**Killara HIGH SCHOOL**

**STUDENT NUMBER**

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2011

TRIAL HIGHER SCHOOL CERTIFICATE

EXAMINATION

# Mathematics Extension 1

**General Instructions**

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

**Total marks – 84**

- Attempt Questions 1–7
- All questions are of equal value

| QUESTION | MARK |
|---|---|
| 1 | /12 |
| 2 | /12 |
| 3 | /12 |
| 4 | /12 |
| 5 | /12 |
| 6 | /12 |
| 7 | /12 |
| TOTAL | /84 |

***THIS QUESTION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM***

*This assessment task constitutes* 40% *of the Higher School Certificate Course Assessment.*

**Total marks – 84**
**Attempt Questions 1–7**
**All questions are of equal value**

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

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## Question 1 (12 marks) Use a SEPARATE writing booklet.

**Marks**

(a) Evaluate $\lim_{x\to 0}\frac{\sin 4x}{3x}$ **1**

(b) Differentiate $\tan^{-1}(\sin x)$ **2**

(c) Use the table of standard integrals to find $\int \frac{1}{\sqrt{x^2-16}}\,dx$ **1**

(d) Solve the inequation : $\frac{3x}{2-x}\le 2$ **3**

(e) The point $P$ divides the interval AB externally in the ratio 5:2. **2**
Given that A and B have coordinates (-4, 2) and (2, -1) respectively,
find the coordinates of P.

(f) Use the substitution $u = 25 - x^2$ to evaluate $\int_3^4 7x\sqrt{25-x^2}\,dx$ **3**

**Question 2** (12 marks) Use a SEPARATE writing booklet.

**Marks**

(a) Solve $3^x = 5$. **2**

Express your answer correct to two decimal places.

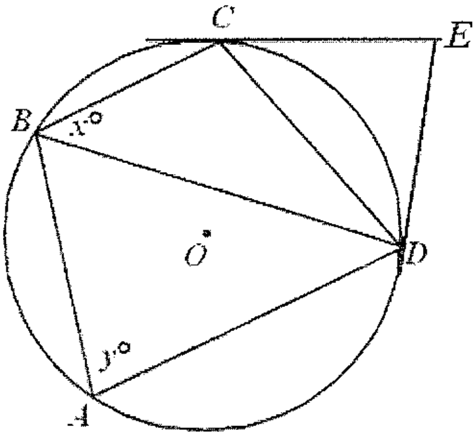
(b) Evaluate: $2\int_0^{\frac{\pi}{8}} \cos^2 4x\,dx$ **3**

(c) Find the general solution to $\sin 2x + \sin x = 0$. **3**

Express your answer in terms of $\pi$.

(d) The circle ABCD has centre O. Tangents are drawn from the external point E to contact the circle at C and D.

$\angle CBD = x^0$ and $\angle BAD = y^0$.



(i) Explain why $\angle CED = (180 - 2x)^0$. **2**

(ii) Show that $\angle BDC = (y - x)^0$. **2**

**Question 3** (12 marks) Use a SEPARATE writing booklet.

**Marks**

(a) (i) State the domain and range of the function $f(x) = 2\cos^{-1}\left(\frac{x}{3}\right)$ **2**

(ii) Sketch the function $f(x) = 2\cos^{-1}\left(\frac{x}{3}\right)$ **1**

(b) Consider the function $f(x) = x^3 - \log_e(x+1)$

(i) Show that the function has a root between 0·8 and 0·9. **1**

(ii) Using the value 0·8 and one application of Newton's method, find a better approximation for this root, correct to two decimal place. **3**

(c) Gary and George are cooking a 2 kilogram roast, which has an initial temperature of 10°C. They place it in an oven which has been preheated to 190°C at 5:00pm. It is found that the temperature of the roast increases to 50°C after 75 minutes.

A roast is considered medium rare when it reaches a temperature of 65°C. At time $t$ its temperature $T$ increases according to the equation

$$\frac{dT}{dt} = -k(T - 190), \text{ where } k \text{ is a positive constant.}$$

(i) Show that $T = 190 + Ae^{-kt}$ is a solution to the equation, where A is a constant **1**

(ii) Find the values of $A$ and $k$ **2**

(iii) At what time will the roast be medium rare? (answer correct to the nearest minute) **2**

**Question 4** (12 marks) Use a SEPARATE writing booklet.

**Marks**

(a) Use the principle of mathematical induction to show that $5^n + 2 \times 11^n$ is divisible by 3 for all positive integers *n*. **3**

(b) The velocity $v\ ms^{-1}$ of a particle moving in a straight line along the x-axis is given by $v^2 = 12 + 4x - x^2$.

(i) Show that the particle is moving in simple harmonic motion. **2**

(ii) Between which two points is the particle oscillating? **2**

(iii) What is the amplitude of the motion? **1**

(c) When the polynomial $P(x)$ is divided by $(x+2)(x-3)$, the quotient is $Q(x)$ and the remainder is $R(x)$.

(i) Explain why the most general form of $R(x)$ is given as: $R(x) = ax + b$, where $a$ and $b$ are constants. **1**

(ii) Given that $P(3) = 1$, show that $R(3) = 1$. **1**

(iii) Find $R(x)$ if $P(3) = 1$, $R(3) = 1$ and when $P(x)$ is divided by $(x+2)$ the remainder is 6. **2**

**Question 5** (12 marks) Use a SEPARATE writing booklet. **Marks**

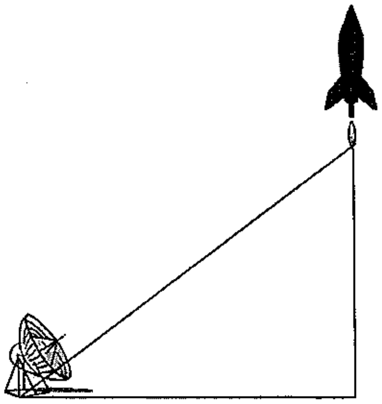
(a) The function $f(x) = cosec\, x$ for $\frac{\pi}{2} \leq x < \pi$

(i) State the domain of the inverse function $f^{-1}(x)$ **1**

(ii) Show that $f^{-1}(x) = \sin^{-1}\left(\frac{1}{x}\right)$ **2**

(iii) Hence find $\frac{d}{dx}f^{-1}(x)$ **1**

(b) A rocket is launched vertically and is tracked by a radar station, which is located on the ground 3km from the launch site. **3**



What is the vertical speed of rocket at the instant when its distance from the radar station is 5km and this distance is increasing at the rate of 5000 km per hour?

(c) (i) Show that the equation of the normal at R$(10q,\ 5q^2)$ on the parabola $x^2 = 20y$ is given by $x + qy = 5q^3 + 10q$. **2**

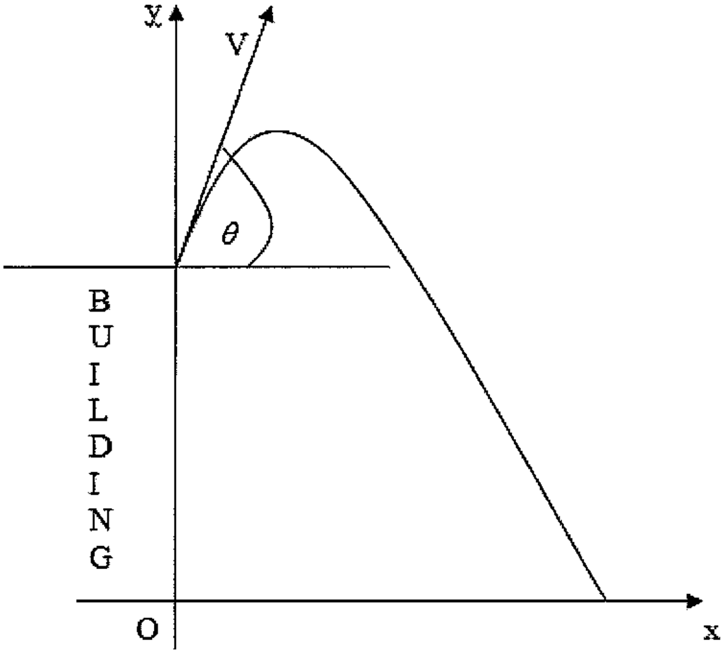
(ii) The normal intersects the y-axis at T. Find the coordinates of T and hence show that the coordinates of S, the midpoint of RT, is given by S$(5q,\ 5q^2 + 5)$. **2**

(iii) Hence find the Cartesian equation of the locus of S. **1**

**Question 6** (12 marks) Use a SEPARATE writing booklet.

**Marks**

(a) Jackson throws a ball from the top of an 18 metre high building with a velocity (V) of 12 m $s^{-1}$. The angle of projection ($\theta$) is $60^0$ to the horizontal.



Assume that the equations of motion for the ball are $\ddot{x} = 0$ and $\ddot{y} = -10$, and that air resistance is negligible.

(i) Let (x, y) be the position of the ball at time $t$ seconds after it was thrown and before it hits the ground. **3**

Show that $x = 6t$ and $y = -5t^2 + 6\sqrt{3}t + 18$.

(ii) Calculate the time that has elapsed before the ball hits the ground. **2**

(iii) What is the maximum height reached by the ball. **2**

*Question 6 continues on page 8*

*Question 6 (continued)*

(b) If $f(x) = g(x) - \ln[g(x)+1]$

(i) Prove $f'(x) = \dfrac{g(x)g'(x)}{g(x)+1}$. **2**

(ii) Hence evaluate $\displaystyle\int_{\frac{\pi}{12}}^{\frac{\pi}{4}} \frac{\sin 2x \cos 2x}{\sin 2x + 1}\, dx$. **3**

**End of Question 6**

**Question 7** (12 marks) Use a SEPARATE writing booklet.

**Marks**

(a) (i) Show that $\tan(rx)\tan(r+1)x = \dfrac{\tan(r+1)x}{\tan x} - \dfrac{\tan(rx)}{\tan x} - 1$ **2**

(ii) Hence, or otherwise, show that **2**

$$\tan 20° \tan 40° + \tan 40° \tan 60° + \ldots.. + \tan 180° \tan 200° = -9$$

(b) A river running due east has straight parallel banks. A vertical post stands with its base, $P$, on the north side of the river. On the south bank are two surveyors, $A$ who is to the east of $B$ and $B$ who is to the west of the post.

$A$ & $B$ are at a distance $\dfrac{2a}{7}$ apart and the $\angle APB = 150°$.

The angles of elevation from $A$ and $B$ to the top $Q$, of the post are 45° and 30° respectively.

(i) Draw a diagram to show this information.

(ii) Find in terms of $a$,

(α) the height of the post. **2**

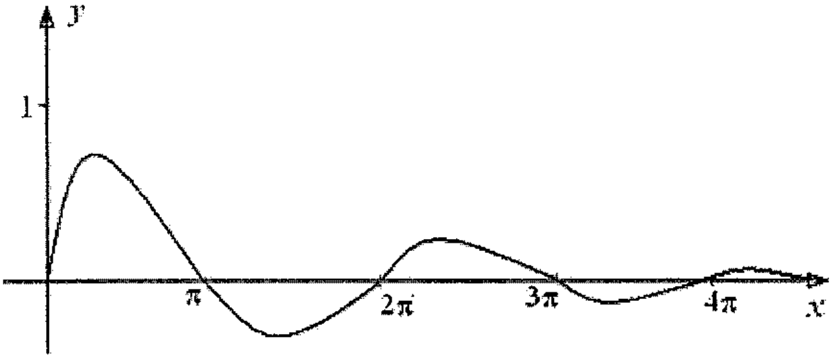
(β) the width of the river **2**

*Question 7 continues on page 10*

***Question 7 (continued)***

(b) (i) Show that $y = e^{-x}\sin x$ has infinitely many local maxima/minima. **1**

A sketch of the graph of $y = e^{-x}\sin x$, for $x \ge 0$ is given below:



(ii) The area between the curve $y = e^{-x}\sin x$, the $x$ axis, and the ordinates between $x = a$ and $x = b$ is given by

$$A = \frac{1}{2}\left|\left[e^{-x}(\cos x + \sin x)\right]_a^b\right| \quad \textit{(You do not need to prove this)}$$

($\alpha$) Using the formula given above, prove that the area enclosed between the curve, the $x$ axis, and the ordinates $x = 0$ and $x = \pi$ is $\frac{1}{2}\left(e^{-\pi} + 1\right)$. **1**

($\beta$) Find similar expressions (as shown in (ii)(α)) for the successive areas enclosed between the curve $y = e^{-x}\sin x$ and the $x$ axis and show that **2**

the sum of those areas for $x \ge 0$ is $\frac{1}{2}\left(\frac{e^{\pi}+1}{e^{\pi}-1}\right)$.

**End of paper**

## Extension 1

(a) $\lim_{x \to 0} \frac{\sin 4x}{3x} = \frac{4}{3} \lim \frac{\sin 4x}{4x}$

$= 4/3$

1 mark.
MUST SHOW ALL WORKING.

b) $\tan^{-1}(\sin x) = \frac{1}{1+\sin^2 x} \cdot \cos x.$

at this point Q is differentiated. no need to go on. $\longrightarrow$ $= \frac{\cos x}{1+\sin^2 x}.$

1 mark for correct differentiation of inverse trig

1 mark for correct answer

c) $\frac{1}{\sqrt{x^2-4^2}}\,dx = \ln\left(x+\sqrt{x^2-16}\right)+C$

1 mark. many student ~~missed this!~~

many students missed this!

d) $\frac{3x}{2-x} \leq 2.$ $\quad x \neq 2$

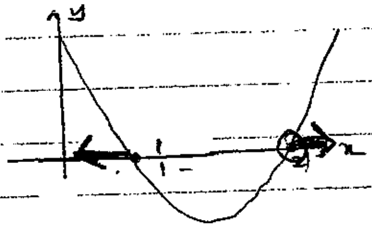
$(2-x)^2 \frac{3x}{2-x} \leq 2(2-x)^2.$

$3x(2-x) \leq 2(4-4x+x^2)$

$6x-3x^2 \leq 8-8x+2x^2.$

$0 \leq 5x^2-14x+8$

$\therefore 0 \leq (5x-4)(x-2)$



$\therefore x \leq \frac{4}{5},\ x > 2$

OR $\frac{3x}{2-x} \leq 2 \quad \therefore x \neq 2.$

If $x<2$, $3x \leq 4-2x.$
$5x \leq 4$
$x \leq \frac{4}{5}$ ✓

$\boxed{\text{If } x>2,}$ $3x \geq 2(2-x)$
$3x \geq 4-2x.$
$5x \geq 4$
$x \geq \frac{4}{5}$ ✗

$\therefore x > 2.$

$\therefore x \leq \frac{4}{5},\ x > 2.$

1 mark for $x \neq 2$ and either × b.s. by $\square^{\text{den}}$ OR considering $x<2$ & $x>2$.

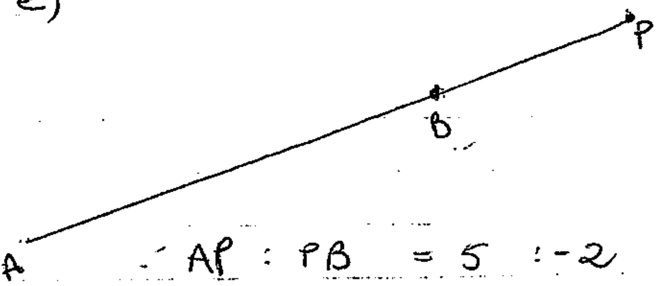
1 mark working. (depending on method.

1 mark for correct soln.

subtract 1 for $x \leq \frac{4}{5},\ x \geq 2$

* Students know the method for this question but have little understand of what they are actually doing

e)



$\therefore AP : PB = 5 : -2$

| A | B |
|---|---|
| $(-4, 2)$ | $(2, -1)$ |
| $5$ | $-2$ |

$\therefore P\left(\frac{5\times 2 - 2\times -4}{5-2}, \frac{5\times -1 + -2\times 2}{5-2}\right)$

$= \frac{18}{3}, \frac{-9}{3}$

$= (6, -3)$

This needs revising for many students

NOT WELL DONE

A wrong formula = zero marks

1 mark for correct substitution

1 mark for correct answer.

f) $u = 25 - x^2$ $\quad x = 4, u = 9$

$du = -2x\,dx$ $\quad x = 3, u = 16$

$\int_{3}^{4} 7x\sqrt{25 - x^2}\,dx$

$= \frac{-7}{2}\int_{3}^{4} -2x\sqrt{25 - x^2}\,dx$

$= -\frac{7}{2}\int_{16}^{9} \sqrt{u}\,du$

$= \frac{-7}{2}\left[\frac{2}{3}u^{3/2}\right]_{16}^{9}$

$= \frac{7}{3}\left[u^{3/2}\right]_{9}^{16}$

$= \frac{7}{3}\left[16^{3/2} - 9^{3/2}\right]$

$= \frac{7}{3}\left[4^3 - 3^3\right]$

$= \frac{259}{3}$

1 mark

Students who found the terminals in terms of 'u' were more successful than those who substituted back for u & used $x = 4$ & $3$.

1 mark.

This needs revising for MANY students

1 mark

## Question 2

(a) $3^x = 5$

$x = \log_3 5$

$= \dfrac{\log 5}{\log 3}$ 1 $\dfrac{\log 5}{\log 3}$

$= 1.464973521$

$= 1.46$ 1 $1.46$

(b) $2\int_0^{\pi/8} \cos^2 4x \, dx = \int_0^{\frac{\pi}{8}} [\cos 8x + 1] \, dx$ 1

$= \left[\frac{1}{8}\sin 8x + x\right]_0^{\frac{\pi}{8}}$ 1

$= \frac{\pi}{8}$

(c) $\sin 2x + \sin x = 0$

$2\sin x \cos x + \sin x = 0$

$\sin x(2\cos x + 1) = 0$

$\sin x = 0, \ \cos x = -\frac{1}{2}$

$x = 0, \pi, \ldots, \quad x = \frac{2\pi}{3}, \frac{4\pi}{3}, \ldots$

$x = n\pi, \quad x = 2n\pi \pm \frac{2\pi}{3}$

* not knowing expansion

* not factor sin x

1 for $\sin x(2\cos x + 1) = 0$

* not know general solution

1 $n\pi$

1 $2n\pi \pm \frac{2\pi}{3}$

(d)

(i) $\angle EDU = x$ (angle in alternate segment is equal to angle between the tangent and chord)

$EC = ED$ (tangents from an external point are equal)

$\angle ECD = x$ (angles opposite equal sides of an isosceles triangle)

$\angle CED = 180 - 2x$ (angle sum of triangle)

*abbreviated statements

* solution did not progress adequately 1

* Isosceles △ just appeared 1

* no such thing as base angl

(ii) $\angle BCD + \angle DAB = 180$ (opposite angles of a cyclic quadrilateral) 1

$y + \angle DAB = 180$

$\angle BAB = 180 - y$

$\angle BDC + x + 180 - y = 180$ (angle sum 1

Question 3.

$f(x) = 2\cos^{-1}\frac{x}{3}$.

(i)

$y=\cos^{-1}x$ Domain $-1 \le x \le 1$

$-1 \le \frac{x}{3} \le 1$

$-3 \le x \le 3$.

$f(x)=\cos^{-1}x$ Range. $0 \le \cos^{-1}x \le \pi$

$0 \le 2\cos^{-1}x \le 2\pi$.

$\therefore$ Domain $-3 \le x \le 3$.

Range. $0 \le y \le 2\pi$

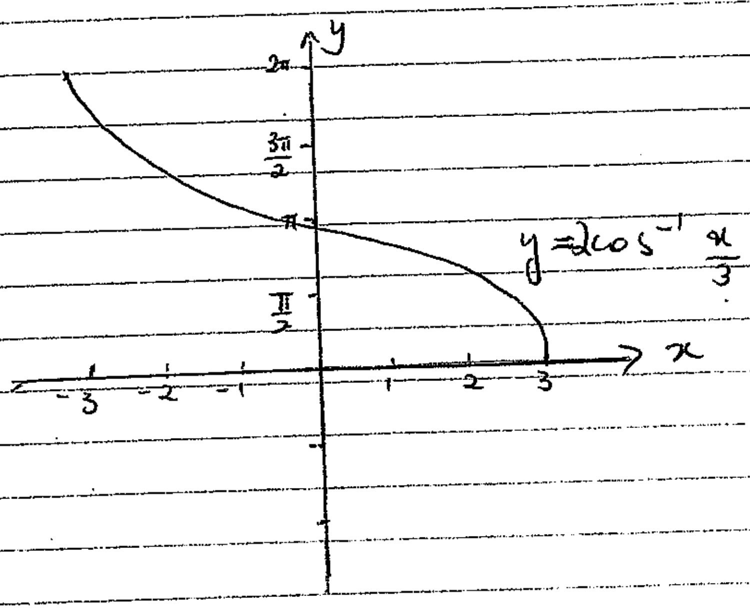
Most students did this well.

All students should know basic inv. trig curves

1 mark

1 mark.

(ii)



1 mark.

[must use ruler & have consistent scale].

b) (i) $f(x) = x^3 - \ln(x+1)$

$f(0.8) = 0.8^3 - \ln(1.8) \doteq -0.0758...$

$f(0.9) = 0.9^3 - \ln(1.9) \doteq 0.0871...$

As $f(0.8)$ & $f(0.9)$ are opp. sign

there is a root between 0.9 & 0.9.

many students didn't explain the opposite sign!

1 mark. must mention opp sign.

(ii) $f(x) = x^3 - \ln(x+1)$ $f(0.8) = -0.0758...$

students who didn't show subst. often got zero

1 mark $f'(0.8)$

$f'(x) = 3x^2 - \frac{1}{x+1}$

$f'(0.8) = 3(0.8)^2 - \frac{1}{0.8+1}$

$\doteq 1.364$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$= 0.8 - \frac{-0.0758}{1.364}$

$= 0.8555.$ $= 0.86$ (2 dp)

1 mark correct sub into Newtons method

1 mark correct answer rounded (not 0.855.. 0.9)

c) Original temp = 10°C.
surrounding = 190°C.
50°C after 75min.

Students who integrated has difficulties with $e^c = A$

(i) $T = 190 + Ae^{-kt} \rightarrow \therefore Ae^{-kT} = T - 190^\circ$

$$\frac{dT}{dt} = -kAe^{-kT}$$
$$= -k(T - 190)$$

This was only worth 1 mark
Differentiate $T = 190 + Ae^{-kt}$.
1 mark

ii) $T = 190 + Ae^{-kt}$.

when $t = 0$, $T = 10$

$\therefore 10 = 190 + Ae^{-k.0}$

$-180 = A$

$\therefore \underline{A = -180}$. → 1 mark for A

Well done

$\therefore T = 190 - 180e^{-kt}$.

when $t = 75$, $T = 50$

$\therefore 50 = 190 - 180e^{-75k}$.

$-140 = -180e^{-75k}$.

$\frac{7}{9} = e^{-75k}$. $\therefore \underline{k = \frac{-1}{75}\ln\frac{7}{9}}$ → 1 mark for k.

$k = 0.003$

Few transcription errors READ CAREFULLY!

iii) $T = 190 - 180e^{-kt}$.

$65 = 190 - 180e^{-kt}$

$-125 = -180e^{-kt}$

$\frac{25}{36} = e^{-kt}$.

$\ln\frac{25}{36} = -kt$.

$\therefore t = \ln\frac{25}{36} \div +\frac{1}{75}\ln\frac{7}{9}$ → 1 mark

$t = 108.8207\ldots$

$t = 109$ min (to nearest min)

$\therefore$ Roast is ready at 6.49pm. → 1 mark.

Many students neglected to show time

Make sure you ANSWER THE QUESTION!

## Question 4

(a) For $n=1$

$5^1 + 2\times 11^1 = 27$ which is divisible by 3.

$\therefore$ true for $n=1$ 1

Assume true for $n=k$.

That is, $5^k + 2\times 11^k = 3m$ for integer $m$ } 1

ie, $5^k = 3m - 2\times 11^k$

For $n=k+1$

$5^{k+1} + 2\times 11^{k+1}$

$= 5^k . 5 + 2\times 11^{k+1}$

$= (3m - 2\times 11^k)5 + 2\times 11^{k+1}$

$= 15m - 10\times 11^k + 2\times 11\times 11^k$ 1

$= 15M + 12\times 11^k$

$= 3\times(5m + 4\times 11^k)$ which is divisible by 3.

* substituting for $5^k$ and $11^k$
* trying to fudge result

Therefore, if it is true for $n=k$ then it is true for $n=k+1$

* no final statement

It is true for $n=1$, $\therefore$ true for $n=1+1=2$

True for $n=2$, $\therefore$ true for $n=2+1=3$

and so on.

$\therefore$ true for all integer (positive) $n$.

(b) i $v^2 = 12 + 4x - x^2$

$\frac{1}{2}v^2 = 6 + 2x - \frac{1}{2}x^2$

$\frac{d(\frac{1}{2}v^2)}{dx} = 2 - x$ 1

$\ddot{x} = -(x-2)$ which is of the form $a = -n^2(x)$ or $\therefore$ acceleration is directly proportional to displacement 1

no final statement

Question 4

(b) ii $v^2 = 12 + 4x - x^2$

$= (6-x)(2+x)$

$v = 0$ (at the end points)

$\therefore (6-x)(2+x) = 0$ ← 1

$x = -2$ and $6$.

∴ the particle oscillates between $x = 2$ and $6$. ← 1

iii Amplitude $= \frac{8}{2} = 4$ 1

(c) i $P(x) = (x+2)(x3)\,Q(x) + R(x)$

* this equation needs to be formed, particular for ii

$$\frac{P(x)}{(x+2)(x-3)} = Q(x) + \frac{R(x)}{(x+2)(x-3)}$$

The degree of $R(x)$ is less than the degree of the divisor ← 1

∴ $R(x)$ is of form $ax+b$

(ii) $P(3) = (3+2)(3-3)\,Q(3) + R(3)$ ← 1

$1 = 0 \times Q(3) + R(3)$

$\therefore R(3) = 1$.

iii $R(x) = ax+b$ and $R(3) = 1$

$\therefore 3a + b = 1$ —— ①

$R(-2) = 6$

$\therefore -2a + b = 6$ —— ②

*often equations were not formed and thus solved

1

solving ① and ② gives

$a = -1$, $b = 4$.

$\therefore R(x) = -x + 4$ ← 1

5. a)(i) Range $f(x) : f(x) \geq 1$

$\therefore$ Domain $f^{-1}(x) = x \geq 1$

(ii) $y = \operatorname{cosec} x$

SHOW

$x = \operatorname{cosec} y \longrightarrow$

$x = \dfrac{1}{\sin y}$

$\therefore \sin y = \dfrac{1}{x}$

$\therefore y = \sin^{-1}\left(\dfrac{1}{x}\right)$

$f^{-1}(x) = \sin^{-1} \dfrac{1}{x} \longrightarrow$

(iii) $\dfrac{d}{dx} f^{-1}(x) = \dfrac{d}{dx}\left(\sin^{-1}\left(\dfrac{1}{x}\right)\right)$

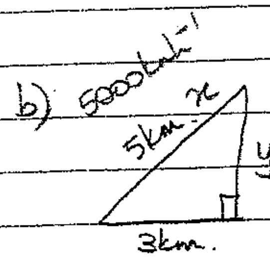
$= \dfrac{1}{\sqrt{1-\left(\frac{1}{x}\right)^2}} \cdot -x^{-2}$

$= \dfrac{-1}{\sqrt{\frac{x^2-1}{x^2}}} \cdot \dfrac{1}{x^2}$

$= \dfrac{-x}{\sqrt{x^2-1}} \cdot \dfrac{1}{x^2}$

$= \dfrac{-1}{x\sqrt{x^2-1}} \longrightarrow$

b)



$\dfrac{dy}{dt} = ?$

$x^2 = 9 + y^2$

$y = \sqrt{x^2-9}$

$\dfrac{dy}{dx} = \dfrac{1}{2}(x^2-9)^{-\frac{1}{2}} 2x$

$\dfrac{dy}{dx} = \dfrac{x}{\sqrt{x^2-9}}$

$\dfrac{dy}{dt} = \dfrac{dy}{dx} \cdot \dfrac{dx}{dt}$

$= \dfrac{x}{\sqrt{x^2-9}} \cdot 5000$

When $x = 5$, $\dfrac{dy}{dt} = \dfrac{5}{\sqrt{11}} \cdot 5000 = 6250\text{kmh}^{-1}$ 1mark

This was poorly done Students who took the time to sketch $y = \operatorname{cosec} x$ did this Q with ease.

1 ~~mark~~

1 mark This was a mark many students got

This was well don by most. Students are reminded that

1 mark SHOW means all working must be there.

Students were not required to simplify to final answer.

Some students were saying the $\frac{d}{dx}\left(\frac{1}{x}\right) = \ln x$ Don't confuse differentiating with integrating.

1 mark.

This was poorly done Students who drew the diagram were more sucessfull.

Students do not seem to understand that the only fixed length is 3

1mark.

1mark

:) (i) $y = \frac{x^2}{20}$

$\frac{dy}{dx} = \frac{x}{10}$

at $R(10q, 5q^2)$ $\frac{dy}{dx} = \frac{10q}{10} = q$

$m_{tang} = q$

$m_{norm} = -\frac{1}{q}$

This was well done but again students are reminded that in a SHOW Q, all working must be shown.

1 mark

$y - 5q^2 = -\frac{1}{q}(x - 10q)$

HOW $qy - 5q^3 = -x + 10q$

$10q + 5q^3 = x + qy$

1 mark

ii) At T, $x = 0$

$10q + 5q^3 = 0 + qy$

$y = 10 + 5q^2$

$\boxed{T = (0, 5q^2 + 10)}$

Well done.

1 mark

$R(10q, 5q^2)$

$S = \left(\frac{10q + 0}{2}, \frac{5q^2 + 5q^2 + 10}{2}\right)$

HOW $S = (5q, 5q^2 + 5)$

Well done.

1 mark

ii) $x = 5q$ — ①

$y = 5q^2 + 5$ — ②

om ① $q = \frac{x}{5}$ $\quad y = 5\left(\frac{x}{5}\right)^2 + 5$

$y = \frac{x^2}{5} + 5$

Well done.

1 mark.

Students were not asked to DESCRIBE the locus, just find the eq'n. To find the focus & vertex & describe WASTES TIME

Question 6.

(a) i Initially

Given
$V = 12$
$\theta = 60$

$\therefore \dot{y} = 12 \times \sin 60$
$= 6\sqrt{3}$ ← 1

* no setting up of initial conditions.
* inadequate reasoning/development of results

$\ddot{x} = 0$
$\dot{x} = k_1$
$= V\cos\theta$
$= 12 \times \cos 60$
$\dot{x} = 6$
$x = 6t + k_2$
$x = 6t \quad (t=0, x=0)$
$k_2 = 0$

$\ddot{y} = -10$
$\dot{y} = -10t + C_1$
$6\sqrt{3} = 0 + C_1 \quad (t=0)$
$\therefore C_1 = 6\sqrt{3}$
$\dot{y} = -10t + 6\sqrt{3}$ ← 1
$y = -5t^2 + 6\sqrt{3}t + C_2$
$18 = 0 + 0 + C_2$
$y = -5t^2 + 6\sqrt{3}t + 18.$

(1 for $\dot{x} = 6$).

ii

$y = 0$
$5t^2 - 6\sqrt{3}t - 18 = 0$

$t = \dfrac{6\sqrt{3} \pm \sqrt{468}}{10}$

$= \dfrac{3\sqrt{3} \pm 3\sqrt{13}}{5}$

$\doteqdot 3.2$ seconds

← 1

*not understanding relationship between the diagram and the equation y

iii Max height occurs when $y' = 0$

ie $-10t + 6\sqrt{3} = 0$

$t = \dfrac{3\sqrt{3}}{5}$ 1 for t

Max height $= -5\left(\dfrac{3\sqrt{3}}{5}\right)^2 + 6\sqrt{3}\left(\dfrac{3\sqrt{3}}{5}\right) + 18$

$= 23.4$ m 1 (answer)

# Question 6

(b) i

$$f(x) = g(x) - \ln[g(x)+1]$$

* Those 2 steps need to be given.

$$f'(x) = g'(x) - \frac{g'(x)}{g(x)+1} \quad \leftarrow 1$$

$$= \frac{g(x)g'(x)+g'(x)-g'(x)}{g(x)+1} \quad \leftarrow 1$$

$$= \frac{g(x)\,g'(x)}{g(x)+1}$$

ii

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{4}} \frac{\sin 2x \cos 2x}{\sin 2x + 1}\,dx$$

*Many did not understand the need for $\frac{1}{2}$ and 2

$\left(\begin{array}{l} g(x) = \sin 2x \\ g' = 2\cos 2x \end{array}\right)$

$$= \frac{1}{2}\int_{\frac{\pi}{10}}^{\frac{\pi}{4}} \frac{(\sin 2x)(2)(\cos 2x)}{\sin 2x + 1}\,dx \quad \leftarrow 1$$

$$= \frac{1}{2}\Big[\sin 2x - \ln[\sin 2x + 1]\Big]_{\frac{\pi}{12}}^{\frac{\pi}{4}} \quad \leftarrow 1$$

$$= \frac{1}{2}\left\{\sin\frac{\pi}{2} - \ln\left(\sin\frac{\pi}{2}+1\right) - \left[\sin\frac{\pi}{6} - \ln\left(\sin\frac{\pi}{6}+1\right)\right]\right\}$$

$$= \frac{1}{2}\left\{1 - \ln 2 - \frac{1}{2} + \ln\frac{3}{2}\right\}$$

$$= \frac{1}{2}\left(\frac{1}{2} + \ln\frac{3}{4}\right)$$

1 (answer)

| | | |
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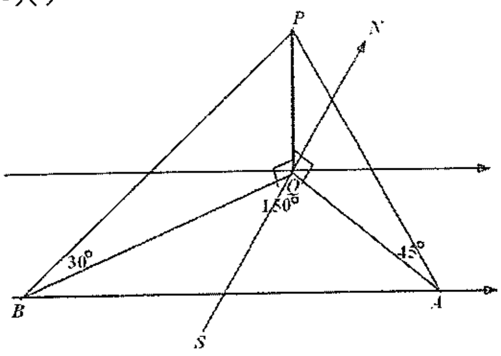
| a)(i) $\tan x = \tan\{(r+1)x - rx\}$<br>$= \frac{\tan(r+1)x - \tan(rx)}{1+\tan(r+1)x.\tan(rx)}$<br>$\therefore\ 1+\tan(r+1)x.\tan(rx) = \frac{\tan(r+1)x - \tan(rx)}{\tan x}$<br>ie. $\tan(rx)\tan(r+1)x = \frac{\tan(r+1)x}{\tan x} - \frac{\tan(rx)}{\tan x} - 1$ | 2 marks for correct proof<br>1 mark: progress towards the solution involving the use of tan (A – B) expansion. | *Many students wasted too much time in this question, attempting it 2 or 3 times. As an exam strategy, it is always better to move to the next question which is just an application of this result.* |
| (ii) Tan 20° tan 40° + an 40° tan 60° + ….. + Tan 180° tan 200° =<br>$\frac{\tan 40}{\tan 20} - \frac{\tan 20}{\tan 20} - 1 + \frac{\tan 60}{\tan 20} - \frac{\tan 40}{\tan 20} - 1 + \ldots. +$<br>$\frac{\tan 200}{\tan 20} - \frac{\tan 180}{\tan 20} - 1$ **1 mark**<br>$= \frac{\tan 200}{\tan 20} - \frac{\tan 20}{\tan 20} - 9 \times 1$<br>since, tan(180 + θ) = tanθ,<br>$= \frac{\tan 20}{\tan 20} - \frac{\tan 20}{\tan 20} - 9 \times 1$<br>$= -9$ | 1 mark: applies the result from (i)<br>1 mark: simplifies and uses the fact tan(180 + θ) = tanθ to get the required result. | *Many students failed to realise this was an application of the previous question.*<br>*See in the marking scheme, simply substituting gets you 1 mark.* |
| b)(i)<br><br> | | *Poor drawing skills.*<br>*Steps to draw a 3D diagram*<br>• *Draw the river*<br>• *Draw the North-South line in approximately 22 ½ °.*<br>• *Erect the post vertically.*<br>• *Now, complete all the triangles.*<br>• *Make sure you show right angle sign in the elevation triangles* |

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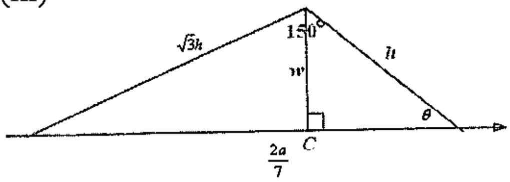
| (ii) Let $h$ be the height of the post<br>In ΔPAQ, PA = PQ = $h$ since tan 45 = 1<br>In ΔPBQ, tan 30 = $\frac{PQ}{PB}$<br>∴ PB = $\sqrt{3}h$ **1 mark**<br>IN ΔAQB, using cosine rule,<br>$\left(\sqrt{3}h\right)^2 + h^2 - 2\sqrt{3}h^2\cos 150 = \frac{4a^2}{49}$<br>$\therefore 3h^2 + h^2 - 2\sqrt{3}h^2 \times \frac{-\sqrt{3}}{2} = \frac{4a^2}{49}$<br>$7h^2 = \frac{4a^2}{49}$<br>$\therefore h = \frac{2a}{7\sqrt{7}}$ **1 mark** | 1 mark: finds the relationship between PA and PB<br>1 mark: uses cosine rule to evaluate the height of the post. | *Some mistakes came in the evaluation process of this cosine rule.* |
| (iii)<br><br> <br>In ΔPAB,<br>$\frac{\sin 150°}{2a/7} = \frac{\sin\theta}{\sqrt{3}h}$<br>$\frac{\sin 150°}{2a/7} = \frac{\sin\theta}{\sqrt{3} \times 2a/7\sqrt{7}}$<br>$\frac{1}{2} = \frac{\sqrt{7}\sin\theta}{\sqrt{3}}$<br>$\therefore \sin\theta = \frac{\sqrt{3}}{2\sqrt{7}}$ **1 mark**<br>In ΔPAC, $\sin\theta = \frac{w}{h}$<br>$\therefore w = h\sin\theta$<br>$= \frac{2a}{7\sqrt{7}} \times \frac{\sqrt{3}}{2\sqrt{7}} = \frac{\sqrt{3}a}{49}$<br>**1 mark** | 1 mark: Uses sine rule to find an expression for sin θ<br>1 mark: correctly evaluates width of the river. | *Another easier approach to this question:*<br>*Area of the triangle A =*<br>$\frac{1}{2} \times \left(\sqrt{3}h\right) \times h\sin 150 = \frac{1}{2} \times \frac{2a}{7} \times w$<br>$W = \left(\frac{\left(\sqrt{3}h\right)^2}{4}\right) \times \frac{7}{a}$<br>*Substitute h from (i) gives you*<br>$W = \frac{\sqrt{3}a}{49}$ |

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| b)(i) $y = e^{-x}\sin x$<br>$\frac{dy}{dx} = e^{-x}\cos x + \sin x.\left(-e^{-x}\right)$<br>$= e^{-x}(\cos x - \sin x)$<br>At stationary points, $\frac{dy}{dx} = 0.$<br>ie. $e^{-x}(\cos x - \sin x) = 0$<br>since, $e^{-x} \neq 0$, then $\cos x = \sin x$.<br>ie. $\tan x = 0$<br>$\therefore x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \ldots..$<br>max.. or min. when $x = \frac{(4n+1)\pi}{4}$,<br>$n = 0, 1, 2, \ldots..$<br>ie. infinite number of local maxima. | 1 mark: For the proof. | *Cos* $x - \sin x = 0$<br>*Cos* $x = \sin x$<br>*Tan* $x = 0$ *(not 1 !)*<br>*was a very common response.*<br><br>*You must get the stationary points to get this mark.* |
| (iii) (α) Area $A_1$ =<br>$\int_0^\pi e^{-x}\sin x dx = \frac{1}{2}\left[e^{-x}(\cos x + \sin x)\right]_\pi^0$<br>$= \frac{1}{2}\left[(1+0) - \left(-e^{-\pi} + 0\right)\right]$<br>$= \frac{1}{2}\left(e^{-\pi} + 1\right)$ | 1 mark: for proof | *Well done* |
| (β) Area $A_2$ =<br>$\left\lvert \frac{1}{2}\left[e^{-x}(\cos x + \sin x)\right]_{2\pi}^{\pi} \right\rvert$<br>$= \left\lvert \frac{1}{2}\left[e^{-\pi}(-1+0) - e^{-2\pi}(1+0)\right] \right\rvert$<br>$= \frac{1}{2}\left\lvert\left(-e^{-\pi} - e^{-2\pi}\right)\right\rvert$<br>$= \frac{1}{2}\left(e^{-2\pi} + e^{-\pi}\right)$<br>Successive areas will be<br><br>$A_1 + A_2 + A_3 + A_4 + \ldots\ldots$<br>ie. $\frac{1}{2}\left(e^{-\pi} + 1\right) + \frac{1}{2}\left(e^{-2\pi} + e^{-\pi}\right) +$<br>$\frac{1}{2}\left(e^{-3\pi} + e^{-2\pi}\right) + \frac{1}{2}\left(e^{-4\pi} + e^{-3\pi}\right) +$<br>$\ldots\ldots\ldots\ldots\ldots\ldots$ | 1 mark: Develops the sequence for the sum of areas. | *The sequence must be developed to get 1 mark.*<br><br>*No pattern can be identified if you are just given the first two terms.*<br><br>*This is what a lot of students had done. Gave the first two terms and claimed that it is a GP and surprise, surprise, they can even find the common ratio!* |

| | | |
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| $= \frac{1}{2}[\ (e^{-\pi}+1) + e^{-\pi}(e^{-\pi}+1) +$<br>$e^{-2\pi}(e^{-\pi}+1) + e^{-3\pi}(e^{-\pi}+1) + \ldots\ldots]$<br>$= \frac{1}{2}(e^{-\pi}+1)[\ 1 + e^{-\pi} + e^{-2\pi} +$<br>$e^{-3\pi} \quad + \ldots\ldots\ldots]$<br>Now, $1 + e^{-\pi} + e^{-2\pi} + e^{-3\pi} + \ldots\ldots\ldots$<br>is a G.P. where a = 1, r = $e^{-\pi}$<br>Limiting sum exists, since $\lvert r \rvert < 1$.<br>Now, $S_\infty = \frac{1}{2}(e^{-\pi}+1)\left(\frac{a}{1-r}\right)$<br>$= \frac{1}{2}(e^{-\pi}+1)\left(\frac{1}{1-e^{-\pi}}\right)$<br>$= \frac{1}{2}\left(\frac{e^{-\pi}+1}{1-e^{-\pi}}\right)$<br>$= \frac{1}{2}\left(\frac{\frac{1}{e^{\pi}}+1}{1-\frac{1}{e^{\pi}}}\right)$<br>$= \frac{1}{2}\left(\frac{\frac{1+e^{\pi}}{e^{\pi}}}{\frac{e^{\pi}-1}{e^{\pi}}}\right)$<br>$= \frac{1}{2}\left(\frac{e^{\pi}+1}{e^{\pi}-1}\right)$ as required | 1 mark: applies the sum of GP to get the required result | *Fudging was the order of the day with this part of the question!!* |