Student Name: ____________________

Maths class: ____________________

James Ruse Agricultural High School

2020 **YEAR 12 Trial HSC Examination**

Mathematics Extension 1

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- In Questions 11–14, show relevant mathematical reasoning and/ or calculations

Total marks: 70

Section I – 10 marks (pages 2–4)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 5-9)

- Attempt Questions 11–14
- Allow about 1 hour and 45 minutes for this section

Section I

Write your answers on the multiple choice answer sheet provided.

1. The probability of success in a Bernoulli trial is 0.3. What is the variance?

   (A) 0.09
   (B) 0.7
   (C) 0.3
   (D) 0.21

2. Which one of the following is a first-order linear differential equation?

   (A) $xy' = 14x^2 + 9y$
   (B) $y'' + 2y' - 8y = 0$
   (C) $y^2 + y + x = 0$
   (D) $y'y - 2x^2y^2 = 14x$

3. How many arrangements of the letters of the word 'OLYMPIC' are possible if the C and the L are to be together?

   (A) 120
   (B) 720
   (C) 240
   (D) 1440

4. If $P(x) = x^3 - 6x^2 + 9x + k$ has a root of multiplicity 2 then a possible value of k is which of the following?

   (A) $k = 4$
   (B) $k = 1$
   (C) $k = -4$
   (D) $k = -1$

5. The derivative of $\sin^{-1}\left(\frac{3x}{4}\right)$ is which of the following?

(A) $\frac{0.75}{\sqrt{1-3x^2}}$

(B) $\frac{3}{\sqrt{16-9x^2}}$

(C) $\frac{3}{\sqrt{1-\frac{9x^2}{16}}}$

(D) $\frac{1}{\sqrt{16-9x^2}}$

6. A random variable X is defined by $P(X = k) = \binom{15}{\mathrm{k}}\,(0.29)^k(0.71)^{15-k}$ for $k = 0, 1, 2, \ldots, 15.$
What is the mean of X?

(A) 0.29

(B) 0.71

(C) 4.35

(D) 10.65

7. How many of the following statements are true?

$|\vec{a}| + |\vec{b}| = |\vec{a} + \vec{b}|$ means that $\vec{a}$ and $\vec{b}$ have the same direction.

$|\vec{a}| + |\vec{b}| = |\vec{a} - \vec{b}|$ means that $\vec{a}$ and $\vec{b}$ have the opposite directions.

$|\vec{a}| + |\vec{b}| = |\vec{a} - \vec{b}|$ means that $\vec{a}$ and $\vec{b}$ have the same magnitude.

$|\vec{a}| - |\vec{b}| = |\vec{a} - \vec{b}|$ means that $\vec{a}$ and $\vec{b}$ have the same direction.

(A) 1

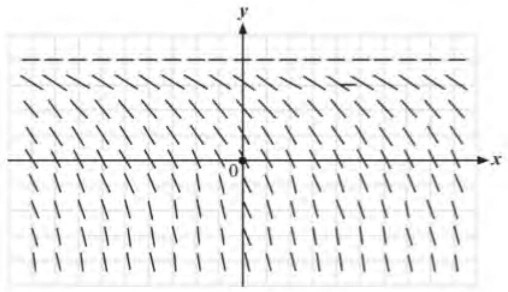
(B) 2

(C) 3

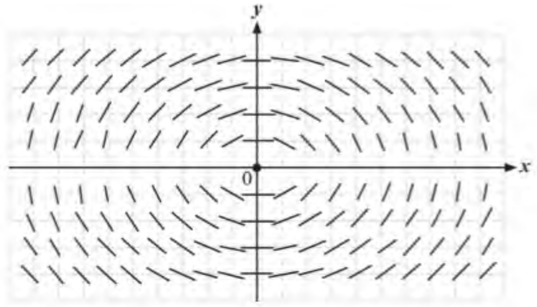
(D) 4

8. The slope field that represents the differential equation $\frac{dy}{dx} = \frac{x}{2y}$ is which of the following?

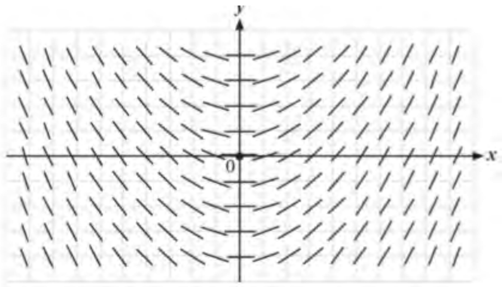
(A)



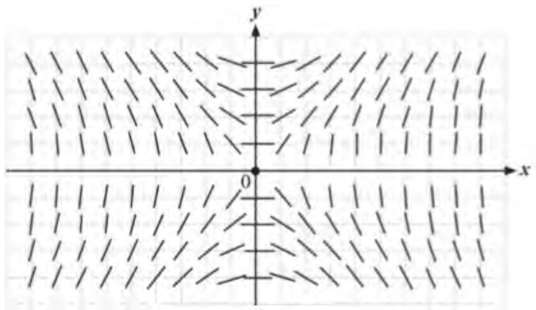
(B)



(C)



(D



9. If the point $P(4, \text{a})$ is on the graph of parametric equations $x = \frac{t}{2}, y = 2\sqrt{t}$, there is another point F(2,0), such that $|\overrightarrow{PF}|$ is which of the following?

(A) 4

(B) 5

(C) 6

(D) 7

10. It is given that $\vec{a} = (-3, \text{m}), \vec{b} = (4, 3)$. If the angle between vector $\vec{a}$ and $\vec{b}$ is an obtuse angle, what is true for the value of m?

(A) $\text{m} < 4$

(B) $\text{m} < 4$ and $\text{m} \neq -\frac{9}{4}$

(C) $\text{m} > 4$

(D) $\text{m} \neq 4$ and $\text{m} > -\frac{9}{4}$

Section II begins on the next page

Section II

60 marks

Attempt Questions 11-14

Allow about 1 hour and 45 minutes for this section

Start each question on a new page. Extra paper is available.

In Questions 11-14, your responses should include relevant mathematical reasoning and/ or calculations.

Question 11 (15 marks) **Start a new page.**

a) Find $\int \frac{5}{2+x^2}\,dx$. **2**

b) In a dice game, success is defined as obtaining a total of 9 when throwing two dice. Six rounds of the game are played. Find the probability of at least five successes. Give your answer to 3 significant figures. **3**

c) (i) Express $3\cos\theta + 4\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. **2**

(ii) Hence find, without the use of calculus, the coordinates of the turning point(s) of the curve **2**

$$y = \frac{2}{3\cos\theta + 4\sin\theta}$$

in the interval $[0,2\pi]$.

(iii) The function f is defined by $f(\theta) = 1 - 3\cos 2\theta - 4\sin 2\theta$, $\theta \in \mathbb{R}$, $0 < \theta < \pi$.

α) State the range of f. **1**

β) Solve the equation $f(\theta) = 0$. **2**

d) (i) Given A (-2, 3), B (4, -5), C (-7, -6) and D (-5, -2), find the vector projection of $\overrightarrow{AB}$ on to $\overrightarrow{CD}$. **2**

(ii) What is the vector component of $\overrightarrow{AB}$ perpendicular to $\overrightarrow{CD}$? **1**

Question 12 (15 marks) **Start a new page.**

a) Consider the parabola $x^2 = 8(y-3)$.

(i) Sketch the parabola labelling its vertex. **1**

(ii) The area bounded by the parabola and the line $y = 5$ is rotated about the x-axis. **3**
Find the volume of the solid formed.

b) It is known that 80% of patients with a certain disease can be cured with a certain drug. **2**
What is the probability that amongst 150 patients with the disease, at most 37 of them cannot be cured with the drug? You must justify the use of the normal approximation.

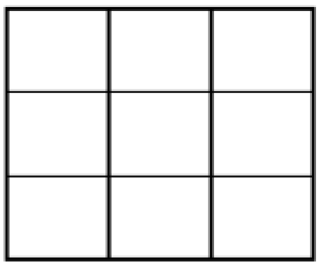
c) (i) Show that **1**

$$\tan\left(\theta + \frac{\pi}{6}\right) = \frac{1 + \sqrt{3}\tan\theta}{\sqrt{3} - \tan\theta}$$

(ii) Hence, or otherwise, solve for $0 \leq \theta \leq \pi$, **2**

$$1 + \sqrt{3}\tan\theta = \left(\sqrt{3} - \tan\theta\right)\tan(\pi - \theta)$$

d) A 3×3 grid is to be filled with numbers from the set $\{-1, 0, 1\}$. Prove that among the **2**
sums by rows, columns and diagonals, there are at least 2 of these sums equal.



e) In the isosceles triangle ABC, $|\overrightarrow{AB}| = |\overrightarrow{AC}|$. D is the midpoint of side AB and E is the midpoint of side AC. $\overrightarrow{CD}$ is perpendicular to $\overrightarrow{BE}$.

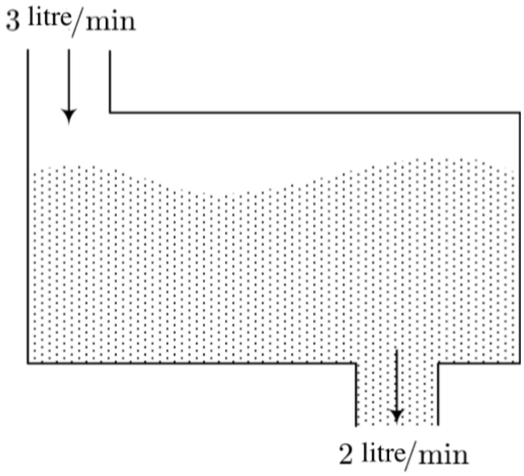
(i) Draw the diagram and label $\angle BAC = \theta$ and $|\overrightarrow{AD}| = r$. **1**

(ii) Hence noting that $\overrightarrow{CD}$ may be written as $\overrightarrow{AD} - \overrightarrow{AC}$, or otherwise, use vector **3**
methods to find the value of $\angle BAC$.

Question 13 (15 marks) **Start a new page.**

a) Use the substitution $u = \sqrt{x}$ to find $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$. **2**

b) A tank contains a saltwater solution consisting initially of 20 kg of salt dissolved into 10 L of water. Fresh water is being poured into the tank at a rate of 3 L/min and the solution (kept uniform by stirring) is flowing out at 2 L/min.



(i) Show that Q, the amount of salt (in kilograms), at time t (in minutes) satisfies the equation **2**

$$\frac{dQ}{dt} = -\frac{2Q}{(10+t)}$$

(ii) Solve the differential equation given in (i) to find the amount of salt in the tank after 5 minutes. Answer in kilograms correct to 2 decimal places. **3**

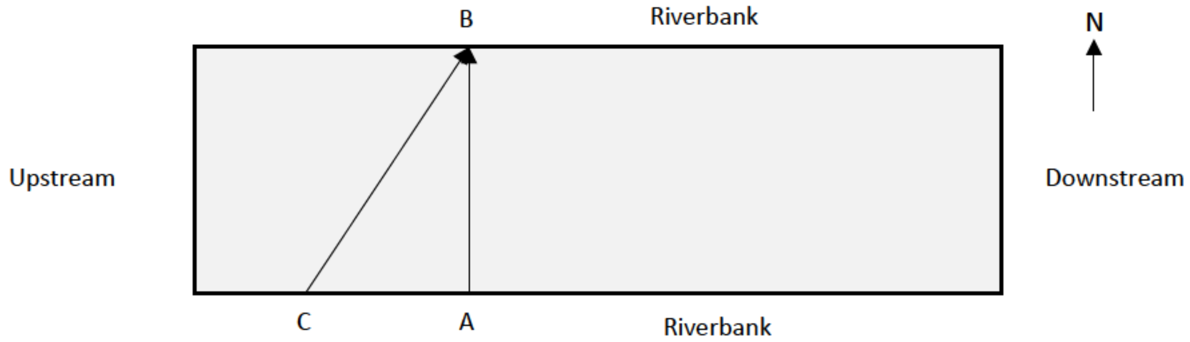
c) (i) Show that $\sin 2\theta + \sin 4\theta - \sin 6\theta = 4\sin 3\theta \sin 2\theta \sin\theta$. **2**

(ii) Hence, solve $\sin 2\theta + \sin 4\theta = \sin 6\theta$ for $0 \le \theta \le \pi$. **2**

Question 13 continues on the next page

Question 13 (continued)

d) Danny wants to swim across the river from position A to position B shown on the diagram below. The riverbanks are parallel straight lines, and $\overrightarrow{AB}$ is perpendicular to the riverbanks. The speed of the river flow is 30 m/min from west to east and the river is 240 metres wide. Assume Danny swims at a constant speed of 60 m/min .

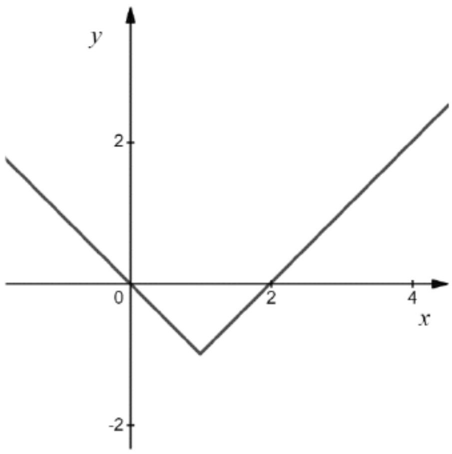


(i) In which direction should Danny aim to swim, in order to move from point A to point B in a straight line under the influence of the current? Draw a vector diagram to demonstrate the velocities and show all necessary working out. **1**

(ii) Gabriel also swims at 60 m/min. but he is to start at position C, which is 60 metres upstream from position A as shown in the diagram. In which direction should Gabriel aim to swim in order to swim directly to position B under the influence of the current? (Draw a vector diagram to demonstrate the velocities and show all necessary working out. Round your answer to one decimal place.) **3**

Question 14 (15 marks) **Start a new page.**

a) Given the graph of $f(x) = |x - 1| - 1$ below, on separate axes sketch of each of the following.



(i) $y = \sqrt{f(x)}$ **2**

(ii) $y = \frac{1}{f(x)}$ **2**

Question 14 continues on the next page.

b) Consider

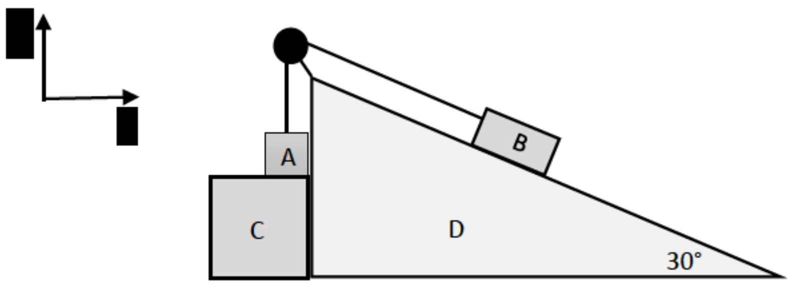
$$f(x) = 4\sin^2\left[2\left(x + \frac{\pi}{8}\right)\right] - 2, \ \ 0 < x < \frac{\pi}{4}.$$

(i) Verify that $f(x)$ could be a probability density function. **3**

(ii) Find the mode of such a distribution. **1**

c) 12 cards are drawn from a regular deck of 52 playing cards which includes four kings and four queens. It is known that out of these 12 cards, exactly 2 are kings. **3**
What is the probability that there will be more queens than kings?

d) The diagram below shows object A of mass 8 kg and object B of mass 5 kg. Both objects are attached to the ends of a light inextensible string that passes over a smooth pulley. Object C is fixed to the ground, and D is a fixed plane at an inclination of 30 degrees as shown.
All objects are at rest, and all surfaces are smooth. g is 10 ms^{-2}.
The basis vectors are shown below.



Let $\underset{\sim}{N_1}$ be the normal force of object C acting on object A and let $\underset{\sim}{N_2}$ be the normal force of the inclined plane acting on object B. Let $\underset{\sim}{T_1}$ and $\underset{\sim}{T_2}$ be the tension forces in the string applied to object A and object B, respectively.

Given $\left|\underset{\sim}{T_1}\right| = t$ Newtons and $\underset{\sim}{N_2} = \begin{bmatrix} \frac{n}{2} \\ \frac{\sqrt{3}}{2}n \end{bmatrix}$

(i) Express $\underset{\sim}{T_2}$ in terms of t in column vector form. **1**

(ii) By setting up vector equations, find the values of t and n. **2**

(iii) Hence, find the components of $\underset{\sim}{N_1}$. **1**

END OF EXAMINATION

MATHEMATICS Extension 1: Multiple Choice __		
Suggested Solutions	**Marks**	**Marker's Comments**
1. D 2. A 3. D 4. C 5. B 6. C 7. C 8. D 9. C 10. B		

Ext 1 Trials 2020.

MATHEMATICS Extension 1: Question 11 — 1/3

Suggested Solutions	Marks	Marker's Comments

a) $\int \frac{5}{2+x^2}\,dx$

$= \int \frac{5\sqrt{2}}{\sqrt{2}^2+x^2}\,dx$

$= \frac{5}{\sqrt{2}} \tan^{-1} \frac{x}{\sqrt{2}} + C$ (1) (1)

Marker's comments: – most did well. – can't use ln. – its in reference sheet, easy question

b) $X \sim Bin\left(6, \frac{1}{9}\right)$

$P(x \geq 5) = P(x=5) + P(x=6)$

${}^6C_5\left(\frac{1}{9}\right)^5\left(\frac{8}{9}\right) + {}^6C_6\left(\frac{1}{9}\right)^6$

$= 0.0000922$

	1	2	3	4	5	6
1						
2						
3						✓
4					✓	
5				✓		
6			✓			

Total of $9 = \frac{1}{9}$ — (1)

correlates to ${}^6C_5$ and ${}^6C_6$ — (1)

– answer — (1)

if P(success) was wrong 2nd mark can only be obtained if students showed an understanding of ${}^6C_5$ & ${}^6C_6$

c) i) $3\cos\theta + 4\sin\theta = R\cos(\theta - \alpha)$

$= R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$

equate coefficient.

$\cos\theta = R\cos\alpha = 3$ — i

$\sin\theta = R\sin\alpha = 4$ — ii

$(i + ii)^2$

$R^2(\cos^2\alpha + \sin^2\alpha) = 25$

$R = \pm 5$ $\quad R>0 \therefore R=5$ — (1)

key pts given in question

$\therefore 5\cos\alpha = 3$ $\quad 5\sin\alpha = 4$

$\cos\alpha = \frac{3}{5}$ $\quad \sin\alpha = \frac{4}{5}$

$= 0.927$ $\quad = 0.927$ — (1)

or

$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{4}{3}$

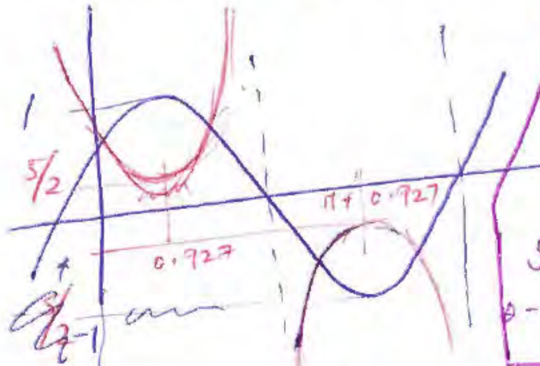
$\alpha = 0.927$ $\quad \left(0 < \alpha < \frac{\pi}{2}\right)$

key pts given in que

other forms accepted $\tan^{-1}\left(\frac{4}{3}\right)$, 53.13°, 0.295π

MATHEMATICS Extension 1: Question 11 — 2/3

Suggested Solutions	Marks	Marker's Comments
c) ii) without using calculus		



$\frac{2}{3\cos\theta + 4\sin\theta}$

$= \frac{2}{5\cos(\theta - \tan^{-1}(\frac{4}{3}))}$ — (2)

$\cos\theta$ $-1 \le k \le 1$

$\cos(\theta - \tan^{-1}(\frac{4}{3})) = \pm 1$

$y_{max} = \frac{2}{5}$ $\quad y_{min} = -\frac{2}{5}$

$\theta - \tan^{-1}(\frac{4}{3}) = \cos^{-1}(1)$ $\quad \theta - \tan^{-1}(\frac{4}{3}) = \cos^{-1}(-1)$

$\theta = 0.927$ $\quad \theta = 4.069$

$\therefore$ tp $(0.927, \frac{2}{5})$ or $(4.069, -\frac{2}{5})$ — (1)

other answers accepted

$\rightarrow (\tan^{-1}(\frac{4}{3}), \frac{2}{5})$ or $(\tan^{-1}(\frac{4}{3}) + \pi, -\frac{2}{5})$

$\rightarrow (0.295\pi, \frac{2}{5})$ or $(1.295\pi, -\frac{2}{5})$

$\rightarrow (53.13°, \frac{2}{5})$ or $(233.13°, -\frac{2}{5})$

Marker's comments: mark was awarded if both x and y coordinates correct for both pts (2). (1): either both x or 'both y' or both correct x & y coordinates of 1 pt.

c) iii) α) $f(\theta) = 1 - 3\cos 2\theta - 4\sin 2\theta$

$= 1 - 5\cos\theta(2\theta - 0.927)$

$\therefore -5 \le 5\cos(2\theta - 0.927) \le 5$

$-4 \le 1 - 5\cos(2\theta - 0.927) \le 6$

Range: $-4 \le f(\theta) \le 6$ — (1)

Marker's comments: is 1 mark got to have both intervals correct.

β) $f(\theta) = 1 - 5\cos(2\theta - \tan^{-1}(\frac{4}{3})) = 0$

$\cos 2\theta - \tan^{-1}(\frac{4}{3}) = \frac{1}{5}$

$2\theta - \tan^{-1}(\frac{4}{3}) = \cos^{-1}(\frac{1}{5})$

$= 1.3696, 4.9137$

$2\theta = 2.29, 5.84$

$\theta = 1.14, 2.92$ — (1) (1)

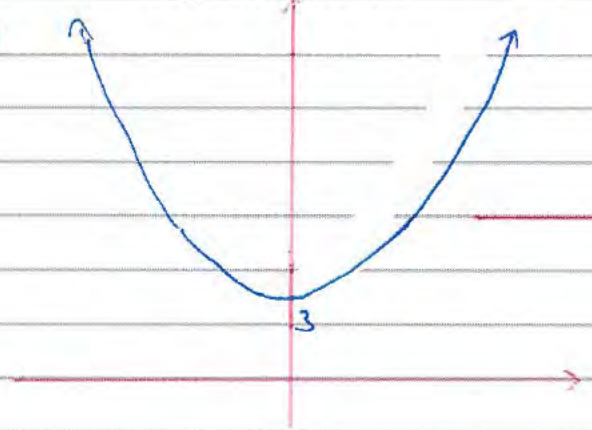
Marker's comments: 1 mark for each value of θ

MATHEMATICS Extension [redacted]: Question 11		3/3
Suggested Solutions	**Marks**	**Marker's Comments**
d) i) (M1) $\vec{AB} = \begin{pmatrix}4\\-5\end{pmatrix} - \begin{pmatrix}-2\\3\end{pmatrix} = \begin{pmatrix}6\\-8\end{pmatrix}$		
$\vec{CD} = \begin{pmatrix}-5\\-2\end{pmatrix} - \begin{pmatrix}-7\\-6\end{pmatrix} = \begin{pmatrix}2\\4\end{pmatrix}$ $\quad |\vec{AB}| = \sqrt{6^2+(-8)^2} = 10$ $\quad |\vec{CD}| = \sqrt{2^2+4^2} = 2\sqrt{5}$		
$proj_{\vec{CD}} \vec{AB} = \frac{\vec{AB}\cdot\vec{CD}}{|\vec{AB}|\cdot|\vec{CD}|} \times \frac{\vec{CD}}{|\vec{CD}|} \cdot |\vec{AB}|$	1	proj correctly stated
$= \frac{\begin{pmatrix}6\\-8\end{pmatrix} \times \begin{pmatrix}2\\4\end{pmatrix}}{2\sqrt{5}} \times \frac{\begin{pmatrix}2\\4\end{pmatrix}}{2\sqrt{5}}$	(1)	
$= \frac{(6\times2)+(-8\times4)}{20} \times \begin{pmatrix}2\\4\end{pmatrix}$		
$= \frac{-20}{20}\begin{pmatrix}2\\4\end{pmatrix} = \begin{pmatrix}-2\\-4\end{pmatrix}$	(1)	correct answer
(M2) $\vec{AB} = \begin{pmatrix}6\\-8\end{pmatrix}$ $\quad proj_{\vec{CD}} \vec{AB} \quad \frac{\vec{AB}\cdot\vec{CD}}{|\vec{CD}|^2} \cdot \vec{CD}$	(1)	
$\vec{CD} = \begin{pmatrix}2\\4\end{pmatrix}$ $\quad = \begin{pmatrix}-2\\-4\end{pmatrix}$	(1)	
ii) perp: $\vec{AB}$ − perp — from pt (i)		
$\begin{pmatrix}6\\-8\end{pmatrix} - \begin{pmatrix}-2\\-4\end{pmatrix}$		
$= \begin{pmatrix}8\\-4\end{pmatrix}$ — (1)		
or		
component of $\vec{AB}$ perp to $\vec{CD} \Rightarrow \vec{AB} - proj_{\vec{CD}} \vec{AB}$		
$\begin{pmatrix}6\\-8\end{pmatrix} - \begin{pmatrix}-2\\-4\end{pmatrix} = \begin{pmatrix}8\\-4\end{pmatrix}$		(1)

MATHEMATICS Extension 1 : Question 12

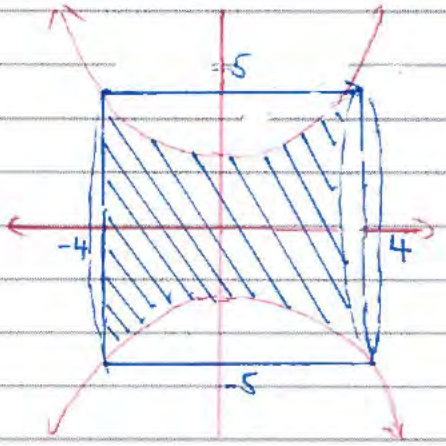
Suggested Solutions	Marks	Marker's Comments

12. a) i)



① Concave up with a vertex

ii)



$V = V_{cylinder} - \pi \int_{-4}^{4} y^2 \, dx$ $\qquad x^2 = 8(y-3)$

$\qquad y - 3 = \frac{x^2}{8}$

$= \pi(5)^2(8) - \pi \int_{-4}^{4} \left(\frac{x^2}{8} + 3\right)^2 dx$ $\qquad y = \frac{x^2}{8} + 3$

① Recognising the volume is obtained by a subtraction between 2 solids

$= 200\pi - \pi \int_{-4}^{4} \left(\frac{x^4}{64} + \frac{3x^2}{4} + 9\right) dx$

$= 200\pi - 2\pi \int_{0}^{4} \left(\frac{x^4}{64} + \frac{3x^2}{4} + 9\right) dx$

$= 200\pi - 2\pi \left[\frac{x^5}{320} + \frac{x^3}{4} + 9x\right]_0^4$

① Integrating

$= 200\pi - 2\pi \left[\frac{16}{5} + 16 + 36\right]$

$= \frac{448\pi}{5}$

① Evaluating

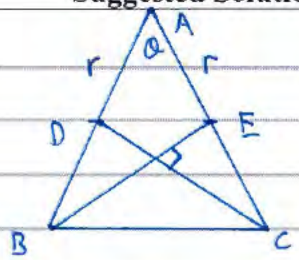
Note: Failure to set up the initial integral properly will result in 1/3 maximum, you do not get 2/3 just for Evaluating any ordinary definite integral.

MATHEMATICS Extension 1 : Question 12

Suggested Solutions	Marks	Marker's Comments
b) $n = 150 \quad p = 0.8, \quad q = 0.2$		
$np = 120 > 10 \qquad nq = 30 > 10$		
$\therefore$ probability can be approximated using the normal distribution.		(1)
Method 1:		
Solve $P(X \geq 113)$ where X is the random variable representing the number of people cured.		
$\sigma = \sqrt{150 \times 0.8 \times 0.2} = \sqrt{24} \qquad E(X) = np = 120$		
$\therefore z_{37} = \dfrac{113 - 120}{\sqrt{24}} = -1.43$		
$\therefore P(Z \geq -1.43) = 1 - 0.0764 = 0.9236 = 92.36\%$		(1) for answer
Method 2:		
Solve $P(\hat{p} \geq \frac{113}{150})$, $\quad \sigma = \sqrt{\dfrac{0.8 \times 0.2}{150}} = 0.033$		
$E(\hat{p}) = 0.8$		
$\therefore z = \dfrac{\frac{113}{150} - 0.8}{0.033} = -1.43.$		
$\therefore P(Z \geq -1.43) = 0.9236 = 92.36\%$		(1) for answer
Note Those who used the continuity correction hence $X = 112.5$ will get $z = -1.53$ and their final answer will be $0.937 = 93.7\%$.		It is not recommended to use the continuity correction in the HSC!

MATHEMATICS Extension 1 : Question 12		
Suggested Solutions	**Marks**	**Marker's Comments**
c) i $\tan\left(\theta+\frac{\pi}{6}\right)=\dfrac{\tan\theta+\tan\frac{\pi}{6}}{1-\tan\theta\tan\frac{\pi}{6}}$ $=\dfrac{\tan\theta+\frac{1}{\sqrt{3}}}{1-\frac{\tan\theta}{\sqrt{3}}}$ $=\dfrac{1+\sqrt{3}\tan\theta}{\sqrt{3}-\tan\theta}$		(1) Must show at least these lines of working.
ii) $1+\sqrt{3}\tan\theta=(\sqrt{3}-\tan\theta)\tan(\pi-\theta)$ $\therefore \dfrac{1+\sqrt{3}\tan\theta}{\sqrt{3}-\tan\theta}=\tan(\pi-\theta)$ $\tan\left(\theta+\frac{\pi}{6}\right)=\tan(\pi-\theta)$ (from i) $\theta+\frac{\pi}{6}=\pi-\theta$ or $\theta+\frac{\pi}{6}=\pi+(\pi-\theta)$ $2\theta=\pi-\frac{\pi}{6}$ or $2\theta=2\pi-\frac{\pi}{6}$ $2\theta=\frac{5\pi}{6}$ or $2\theta=\frac{11\pi}{6}$ $\theta=\frac{5\pi}{12}$ or $\frac{11\pi}{12}$		(1) (1)
d) There are 7 possible sums {-3, -2, -1, 0, 1, 2, 3} These are the pigeonholes <u>or</u> There are 3 rows, 3 columns and 2 diagonals will result in 8 calculations. These are the pigeons.		(1)
∴ By the pigeonhole principle, at least 2 calculations will result in the same sum.		(1)

MATHEMATICS Extension 1 : Question……..

Suggested Solutions	Marks	Marker's Comments
e) i) 	①	
ii) $\vec{CD} = \vec{AD} - \vec{AC}$ $\vec{BE} = \vec{AE} - \vec{AB}$		
$(\vec{CD}) \cdot (\vec{BE}) = 0$ (Perpendicular vectors)		
$(\vec{AD} - \vec{AC}) \cdot (\vec{AE} - \vec{AB}) = 0$	①	Using the perpendicular property and rephrasing $\vec{CD}$ and $\vec{BE}$ in the same line. $(\vec{AE} - \vec{AB})$
$\vec{AD} \cdot \vec{AE} - \vec{AD} \cdot \vec{AB} - \vec{AC}\vec{AE} + \vec{AC} \cdot \vec{AB} = 0$		
$\vec{AD} \cdot \vec{AE} + \vec{AC} \cdot \vec{AB} = \vec{AD} \cdot \vec{AB} + \vec{AC} \cdot \vec{AE}$		
$\vec{AD} \cdot \vec{AE} + \vec{AC} \cdot \vec{AB} = \vec{AD} \cdot 2\vec{AD} + 2\vec{AE} \cdot \vec{AE}$ $= 2r^2 + 2r^2$ $= 4r^2$		
$\lvert\vec{AD}\rvert\lvert\vec{AE}\rvert \cos\theta + \lvert\vec{AC}\rvert\lvert\vec{AB}\rvert \cos\theta = 4r^2$ $r^2 \cos\theta + 4r^2 \cos\theta = 4r^2$ $\cos\theta\,(5r^2) = 4r^2$ $\cos\theta = \frac{4}{5}$ $\theta = \cos^{-1}\frac{4}{5}$	①	Use of dot product formula to introduce θ
$\doteqdot 36°52'$	①	Answer.

Mathematics Extension 1, Question 13		
Suggested Solutions	**Marks**	**Marker's Comments**

(a)

$$I := \int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$$

Let $u = \sqrt{x}$, then $du = \frac{1}{2\sqrt{x}}dx = \frac{1}{2u}dx \to dx = 2u\,du$.

So

$$I = \int \frac{\sin u}{u} 2u\,du = 2\int \sin u\ du$$ **1**

$$= -2\cos u + C$$

$$= -2\cos\sqrt{x} + C$$ **1**

□

(b) (i)

Implicit assumption is that addition of salt contributes to change in volume negligibly.

Looking for $\frac{dQ}{dt}$, the change in **mass** of salt per unit time.

Now, change in salt in system over time δt is

$$\delta Q = (\text{concentration at time } t) \times (\text{volume out})$$

$$= \frac{Q}{V} \times \delta V$$

$$= \frac{Q}{V}\left(-\frac{2L}{\text{min}}\right)\delta t \ldots (1)$$ **1**

Now,

$$\delta V = (V_{\text{in}} - V_{\text{out}}) = \left(3\frac{\text{L}}{\text{min}} - 2\frac{\text{L}}{\text{min}}\right)\delta t = 1\text{L/min}\ \delta\text{t}$$

So

$$\frac{dV}{dt} = \frac{1\text{L}}{\text{min}} \to V = \left(\frac{1\text{L}}{\text{min}}\right)t + C$$

When $t = 0$ minutes, $V = 10$ L, so $C = 10$ L.

Hence $$V = 1\frac{\text{L}}{\text{min}}t + 10\text{ L} \ldots (2)$$ Combining (1) and (2): $$\delta Q = \frac{Q}{\left(1\frac{\text{L}}{\text{min}}t + 10\text{ L}\right)}\left(-\frac{2L}{\text{min}}\right)\delta t$$ $$\frac{\delta Q}{\delta t} = -\frac{2Q}{t\frac{\text{L}}{\text{min}} + 10\text{ L}}$$ Assuming the units now implied,		
$$\frac{dQ}{dt} = \lim_{\delta t\to 0}\frac{\delta Q}{\delta t} = -\frac{2Q}{10+t}$$	**1**	
(ii) The equation is separable, and assuming the units of Q to be kg, we have		
$$\int_{20}^{Q}\frac{dQ}{Q} = -2\int_{0}^{5}\frac{dt}{10+t}$$	**1**	
$$\log Q - \log 20 = -2(\log(15) - \log 10)$$	**1**	
$$\log Q = \log\frac{80}{9}$$		
So, $Q \approx 8.89$ kg. □	**1**	

(c) (i) We can aim to use sums to products: $\sin 2\theta + \sin 4\theta - \sin 6\theta$ $= \sin(3\theta - \theta) + \sin(3\theta + \theta) - \sin 6\theta$ $= 2\sin 3\theta \cos\theta - \sin 6\theta$ sum to product $= 2\sin 3\theta \cos\theta - 2\sin 3\theta \cos 3\theta$ double angle $= 2\sin 3\theta\,(\cos\theta - \cos 3\theta)$ $= 2\sin 3\theta\,(\cos(2\theta - \theta) - \cos(2\theta + \theta))$ $= 2\sin 3\theta\,(2\sin 2\theta \sin\theta)$ sum to product $= 4\sin 3\theta \sin 2\theta \sin\theta$ (ii) Solve $\sin 2\theta + \sin 4\theta = \sin 6\theta$ for $0 \leq \theta \leq \pi$: We have $\sin 2\theta + \sin 4\theta = \sin 6\theta$ $\sin 2\theta + \sin 4\theta - \sin 6\theta = 0$ $4\sin 3\theta \sin 2\theta \sin\theta = 0$ by (i) Hence either: $\sin\theta = 0, \sin 2\theta = 0, \sin 3\theta = 0$ If $\sin\theta = 0$, then $\theta = 0, \pi$ If $\sin 2\theta = 0$, then $2\theta = n\pi$ $(n \in \mathbb{Z})$, so $\theta = \frac{n\pi}{2} \to \theta = 0, \frac{\pi}{2}, \pi$ If $\sin 3\theta = 0$, then $3\theta = n\pi$ $(n \in \mathbb{Z})$, so	1 1 1	

$$\theta = \frac{n\pi}{3} \to \theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi$$

Hence the solution set is

$$\theta = 0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \pi$$

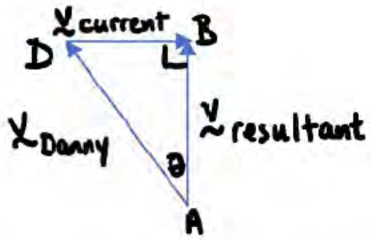
1

□

(d) (i)

Current is moving West to East, hence there'll be a component of velocity in the direction parallel to the bank.

If Danny is to from A to B, he must head **into** the current, so that velocities reconcile as:



If t minutes is the time taken to make the journey, then we should have for displacement:

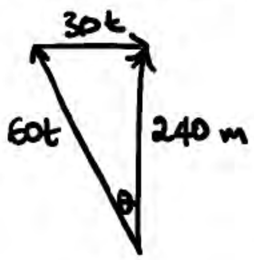
DB:

$$\mathbf{v}_{\text{current}} \times t = 30t \text{ metres}$$

AD:

$$\mathbf{v}_{\text{Danny}} \times t = 60t \text{ metres}$$

And we're given that AB is 240 metres. Hence

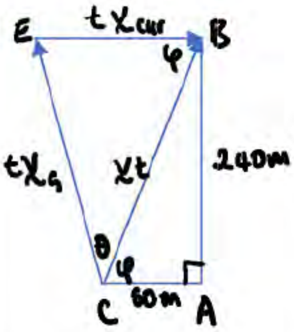


$$\sin\theta = \frac{30t}{60t} = \frac{1}{2} \to \theta = 30^\circ$$

So, Danny must swim N30°W.

1

(ii) Gabriel swims at 60 m/min and starts at position C, to the West of A. By similar reasoning to (i), if the journey is completed in t minutes, we want:



1

where $t\mathbf{v}_{\mathbf{current}} = 30t$ and $t\mathbf{v}_{\mathbf{Gabriel}} = 60t$.

We need $\angle BCE + \angle BCA \equiv \theta + \psi$.

From the construction, $\tan\psi = \frac{240}{60} = 4$. So,

$$\psi = \tan^{-1} 4 \approx 76^\circ$$

Also, $\angle BCA = \angle EBC = \psi$ (alternate angles on parallel lines are equal).

By law of sines,

1

$$\frac{\sin\theta}{30t} = \frac{\sin\psi}{60t} \rightarrow \sin\theta = \frac{1}{2}\sin(\tan^{-1} 4)$$

Hence

$$\theta = \sin^{-1}\left(\frac{1}{2}\sin(\tan^{-1} 4)\right) \approx 29^\circ$$

So, $\theta + \psi \approx 29^\circ + 76^\circ = 105^\circ$.

1

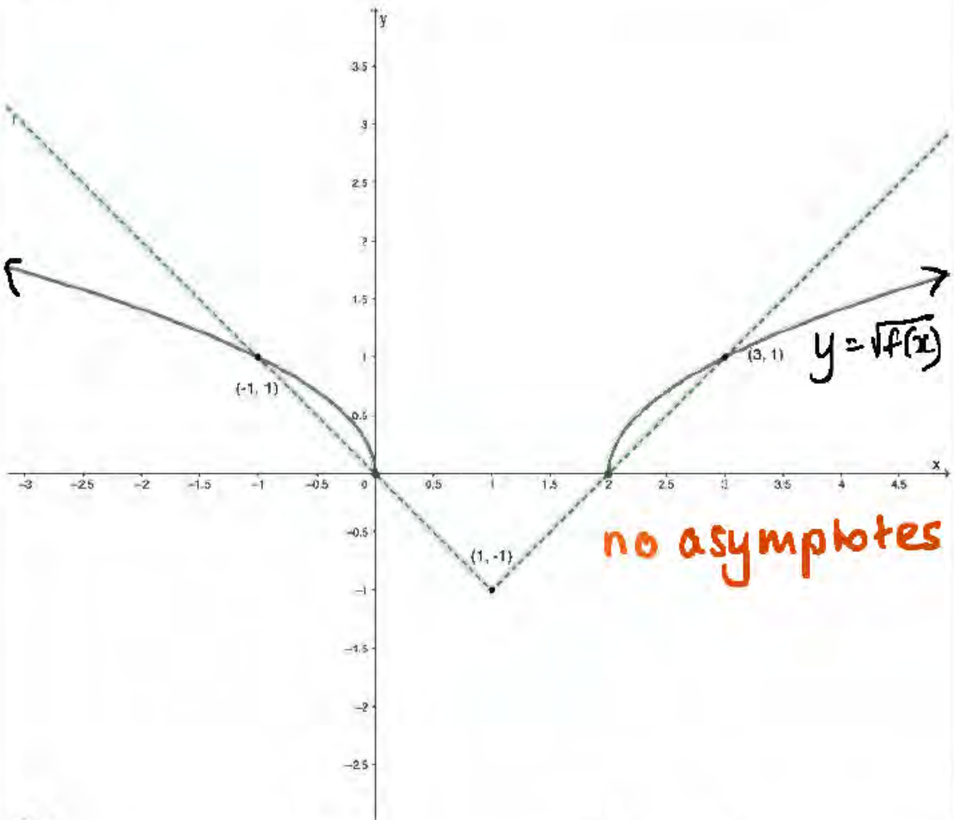
Subtracting 90°, we find Gabriel must swim N15°W.

□

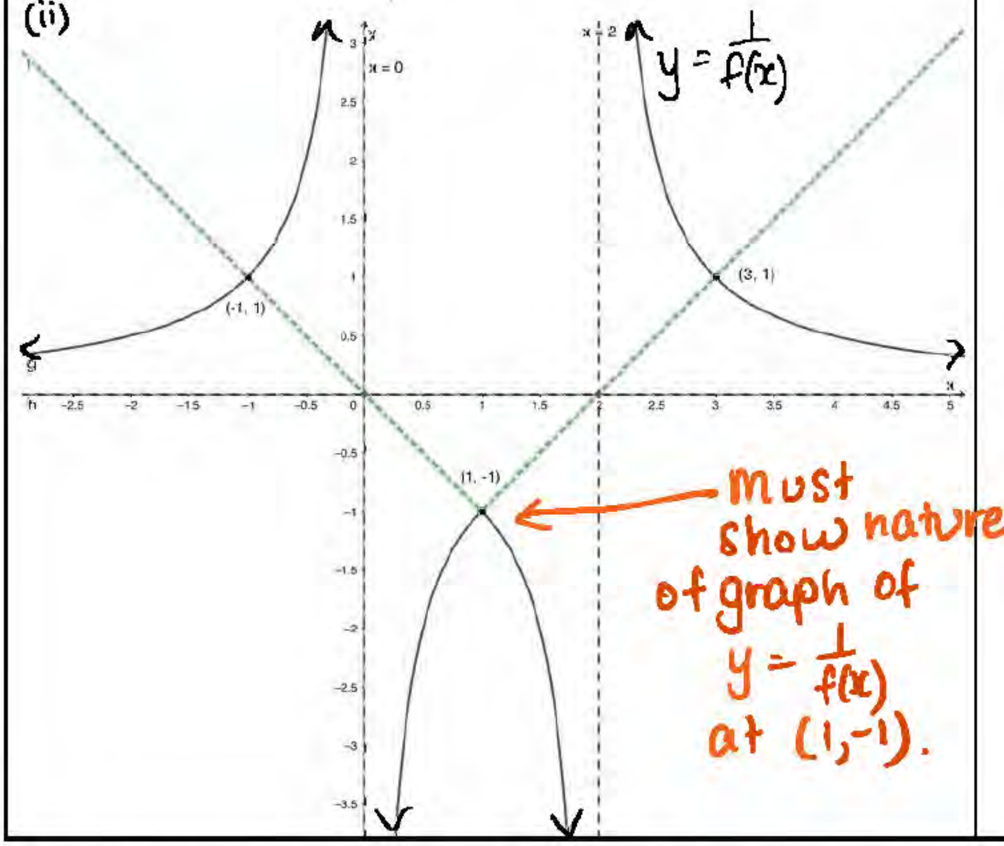
MATHEMATICS Extension 1: Question 14___

Suggested Solutions	Marks	Marker's Comments
a) (i)	1	for $(-1,1)$ and $(3,1)$ and x-intercepts
	1	shape
		Many students did not plot a point on each branch. Drawing the graph of $y=f(x)$ first is advisable.
(ii)	1	for $(3,1)$, $(-1,1)$ and shape at $(1,-1)$.
	1	shape of graph and asymptotes

a) (i)

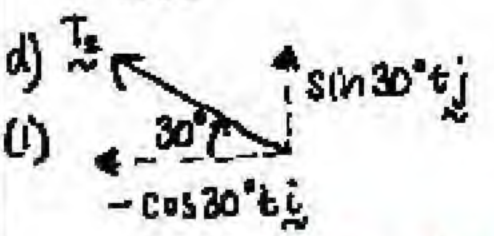
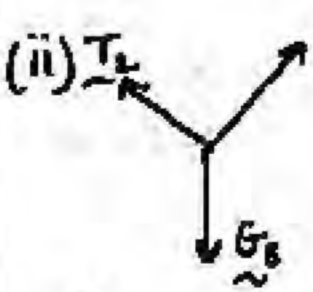


(ii)



b) (i) From Reference Sheet: $\sin^2 2(x+\pi/8) = \frac{1}{2}\left(1-\cos 4(x+\pi/8)\right)$ For probability density function need to show $\int_a^b f(x)\,dx = 1$ and $f(x) \geq$ for $a \leq x \leq b$. $\int_0^{\pi/4} 4\sin^2[2(x+\pi/8)]-2\,dx$ $= \int_0^{\pi/4} \frac{4}{2}[1-\cos 4(x+\pi/8)] - 2\,dx$	1	for changing the form
$= \int_0^{\pi/4} -2\cos 4(x+\pi/8)\,dx$ $= \left[-\frac{1}{2}\sin 4(x+\frac{\pi}{8})\right]_0^{\pi/4}$		Many errors when dealing with -2.
$= -\frac{1}{2}\sin(\pi+\frac{\pi}{2}) + \frac{1}{2}\sin\frac{\pi}{2}$ $= -\frac{1}{2}\times -1 + \frac{1}{2}\times 1$ $= 1$ } must show these steps	1	Evaluating after substituting.
For $0 \leq x \leq \frac{\pi}{4}$: $-\frac{1}{\sqrt{2}} \leq \sin 2(x+\frac{\pi}{8}) \leq 1$ $\frac{1}{2} \leq \sin^2 2(x+\frac{\pi}{8}) \leq 1$ $2 \leq 4\sin^2 2(x+\frac{\pi}{8}) \leq 4$ $0 \leq 4\sin^2 2(x+\frac{\pi}{8})-2 \leq 2$ } must show justification of $f(x) \geq 0$ $\therefore f(x) \geq 0$ for all x in $0 \leq x \leq \frac{\pi}{4}$	1	Many students did not consider $f(x) \geq 0$ as part of the verification process. must show why $f(x) \geq 0$.
(ii) mode = x value to give max $f(x)$. $f(x) = 4\sin^2[2(x+\frac{\pi}{8})]-2$ $= -2\cos 4(x+\frac{\pi}{8})$ ← max value = 2 $\cos 4(x+\pi/8) = -1$ $4(x+\pi/8) = \pi$ $4x = \pi/2$ $x = \pi/8$ Bald answer 0.	1	Many students gave the value of $f(x)$ rather than x.

MATHEMATICS Extension 1: Question 14

Suggested Solutions	Marks	Marker's Comments
c) $P = \dfrac{N(\text{3 Queens \& 2 Kings}) + N(\text{4 Queens \& 2 Kings})}{N(\text{2 Kings})}$ $= \dfrac{({}^4C_2 \times {}^4C_3 \times {}^{44}C_7) + ({}^4C_2 \times {}^4C_4 \times {}^{44}C_6)}{{}^4C_2 \times {}^{48}C_{10}}$ (choose 2 Kings; choose 3 Queens; choose 7 other cards (not K or Q); ${}^{48}C_{10}$ ← choose 10 other cards, not Kings.) $= \dfrac{53}{2163}$ OR/ (2 Queens, 1 Queen, 0 Queens) $P = 1 - \left[\dfrac{({}^4C_2 \times {}^4C_2 \times {}^{44}C_8) + ({}^4C_2 \times {}^4C_1 \times {}^{44}C_9) + ({}^4C_2 \times {}^{44}C_{10})}{{}^4C_2 \times {}^{48}C_{10}}\right]$ $= \dfrac{53}{2163}$	1 1 1	1 for the number of ways of choosing "exactly 2 Kings". 1 for 3Q, 2K 1 for 4Q, 2K Many students incorrectly assumed binomial probability. Many did not consider "exactly 2 Kings."
d) (i)  $\underset{\sim}{T_2} = \begin{bmatrix} -\cos 30° t \\ \sin 30° t \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{3}}{2} t \\ \frac{1}{2} t \end{bmatrix}$ must evaluate sin 30°, cos 30°	1	Many students were not able to correctly identify the components of $\underset{\sim}{T_2}$
(ii)  B is at rest: $\underset{\sim}{T_2} + \underset{\sim}{N_2} + \underset{\sim}{G_B} = \underset{\sim}{0}$ $\begin{bmatrix} -\frac{\sqrt{3}}{2} t \\ \frac{1}{2} t \end{bmatrix} + \begin{bmatrix} \frac{1}{2} n \\ \frac{\sqrt{3}}{2} n \end{bmatrix} + \begin{bmatrix} 0 \\ -50 \end{bmatrix} = 0$ $-\frac{\sqrt{3}}{2} t + \frac{1}{2} n = 0 \Rightarrow n = \sqrt{3} t$ ① $\frac{1}{2} t + \frac{\sqrt{3}}{2} n - 50 = 0$ ②		Many students did not use the vector forms of $\underset{\sim}{T_2}$, $\underset{\sim}{N_2}$, $\underset{\sim}{G_B}$ and the equation

MATHEMATICS Extension 1: Question 14___

Suggested Solutions	Marks	Marker's Comments
$\frac{1}{2}t + \frac{\sqrt{3}}{2} \times \sqrt{3}t - 50 = 0$		relating them.
$\frac{4}{2}t = 50$		
$t = 25\text{ N}$	1	for t value
$n = \sqrt{3}t$		
$= 25\sqrt{3}\text{ N}$	1	for n value
(iii)  A is at rest		Read the question carefully – components of $\underset{\sim}{N_1}$ required.
$\therefore \underset{\sim}{T_1} + \underset{\sim}{N_1} + \underset{\sim}{G_A} = 0$		
$\begin{bmatrix} 0 \\ t \end{bmatrix} + \underset{\sim}{N_1} + \begin{bmatrix} 0 \\ -80 \end{bmatrix} = 0$		
$\begin{bmatrix} 0 \\ 25 \end{bmatrix} + \underset{\sim}{N_1} + \begin{bmatrix} 0 \\ -80 \end{bmatrix} = 0$		
$\underset{\sim}{N_1} = \begin{bmatrix} 0 \\ 55 \end{bmatrix}$	1	