 25}\right)\cos 30$	1	
$\therefore h^2 = \dfrac{70^2}{\frac{1}{\tan^2 20} + \frac{1}{\tan^2 25} - \frac{2\cos 30}{\tan 20 \tan 25}}$ $= 2522.748669\ldots$	1	remember to use degrees!!! on calculator.
$\therefore h = 50.226971\ldots$ $= 50.2$ m (1dp)	1	
b) i) $\sec x = 2\cos x$ $1 = 2\cos^2 x$ $\cos^2 x = \frac{1}{2}$ $\cos x = \pm\frac{1}{\sqrt{2}}$ $x = \frac{\pi}{4}$ for $0 \le x \le \frac{\pi}{2}$	1	
<u>OR</u> when $x = \frac{\pi}{4}$, $\sec\left(\frac{\pi}{4}\right) = \sqrt{2}$ and $2\cos\left(\frac{\pi}{4}\right) = 2 \times \frac{1}{\sqrt{2}} = \sqrt{2}$ $\therefore \sec\frac{\pi}{4} = 2\cos\frac{\pi}{4}$ $\therefore$ x-coordinate of P is $\frac{\pi}{4}$	1	

MATHEMATICS Extension 1 : Question 13

Suggested Solutions	Marks	Marker's Comments
ii) $V = \pi \int_0^{\frac{\pi}{4}} (4\cos^2 x - \sec^2 x)\, dx$	1	
$= \pi \int_0^{\frac{\pi}{4}} (2\cos 2x + 2 - \sec^2 x)\, dx$	1	
as $\cos^2 x = \frac{1}{2}(\cos 2x + 1)$		
$= \pi \left[\sin 2x + 2x - \tan x\right]_0^{\frac{\pi}{4}}$	1	
$= \pi \left[\sin\frac{\pi}{2} + \frac{\pi}{2} - \tan\frac{\pi}{4} - 0\right]$		
$= \pi \left(1 + \frac{\pi}{2} - 1\right)$		
$= \frac{\pi^2}{2}\ u^3$		
c) $1\times 2^0 + 2\times 2^1 + 3\times 2^2 + \dots + n\times 2^{n-1} = 1 + (n-1)\times 2^n$		
Prove true for $n=1$	1	
$LHS = 1\times 2^0$		
$= 1$		
$RHS = 1 + (1-1)\times 2^1$		
$= 1+0$		
$= 1$		
$\therefore LHS = RHS$		
$\therefore$ true for $n=1$		1 for:
Assume true for $n=k$, $k \in \mathbb{Z}^+$		- $k \in \mathbb{Z}^+$
ie. $1\times 2^0 + 2\times 2^1 + \dots + k\times 2^{k-1} = 1 + (k-1)\times 2^k$		- statements of $n=k$, $n=k+1$
Prove true for $n=k+1$		- conclusion
ie. $1\times 2^0 + 2\times 2^1 + \dots + k\times 2^{k-1} + (k+1)\times 2^k = 1 + k\times 2^{k+1}$		- by assumption
$LHS = 1\times 2^0 + 2\times 2^1 + \dots + k\times 2^{k-1} + (k+1)\times 2^k$		
$= 1 + (k-1)\times 2^k + (k+1)\times 2^k$ (by assumption)	1	
$= 1 + 2^k(k-1+k+1)$		
$= 1 + 2^k(2k)$		
$= 1 + 2^{k+1}\times k$		
$= RHS$		
$\therefore$ true for $n=k+1$		
$\therefore$ the statement is true by the process of mathematical induction.		

MATHEMATICS Extension 1 : Question 13

Suggested Solutions	Marks	Marker's Comments
d) i) $P(\text{one } 6) = {}^4C_1 \times \frac{1}{6} \times \left(\frac{5}{6}\right)^3$		
$= \frac{125}{324}$		
$= 0.38580\ldots$		
$= 0.39$ (2dp)	1	
ii) P(two 6 after two rolls)		
= P(1st roll no 6's) × P(2nd roll two 6's)		1 ⇒ one case
+ P(1st roll one 6) × P(2nd roll one 6)		2 ⇒ two cases
+ P(1st roll two 6's) × P(2nd roll no 6)		3 ⇒ all cases plus final solution correct.
$= \left(\frac{5}{6}\right)^4 \times {}^4C_2 \times \left(\frac{1}{6}\right)^2 \times \left(\frac{5}{6}\right)^2$		
$+ {}^4C_1 \times \left(\frac{1}{6}\right) \times \left(\frac{5}{6}\right)^3 \times {}^3C_1 \times \left(\frac{1}{6}\right) \times \left(\frac{5}{6}\right)^2$		
$+ {}^4C_2 \times \left(\frac{1}{6}\right)^2 \times \left(\frac{5}{6}\right)^2 \times \left(\frac{5}{6}\right)^2$		
$= 0.055816329\ldots + \frac{3125}{23328} + \frac{625}{7776}$		
$= 0.270151034\ldots$		
$= 0.27$ (2dp)		

Question 14 (15 marks)

(a)

(i) Let $u = \ln x$ $\qquad \dfrac{du}{dx} = \dfrac{1}{x} \qquad xdu = dx$

$$\int \frac{1}{x\ln x}dx = \int \frac{1}{xu}xdu = \int \frac{1}{u}du = \ln u + C$$

$$\int \frac{1}{x\ln x}dx = \ln(\ln x) + C$$

- Ignore absolute value on the log
- Must use substitution for the mark
- Students must make sure they do not have multiple variables, ie. u and x in the integral.
- Some students still forgetting the +C

1

(ii) $1 = \displaystyle\int_2^P \frac{1}{x\ln x}dx$

$1 = [\ln(\ln x)]_2^P$

$1 = \ln(\ln P) - \ln(\ln 2)$

$1 = \ln\left(\dfrac{\ln P}{\ln 2}\right)$ ← 1st mark for applying log laws

$1 = \ln(\log_2 P)$ (change of base) ← 2nd mark

$e = \log_2 P$ ← 3rd mark

$P = 2^e$ as required

OR

$\dfrac{\ln P}{\ln 2} = e$ ← 2nd mark

$\ln P = e\ln 2$

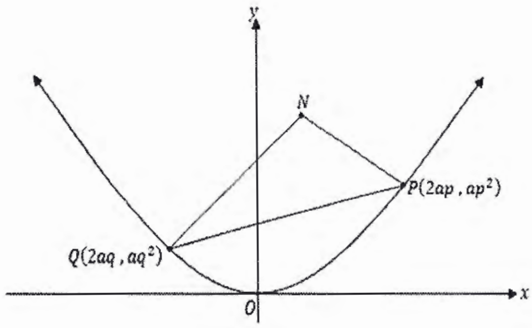
$e^{e\ln} = P$ ← 3rd mark

$(e^{\ln})^e = P$

$2^e = P$

3

(b)



(i) Gradient of the line $y = x$ is 1

$$gradientPQ = \frac{ap^2 - aq^2}{2ap - 2aq}$$

$$gradientPQ = \frac{a(p^2 - q^2)}{2a(p-q)}$$

$$gradientPQ = \frac{a(p-q)(p+q)}{2a(p-q)}$$

$$1 = \frac{p+q}{2}$$

$p + q = 2$ as required

2

(ii) Equation of normals: $y - ap^2 = -\dfrac{1}{p}(x - 2ap)$; $y - aq^2 = -\dfrac{1}{q}(x - 2aq)$ [1st mark]

Solve simultaneously. $y = ap^2 - \dfrac{1}{p}(x - 2ap)$; $y = aq^2 - \dfrac{1}{q}(x - 2aq)$

$ap^2 - \dfrac{1}{p}(x - 2ap) = aq^2 - \dfrac{1}{q}(x - 2aq) \qquad / \times pq$

$ap^3q - q(x - 2ap) = aq^3p - p(x - 2aq)$

3

$$ap^3q - aq^3p = q(x-2ap) - p(x-2aq)$$

$$apq\left(p^2-q^2\right) = x(q-p)$$

$$apq\left(p-q\right)(p+q) = -x(p-q)$$

$-apq(p+q) = x$ use: $p+q=2$

$-2apq = x$ [2^{nd} mark for using $p+q=2$]

substitute in $y = ap^2 - \frac{1}{p}(x-2ap)$ to find y value

$y = ap^2 - \frac{1}{p}(-2apq - 2ap)$

$y = ap^2 + 2aq + 2a$ use: $p = 2-q$

$y = ap(2-q) + 2a(2-p) + 2a$

$y = 2ap - apq + 4a - 2ap + 2a$

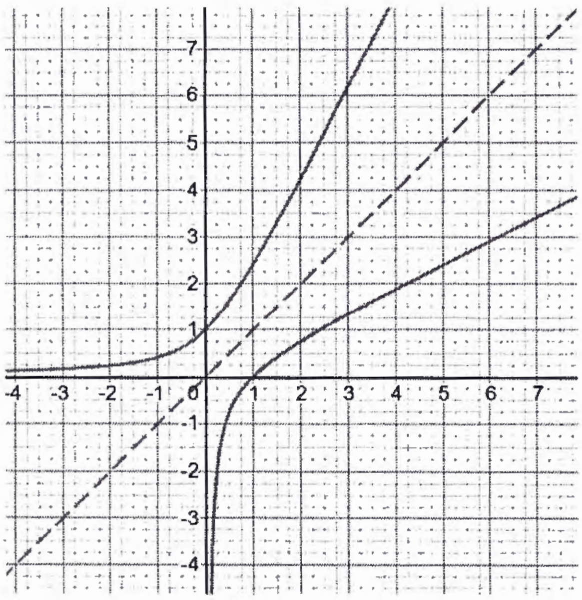
$y = -apq + 6a$ use: $-apq = \frac{x}{2}$

[3^{rd} mark for eliminating p and q and obtaining straight line in form y=mx+b]

$y = \frac{x}{2} + 6a$ as required

(c)

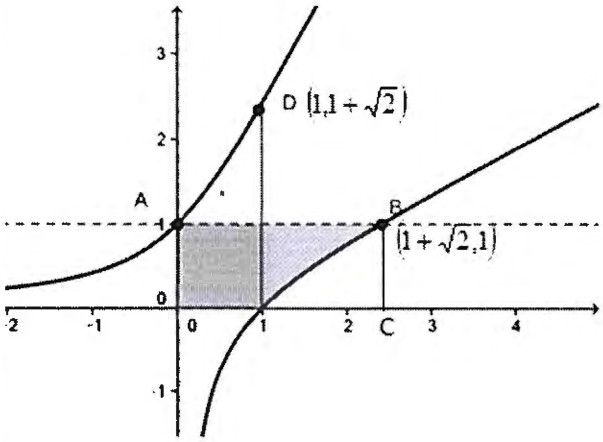
$f(x) = x + \sqrt{x^2+1}$



When asked to sketch the inverse of a function students must ensure that the two graphs are symmetrical. 1

Therefore:

- x and y axes must have the **same scale**
- y = x should also be drawn to show the line of symmetry
- intercepts must be clearly labelled

(i)	$y = x + \sqrt{x^2+1}$ $x = y + \sqrt{y^2+1}$ $x - y = \sqrt{y^2+1}$ $(x-y)^2 = y^2+1$ $x^2 - 2xy + y^2 = y^2 + 1$ $x^2 - 2xy = 1$ $x^2 - 1 = 2xy$ $\frac{x^2-1}{2x} = y$ $\frac{1}{2}\left(\frac{x^2-1}{x}\right) = y$ $f^{-1}(x) = \frac{1}{2}\left(x - \frac{1}{x}\right)$	2
(ii)	 [1st mark for explaining how to use graph to evaluate integral] By inspection of the two graphs: $\int_0^1 \left(x+\sqrt{x^2+1}\right)dx = \text{Area of rectangle ABCO} - \frac{1}{2}\int_1^{1+\sqrt{2}} \left(x - \frac{1}{x}\right)dx$ $A = 1+\sqrt{2} - \frac{1}{2}\left[\frac{x^2}{2} - \ln x\right]_1^{1+\sqrt{2}}$ $A = 1+\sqrt{2} - \frac{1}{2}\left[\frac{(1+\sqrt{2})^2}{2} - \ln(1+\sqrt{2}) - \left(\frac{1}{2}\right)\right]$ [2nd mark for substitution into correct expression]	3

$$A = 1+\sqrt{2} - \left[\frac{(1+\sqrt{2})^2}{4} - \frac{\ln(1+\sqrt{2})}{2} - \frac{1}{4}\right]$$

$$A = 1+\sqrt{2} - \left[\frac{3+2\sqrt{2}-1}{4} - \frac{\ln(1+\sqrt{2})}{2}\right]$$

$$A = \frac{4+4\sqrt{2}}{4} - \frac{2+2\sqrt{2}}{4} + \frac{\ln(1+\sqrt{2})}{2}$$

$$A = \frac{4+4\sqrt{2}-2-2\sqrt{2}}{4} + \frac{\ln(1+\sqrt{2})}{2}$$

$$A = \frac{2+2\sqrt{2}}{4} + \frac{\ln(1+\sqrt{2})}{2}$$

$A = \frac{1+\sqrt{2}}{2} + \frac{\ln(1+\sqrt{2})}{2} = \frac{1}{2}\left[1+\sqrt{2}+\ln(1+\sqrt{2})\right]$ as required [3rd mark for final expression]