Killara HIGH SCHOOL

2008
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

General Instructions

- Reading Time- 5 minutes
- Working Time – 2 hours
- Write using a blue or black pen
- Approved calculators may be used
- A table of standard integrals is provided at the back of this paper.
- All necessary working should be shown for every question.
- Begin each question on a new booklet.

Total marks (**84**)

- Attempt Questions 1-7
- All questions are of equal value

Question 1 (12 Marks) Start a new booklet. **Marks**

(a) Find the acute angle to nearest minute between the lines: **2**

$$x - 2y + 1 = 0$$

$$y = 5x - 4$$

(b) Find $\lim_{x \to 0} \frac{\sin\left(\frac{\pi}{4}x\right)}{2x}$ **2**

(c) Solve the inequality $\frac{2x+3}{x-4} \le 1$ **3**

(d) Differentiate $ln(\sin^{-1} 3x)$ **2**

(e) Find the Cartesian equation of the parabola given $x = t + 4$ and $y = 1 - 2t^2$. **1**

(f) How many arrangements of the word CHARACTERISTIC are there? **2**

End of Question 1

Question 2 (12 Marks) Start a new booklet. **Marks**

(a) Find the remainder when the polynomial $x^4 + 2x^2 - 7$ is divided by $(x+5)$ **1**

(b) Use the process of mathematical induction to show that: **4**

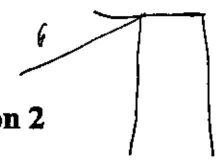
$$1 + 3 + 9 + \dots\dots\dots\dots\dots + 3^{n-1} = \frac{1}{2}\left(3^n - 1\right)$$

(c) i. Prove that $\sin\theta\sec\theta = \tan\theta$ **1**

ii. Hence solve $\sin\theta\sec\theta = \sqrt{3}$. $(0 \le \theta \le 2\pi)$ **1**

(d) Find the coefficient of x^4 in the expansion of $\left(3x - \frac{4}{x^2}\right)^7$. **2**

(e) From the top of a cliff an observer spots two ships out at sea. One is north east with an angle of depression of 6° while the other is south east with an angle of depression of 4°. If the two ships are 200 metres apart, find the height of the cliff, to the nearest metre. **3**



End of Question 2

$\frac{1}{2}3^k - \frac{1}{2} + 3^k = \frac{3^k}{2} \cdot \frac{3}{2} - \frac{1}{2}$

$3^k - 1 + 2 3^k = 3^k . 3 - 1.$

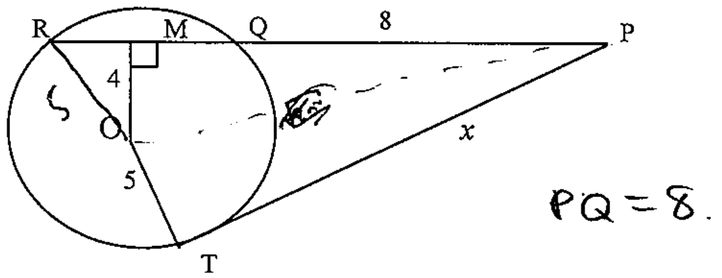
$2 \cdot 3^k + 3^k$

$3^k(2+1) - 1.$

$3^k(3) - 1.$

Question 3 (12 Marks) Start a new booklet. **Marks**

(a) **2**



PT is a tangent to the circle, centre O. OM is perpendicular to the secant RQ. Find the value of x.

(b) If $\alpha = \sin^{-1}\left(\frac{2}{3}\right)$ and $\beta = \sin^{-1}\left(\frac{3}{5}\right)$, find the value of $\sin(\alpha + \beta)$ **3**

(c) Find the value of $1 + \sin^2 x + \sin^4 x + \sin^6 x + \ldots\ldots\ldots\ldots \quad (0 < x < \frac{\pi}{2})$ **2**

(d) Evaluate $\int \cos^2 4x \, dx$ **2**

(e) Using the substitution $u = x - 1$, evaluate $\int_3^5 \frac{(x+1)^2}{(x-1)} dx$ **3**

End of Question 3

Question 4 (12 Marks) Start a new booklet. **Marks**

(a) Prove that $\binom{n}{r} \cdot {}^rP_r = {}^nP_r$ **2**

(b) Find the area under the curve $y = \dfrac{1}{\sqrt{4 - x^2}}$ from $x = 1$ to $x = 2$. **2**

Come Back

(c) Let P($2ap$, $2ap^2$) and Q($2aq$, $2aq^2$) be points on the parabola $y = \dfrac{x^2}{2a}$. **6**

i. Find the equation of the chord PQ.

ii. If PQ is a focal chord, find the relationship between p and q.

iii. Show that the locus of the midpoint of PQ is a parabola.

(d) A first approximation to the solution of the equation $x^3 - 3x^2 + 1 = 0$ is 0·5. Use one application of Newton's method to find a better approximation. Give your answer correct to two decimal places. **2**

End of Question 4

Question 5 (12 Marks) Start a new booklet. **Marks**

(a) The polynomial $x^3 - 4x^2 + 5x - 1 = 0$ has 3 roots, namely α, β and γ.

i. Find the value of $\alpha + \beta + \gamma$ **1**

ii. Find the value of $\alpha\beta\gamma$ **1**

iii. Find the equation of the polynomial with roots 2α, 2β and 2γ. **2**

(b) If $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots\ldots\ldots\ldots + \binom{n}{n}x^n$, show by differentiating both sides

$2\binom{n}{2} + (2\times 3)\binom{n}{3} + (3\times 4)\binom{n}{4} + \ldots\ldots\ldots\ldots + n(n-1)\binom{n}{n} = n(n-1)\,2^{n-2}$ **2**

(c)

i. At the local high school, seven boys and eight girls have nominated for prefect. If three boys and three girls are selected, in how many ways can this be done. **1**

ii. At their first meeting the six prefects sit around a circular table. What is the probability that the two captains do not sit together? **2**

(d) The probability of a person contracting influenza on exposure is 0·65. A family of six has come into contact with a person who has influenza. What is the probability that: **3**

i. The entire family contracts the flu?

ii. Only two members of the family contract the flu?

iii. Less than half the family contracts the flu?

End of Question 5

Question 6 (12 Marks) Start a new booklet. **Marks**

(a) Given $P = 2000 + Ae^{kt}$,

i. Prove that it satisfies the equation $\frac{dP}{dt} = k\left(P - 2000\right)$ **1**

ii. Initially, P = 3000, and when t = 5, P = 8000. Use this information to find the values of 'A' and 'k'. **2**

iii. How long does it take the value of 'P' to double and what is the rate of change of 'P' at this time. **3**

(b) The two equal sides of an isosceles triangle are of length 6cm. If the angle between them is increasing at the rate of 0·05 radians per second, find the rate at which the area of the triangle is increasing when the angle between the equal sides is $\frac{\pi}{6}$ radians. **2**

(c) The volume, V of a large spherical balloon of radius R cm is increasing at a constant rate of 200 cm^3 per second.

i. Find $\frac{dR}{dt}$ in terms of R **1**

ii. Hence find the rate of increase of the surface area of the balloon, when its radius is 1 metre. **3**

End of Question 6

Question 7 (12 Marks) Start a new booklet. **Marks**

(a) A particle moves so that its distance x centimetres from a fixed point O at time t seconds is $x = 8\sin 3t$. **4**

i. Show that the particle is moving in simple harmonic motion.

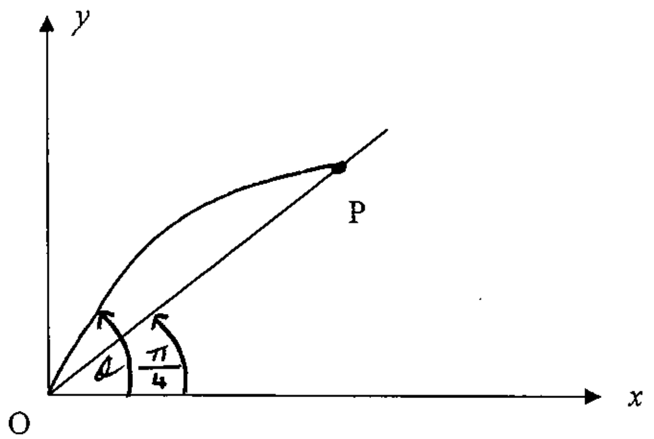
ii. What is the period of the motion?

iii. Find the velocity of the particle when it first reaches 4 centimetres to the right of the origin.

(b) From the bottom of a scope, inclined at $\frac{\pi}{4}$ to the horizontal, a golf ball is hit, with a velocity of 5m/s at an angle of ϑ to the horizontal.

Naturally $\frac{\pi}{4} < \vartheta < \frac{\pi}{2}$

The equations of motion are $\ddot{x} = 0$ and $\ddot{y} = -10$.



i. Find the x and y coordinates of the ball's position at time t. **2**

ii. The ball lands at P. If P is M metres from O, show that $x = y = \frac{M}{\sqrt{2}}$ **1**

iii. Show that $M = 5\sqrt{2}(\cos\vartheta.\sin\vartheta - \cos^2\vartheta)$ **3**

iv. By differentiation, find the exact value of ϑ, in radians, for the ball to achieve maximum distance M. **2**

End of Examination

TRIAL HSC MATHS EXTENSION 1 : 2008 : SOLUTIONS

QUESTION 1:

A. $y = \frac{1}{2}x + \frac{1}{2}$
$y = 5x - 4.$

$\tan\theta = \left|\frac{\frac{1}{2} - 5}{1 + 5\times\frac{1}{2}}\right| = \frac{9}{7}$

$\theta = 52^\circ 8'$ (2)

B. $\lim_{x\to 0} \frac{\sin\frac{\pi x}{4}}{2x} = \lim_{x\to 0} \frac{1}{2}\cdot\frac{\pi}{4}\,\frac{\sin\frac{\pi}{4}x}{\frac{\pi}{4}x}$
$= \frac{\pi}{8}$ (2)

C. $(2x+3)(x-4) \le (x-4)^2$
$(x-4)\left[(x-4)-(2x+3)\right] \ge 0.$
$(x-4)(-x-7) \ge 0$
$-7 \le x < 4 \quad (x \ne 4)$ (3)

D. $\frac{d}{dx}\left(\ln \sin^{-1} 3x\right)$
$= \frac{1}{\sin^{-1} 3x}\cdot\frac{3}{\sqrt{1-9x^2}}$ (2)

E. $x = t + 4$ ~①
$y = 1 - 2t^2$ ~②
From ①, $t = x - 4$
Sub in ② $y = 1 - 2(x-4)^2$
$= 1 - 2(x^2 - 8x + 16)$
$= -2x^2 + 16x - 31$ (1)

F. Arrangements $= \frac{14!}{3!\,2!\,2!\,2!\,2!}$
$= 908,107,200$ (2)

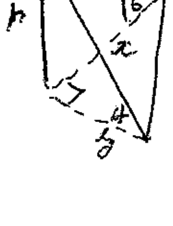
QUESTION 2

A. $f(-5)$ = Remainder
$= (-5)^4 + 2(-5)^2 - 7$
$= 625 + 50 - 7$
$= 668$ (1)

B. Test $n=1$, $3^{1-1} = \frac{1}{2}(3^1 - 1) = 1$ ∴ True
$n = k$, $1 + 3 + 9 + 3^{k-1} = \frac{1}{2}(3^k - 1)$
Prove true for $n = k+1$
$1 + 3 + 9 + \dots + 3^{k-1} + 3^k = \frac{1}{2}(3^k - 1) + 3^{(k+1)-1}$
$= \frac{1}{2}(3^k - 1) + 3^k$
$= \frac{1}{2}(3^k - 1 + 2.3^k)$
$= \frac{1}{2}(3.3^k - 1)$
$= \frac{1}{2}(3^{k+1} - 1)$ (4)
Which is form $\frac{1}{2}(3^n - 1)$ when $n = k+1$
∴ True for $n = k+1$, when true for $n = k$
∴ True for 1 so for all

C. i $\sin\theta.\sec\theta = \sin\theta.\frac{1}{\cos\theta} = \tan\theta.$
ii $\tan\theta = \sqrt{3}$
$\theta = \frac{\pi}{3}, \frac{4\pi}{3}$

D. $T_{k+1} = {}^7C_k (3x)^{7-k}(-4x^{-2})^k$
$= {}^7C_k\, 3^{7-k}(-4)^k\, x^{7-3k}.$
Coeff x^4 when $7 - 3k = 4$ $k = 1.$
Coeff $x^4 = {}^7C_1\, 3^{7-1}(-4)^1$
$= -20,412$ (2)

E.


$\tan 6^\circ = \frac{h}{x}$ $x = \frac{h}{\tan 6}$
$\tan 4 = \frac{h}{y}$ $y = \frac{h}{\tan 4}$
Now $200^2 = \left(\frac{h}{\tan 6}\right)^2 + \left(\frac{h}{\tan 4}\right)^2$
$h^2 = 200^2 \div \left(\frac{1}{\tan^2 6} + \frac{1}{\tan^2 4}\right)$
$h = 11.6438$
$\doteqdot 12$ metres. (3)

QUESTION 3

(A) $RM = \sqrt{5^2 - 4^2} = 3$

SINCE LINE THROUGH CENTRE $\perp$ TO CHORD BISECTS THE CHORD, $RQ = 6$ SO $PR = 14$

$(PT)^2 = PQ.PR$

$x^2 = 8 \times 14 = 112$

$x = 10.6$ (1 DP)

(2)

(B) $\sin\alpha = \frac{2}{3}$ & $\sin\beta = \frac{3}{5}$.

BY PYTHAGORAS

$\cos\alpha = \frac{\sqrt{5}}{3}$ $\quad \cos\beta = \frac{4}{5}$.

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta.\cos\alpha$

$= \frac{2}{3}\times\frac{4}{5} + \frac{3}{5}\times\frac{\sqrt{5}}{3}$

$= \frac{8+3\sqrt{5}}{15}$

(3)

(C) $1 + \sin^2 x + \sin^4 x + \sin^6 x + \cdots$

$a = 1 \quad R = \sin^2 x$

$S_\infty = \frac{1}{1-\sin^2 x} = \frac{1}{\cos^2 x}$

$= \sec^2 x$

(2)

(D) $\int \cos^2 4x \, dx$

Now $\cos 2x = 2\cos^2 x - 1$

$\cos^2 x = \frac{1}{2}(\cos 2x + 1)$

$\cos^2 4x = \frac{1}{2}(\cos 8x + 1)$ (2)

$\frac{1}{2}\int \cos 8x + 1 = \frac{1}{2}\left[\frac{1}{8}\sin 8x + x\right] + c$

$= \frac{\sin 8x}{16} + \frac{x}{2} + c$

(E) $U = x - 1$

$U + 2 = x + 1$

(3)

$\int_2^4 \frac{(U+2)^2}{U} du = \int_2^4 \left(U + 4 + \frac{4}{U}\right) du$

$= \left[\frac{U^2}{2} + 4U + 4\ln U\right]_2^4$

$= 14 + 4\ln 4 - 4\ln 2$

$= 14 + 4\ln 2$

QUESTION 4

(A) $LHS = \binom{N}{R} {}^R P_R = \frac{N!}{(N-R)!\,R!} \times \frac{R!}{(R-R)!}$

$= \frac{N!}{(N-R)!} = {}^N P_R$

(2)

(B) $Area = \int_1^2 \frac{dx}{\sqrt{4-x^2}} \cdots$

$= \left[\sin^{-1}\frac{x}{2}\right]_1^2$

$= \frac{\pi}{2} - \frac{\pi}{6}$

$= \frac{\pi}{3} \; u^2$

(2)

(C) i $M = \frac{2AP^2 - 2AQ^2}{2AP - 2AQ} = P + Q$.

$y - 2AP^2 = (P+Q)(x - 2AP)$

$y = Px + Qx - 2APQ$.

(2)

ii $x^2 = 2Ay$ So $4A = 2a$. $A = \frac{a}{2}$.

$\therefore$ FOCUS $S\ (0, \frac{a}{2})$

So CHORD $y = Px + Qx - 2APQ$ PASSES THE FOC

So $\frac{a}{2} = -2APQ$

$PQ = -\frac{1}{4}$

(2)

iii MID.PT IS $\left(\frac{2A(P+Q)}{2}, \frac{2A(P^2+Q^2)}{2}\right)$

$x = A(P+Q)$

So $P+Q = \frac{x}{A}$.

Now $y = A(P^2+Q^2) = A\left[(P+Q)^2 - 2PQ\right]$

$= A\left[\frac{x^2}{A^2} - 2\left(-\frac{1}{4}\right)\right]$

So $y = \frac{x^2}{A} + \frac{A}{2}$

$x^2 = Ay - \frac{A^2}{2}$

$= A\left(y - \frac{A}{2}\right)$

WHICH IS A PARABOLA V: $0, \frac{A}{2}$ FL $= \frac{a}{4}$.

(2)

(D) $f(x) = x^3 - 3x^2 + 1 \quad f(0.5) = 0.5^3 - 3(0.5)^2 + 1 = 0.375$

$f'(x) = 3x^2 - 6x \quad f'(0.5) = 3(0.5)^2 - 6(0.5) = -2.25$

$A_1 = A_0 - \frac{f(A_0)}{f'(A_0)} = 0.5 - \frac{0.375}{-2.25}$

$= 0.67$

(2)

QUESTION 5

A. i $x^3 - 4x^2 + 5x - 1 = 0$

$\alpha+\beta+\gamma = \frac{-(-4)}{1} = 4$ (1)

ii $\alpha\beta\gamma = -(-1) = 1$ (1)

iii EQTN $x^3 - 2(\alpha+\beta+\gamma)x^2 + 4\alpha\beta + 4\beta\gamma + 4\gamma\alpha)x - 8\alpha\beta\gamma$

$x^3 - 2(\alpha+\beta+\gamma)x^2 + 4(\alpha\beta+\beta\gamma+\gamma\alpha)x - 8\alpha\beta\gamma = 0$

$x^3 - 2\times4x^2 + 4\times5x - 8\times1 = 0$

$x^3 - 8x^2 + 20x - 8 = 0$ (2)

(B) DIFFERENTIATE BOTH SIDES.

$N(1+x)^{N-1} = \binom{N}{1} + 2\binom{N}{2}x + \ldots N\binom{N}{N}x^{N-1}$

DIFF BOTH SIDES AGAIN

$N(N-1)(x+1)^{N-2} = 2\binom{N}{2} + 2\times3\binom{N}{3}x + \ldots + N(N-1)x^{N-2}$

LET $x = 1$ (2)

$\therefore 2\binom{N}{2} + (2\times3)\binom{N}{3} + (3\times4)\binom{N}{4} + \ldots + N(N-1)\binom{N}{N} = N(N-1).2^{N-2}$

(C) i TOTAL WAYS $= {}^7C_3 \times {}^8C_3 = 1960$

ii NO WAY $= 5! = 120$

NO WAYS CAPT. TOGETHER $= 2!\times4! = 48$

NO OF WAY NOT TOGETHER $= 5! - 48 = 72$

$\therefore$ PROB $= \frac{72}{120} = \frac{3}{5}$ (3)

(D) i $P(\text{ENTIRE FAM}) = 0.65^6 = 0.07542$ (1)

ii $P(\text{ONLY 2 WITH FLU}) = {}^6C_2 = 0.65^2 \times 0.35^4 = 0.095102$ (1)

iii $P(<\tfrac{1}{2}\text{ CONTACT FLU}) = (0.35)^6 + {}^6C_1 .65 \times 0.35^5 + {}^6C_2(0.65)^2(0.35)^4 = 0.117424$ (1)

QUESTION 6

A. i $\frac{dP}{dt} = K.Ae^{Kt} = K(P-2000)$ (1)

ii $P = 200 + Ae^{Kt}$ WHEN $t=0$ $P=300$

$\therefore 3000 = 2000 + Ae^0$

$A = 100$ (1)

$P = 2000 + 1000e^{Kt}$

WHEN $t=5$ $P=800$

$\therefore 8000 = 2000 + 1000e^{5K}$

$6 = e^{5K}$

$5K = \ln 6$

$K = 0.358351893$ (1)

iii $P = 2000 + 1000e^{0.358351893t}$

$6000 = P$

SO $4000 = 1000e^{0.358351893}$

$t = \ln 4 \div 0.358351893$

$t = 3.868$

$\doteqdot 3.9$ years. (2)

$\frac{dP}{dt} = K(P-2000)$

$= 0.358351893(6000-2000)$

$= 1433.407572$ (1)

B. 

$\frac{d\theta}{dt} = 0.05$

$A = \tfrac{1}{2}AB\sin C = 18\sin\theta$

$\frac{dA}{d\theta} = 18\cos\theta$. (2)

Now $\frac{dA}{dt} = \frac{dA}{d\theta}\times\frac{d\theta}{dt} = 18\cos\theta\times0.05$

WHEN $\theta = \pi/6$ $\frac{dA}{dt} = 18\cos\frac{\pi}{6}\times0.05 = 0.78\ \text{cm}^2/\text{s}$ (2D.)

C. $\frac{dV}{dt} = 200$ $V = \tfrac{4}{3}\pi R^3$ $\frac{dV}{dR} = 4\pi R^2$

Now $\frac{dV}{dt} = \frac{dV}{dR}\times\frac{dR}{dt} = 4\pi R^2\times\frac{dR}{dt}$ So

$\frac{dR}{dt} = \frac{50}{\pi R^2}$ (1)

Now $S = 4\pi R^2$

$\frac{dS}{dR} = 8\pi R$

$\frac{dS}{dt} = \frac{dS}{dR}\times\frac{dR}{dt} = 8\pi R\times\frac{50}{\pi R^2} = \frac{400}{R}$ (3)

WHEN $R = 1\text{m}$ (100cm)

$\frac{dS}{dt} = \frac{400}{100} = 4\ \text{cm}^3/\text{s}$

QUESTION 7

Ai. $x = 8\sin 3t$

$\dot{x} = 24\cos 3t$

$\ddot{x} = -72\sin 3t$

$= -72x$ (1)

WHICH IS IN FORM $-n^2x$.

ii PERIOD $= \frac{2\pi}{n}$

$= \frac{2\pi}{3}$. (1)

iii $x = 8\sin 3t$

WHEN $x = 4$

$\sin 3t = \frac{1}{2}$

$3t = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6} \ldots$

$t = \frac{\pi}{18}, \frac{5\pi}{18}, \frac{13\pi}{18}, \frac{17\pi}{8} \ldots$ (1)

$\dot{x} = 24\cos 3t$

WHEN $t = \frac{\pi}{18}$

$\dot{x} = 24\cos 3\left(\frac{\pi}{18}\right)$

$= 24\cos\frac{\pi}{6}$

$= 12\sqrt{3}$ cm/s. (1)

B. i $\ddot{x} = 0$

$\dot{x} = c_1$

At $t=0$, $\dot{x} = 5\cos\theta$

$x = 5t\cos\theta + c_2$

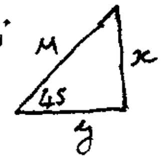
WHEN $t=0, x=0$ SO $\underline{x = 5t\cos\theta}$

$\ddot{y} = -10$

$\dot{y} = -10t + c$ $t=0, \dot{y} = 5\sin\theta$

$\dot{y} = -10t + 5\sin\theta$ (2)

$y = -5t^2 + 5t\cos\theta$ $(c_2 = 0)$

ii 

$x = y$

$\frac{x}{M} = \sin 45 = \frac{1}{\sqrt{2}}$ (1)

$\therefore x = y = \frac{M}{\sqrt{2}}$

QUESTION 7 (CONT).

TO SHOW $M = 5\sqrt{2}(\cos\theta.\sin\theta - \cos^2\theta)$

IF $x = y$, $5t\cos\theta = -5t^2 + 5t\sin\theta$.

SO $5t\cos\theta + 5t^2 - 5t\sin\theta = 0$

$5t(t + \cos\theta - \sin\theta) = 0$

SO $t = 0$, OR $t = \sin\theta - \cos\theta$ (3)

CONSIDERING ONLY $t = \sin\theta - \cos\theta$,

$x = 5(\sin\theta - \cos\theta).\cos\theta$

SO $M = 5\sqrt{2}(\sin\theta.\cos\theta - \cos^2\theta)$

iv. $\frac{dM}{d\theta} = 5\sqrt{2}(\sin\theta.-\sin\theta + \cos\theta.\cos\theta - 2\cos\theta.(-\sin\theta))$

$= 5\sqrt{2}(\cos^2\theta - \sin^2\theta + \sin 2\theta$

$= 5\sqrt{2}(\cos 2\theta + \sin 2\theta)$

IF $\frac{dM}{dt} = 0$, $\cos 2\theta + \sin 2\theta = 0$

$\tan 2\theta = -1$

$2\theta = \frac{3\pi}{4}$

$\theta = \frac{3\pi}{8}$

$\left(\frac{d^2y}{dx^2} = 5\sqrt{2}[-2\sin 2\theta + 2\cos 2\theta)\right.$

$= 10\sqrt{2}(\cos 2\theta - \sin 2\theta$

$\left.\text{AT } \theta = \frac{3\pi}{8}, \frac{d^2M}{dt^2} < 0 \therefore \text{MAX}\right)$