Killara HIGH SCHOOL

STUDENT NUMBER

MATHEMATICS EXTENSION ONE

TRIAL HIGHER SCHOOL CERTIFICATE

WEDNESDAY **18th JULY 2012**

General Instructions

- Reading Time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question
- Start a new booklet for each question

Total Marks - 75

- Attempt questions 1 - 14
- Answer questions 1 – 10 on the multiple choice answer sheet provided
- For questions 11-14, start each question in a new booklet

QUESTION NO	MARK
1 - 10	/10
11	/15
12	/15
13	/15
14	/15
TOTAL	/70

THIS QUESTION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM

This assessment task constitutes 40% of the Higher School Certificate Course Assessment.

Answer questions 1 – 10 on the multiple choice answer sheet provided *1 MARK EACH*

1. The coordinates of the point P that divides the interval joining (-1,3) and (4,8) internally in the ratio 3:2 is

(A) (6 ,2)

(B) (2,6)

(C) (1,5)

(D) (5,1)

2. The value of $\lim_{x\to 0}\frac{\sin 3x}{2x}$

(A) $\frac{2}{3}$

(B) $\frac{1}{2}$

(C) $\frac{1}{6}$

(D) $\frac{3}{2}$

3. The Cartesian equation of the curves whose parametric equations are $x=\sin t$ and $y=\cos^2 t+1$ is

(A) $y=x^2-1$

(B) $y=1-x^2$

(C) $y=2-x^2$

(D) $y=x^2-2$

4. The values of x for which $\frac{x^2-9}{x}<0$ is

(A) $-3<x<3$

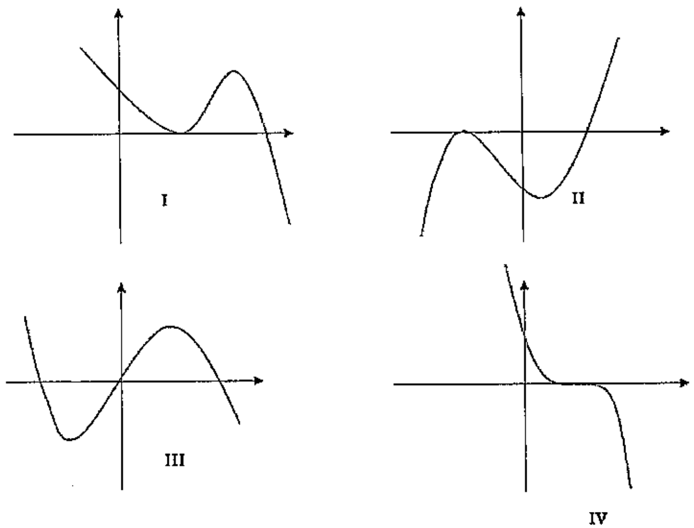
(B) $x<-3$ and $0<x<3$

(C) $x<-3$ and $x>3$

(D) $-3<x<0$ and $x>3$

5. A polynomial equation $f(x) = 0$ has a double root.

Which of the following are possible graphs of $y = f(x)$?



(A) I or II

(B) II or III

(C) III or IV

(D) II or III or IV

6. A function $f(x)$ is given by the equation $f(x) = 2\sin^{-1}(3x)$.
The domain and the range of $f(x)$ are respectively:

(A) $-3 \le x \le 3$ and $-2 \le f(x) \le 2$.

(B) $\frac{-1}{3} \le x \le \frac{1}{3}$ and $\frac{-\pi}{2} \le f(x) \le \frac{\pi}{2}$

(C) $\frac{-1}{3} \le x \le \frac{1}{3}$ and $\frac{-\pi}{4} \le f(x) \le \frac{\pi}{4}$

(D) $\frac{-1}{3} \le x \le \frac{1}{3}$ and $-\pi \le f(x) \le \pi$

7. A **cubic equation** has only one real root.

Which of the following statement(s) is/are true about the equation:

I *It has neither a maximum nor a minimum value*

II *Its maximum and minimum values must have opposite signs*

III *It must have a horizontal point of inflexion*

(A) I

(B) II

(C) III

(D) I or III

8. A projectile is fired from point O with an initial velocity of V ms^{-1} at an angle of θ with the horizontal. At time t seconds its position is given by

$$x = 100t \quad \text{and} \quad y = 173t - 5t^2$$

The initial velocity V (correct to 3 significant figures) is:

(A) 190 ms^{-1}

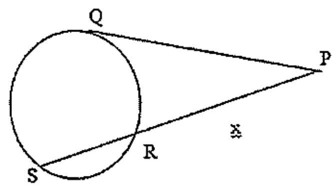
(B) 200 ms^{-1}

(C) 210 ms^{-1}

(D) 215 ms^{-1}

9. PQ is a tangent .to the circle where PRS is a secant intersecting the circle at S and R. It is given that SR = 5 units, QP = 6 units and RP = x units.

The value of x is:



Not drawn to scale

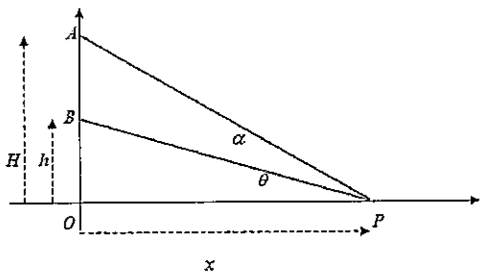
(A) 1 unit

(B) $\sqrt{11}$ units

(C) 4 units

(D) 5 units

10. The diagram shows point P on the horizontal axis, a variable distance x from the origin O. The points A and B are fixed points on the vertical axis, with distance H and h respectively, from the origin O.



Let $\angle BPO = \theta$ and $\angle APB = \alpha$. Then the value of α is?

(A) $\tan^{-1}\left(\frac{x}{H}\right)-\tan^{-1}\left(\frac{x}{h}\right)$

(B) $\tan^{-1}\left(\frac{x}{H}\right)+\tan^{-1}\left(\frac{x}{h}\right)$

(C) $\tan^{-1}\left(\frac{H}{x}\right)-\tan^{-1}\left(\frac{h}{x}\right)$

(D) $\tan^{-1}\left(\frac{H}{x}\right)+\tan^{-1}\left(\frac{h}{x}\right)$

End of multiple choice section

Question 11 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) For the function $f^{-1}(x) = 3\cos^{-1}(1-x)$

(i) write the domain and range **2**

(ii) sketch the graph **2**

(iii) write down the equation of $f(x)$ **2**

(b) Find $\frac{d}{dx}\left(e^{2x}\cos^{-1}x\right)$ **2**

(c) (i) Express $\sin 2x - \cos 2x$ in the form $R\sin(2x-\alpha)$ **2**

(ii) Hence or otherwise, sketch the graph of $y = \sin 2x - \cos 2x$ **2**

for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

(d) Use mathematical induction to prove **3**

$$\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}\dots\dots\dots\dots+\frac{n}{2^n}=2-\frac{n+2}{2^n}$$

End of question 11

Question 12 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Integrate $\int \frac{1}{1+16x^2}\,dx$ **2**

(b) A particles displacement is $x = 6 - 2\cos 2t$, where x is in centimetres and t is in seconds.

(i) Show that the motion is simple harmonic. **2**

(ii) What are the amplitude, period and centre of the motion? **2**

(iii) Find the time and position when the acceleration is first maximum. **2**

(iv) Find first two times its speed is half its maximum speed. **2**

(c) Explain why $\sin^{-1}\sin 2 \neq 2$ **1**

(d) Find the exact value of $\sin^{-1}\left(\sin\frac{4\pi}{3}\right)$ **1**

(e) Show that $y = e^{-x} - 2x$ has only one real root. **3**

The root lies in the interval $0.3 < x < 0.4$. Taking $x_1 = 0.35$ as a first approximation, use one application of Newton's method, show that a better approximation x_2, correct to 3 significant figures, is 0.352

End of question 12

Question 13 (15 marks) Use a SEPARATE writing booklet

(a) Find $\int \sin^2 5x \; dx$ **2**

(b) A function $f(x)$ is defined by $f(x) = x^2 - kx + 3$, where k is a non-zero constant.

(i) Give a reason why the inverse function of $f(x)$ does not exist. **1**

(ii) Given that $f(x) \geq -1$ and that $f^{-1}(x)$ exists if the domain is restricted to the set of negative real numbers, find the value of k. **1**

(iii) Write down the equation of $f^{-1}(x)$. **1**

(c) Carbon-14 is a radioactive isotope. The amount of present day levels of it is used to find the age of the ancient remains. It's rate of disintegration is given by $\frac{dN}{dt} = -kN$, where N is the amount or concentration of Carbon-14 and k is a constant.

(i) Show all steps in solving the differential equation above to give $N = N_0 e^{-kt}$, where N_0 is the initial amount. **2**

(No marks will be given for differentiating the solution to show that it satisfies the differential equation.)

(ii) The half-life period, denoted by $t_{1/2}$ of a radioactive element is the time taken for the amount or concentration of that isotope to fall to half of its original value. **1**

If $t_{1/2}$ of Carbon-14 is 5570 years, find the exact value of k.

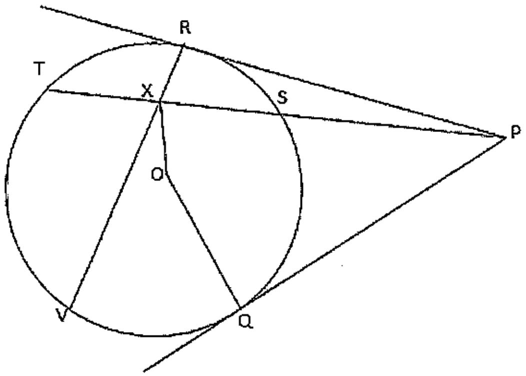
(iii) Central American civilisations include the following: **2**

Mayan 2000 BC – 900 AD, Toltec 900-1200AD, Aztec 1200-1500 AD.Identify the civilisation whose remains contain a Carbon-14 level of 75% of present day levels

Question 13 is continued on the next page

(d) If the roots of the equation $32x^3 - 48x^2 + 22x + 24 = 0$, form consecutive terms of an arithmetic sequence, find one of the roots. **1**

(e)



In the diagram above, O is the centre of the circle. From a point P, tangents are drawn to the circle touching the circle at Q and R. A line through P cuts the circle at S and T and OX bisects ST. RX produced cuts the circle at V.

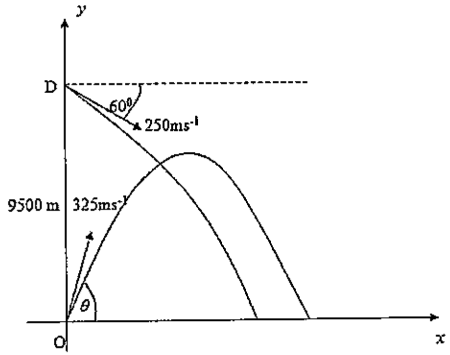
(i) Explain why OXRP is a cyclic quadrilateral. **1**

(ii) Prove that TS is parallel to VQ. **3**

End of question 13

Question 14 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) During an army exercise, a surface to air missile is launched from the point O in order to intercept a dummy bomb that is released from point D.
The point D is 9500 metres directly above O.



The dumby bomb is released at an angle of 60^0 below the horizontal with a velocity of $250\ ms^{-1}$. It can be shown that the equations of motion of the dumby bomb are :

$x_D = 125t$ and $y_D = 9500 - 125\sqrt{3}\,t - 5t^2$ *(Do NOT prove these results)*

(i) Calculate how long it would take the dumby bomb to reach the ground (correct to the nearest second) and where it would strike the ground (correct to the nearest minute). **2**

The missile is launched at the same time as the dumby is released. It is launched with an initial velocity of $325\ ms^{-1}$ and its angle of projection above the horizontal is θ.
The equations of motion of the missile are:

$x_M = 325t\cos\theta$ and $y_M = 325t\sin\theta - 5t^2$ *(Do NOT prove these results)*

(ii) Show that in order for the missile to intercept the dumby bomb it must be launched with an angle of projection of $\theta = \cos^{-1}\left(\frac{5}{13}\right)$. **1**

(iii) How high above the ground, correct to the nearest metre, does the collision occur? **3**

Question 14 is continued on the next page

(b) The acceleration (in ms^{-2}) of a particle P is given by the equation

$$\frac{d^2x}{dt^2} = 2x^3 + 4x.$$

where x is the displacement of P from a fixed point O after t seconds.

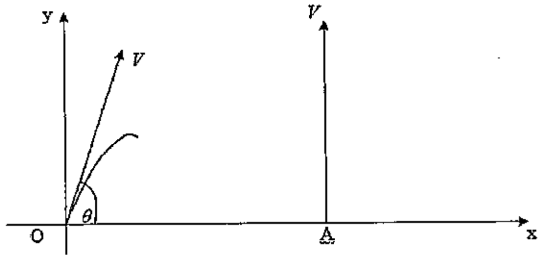
(i) If the particle is initially 2 metres to the right of O travelling with velocity 6 ms^{-1}, find an expression v^2 (the square of the velocity) in terms of x. **2**

(ii) What is the minimum speed of the object? Give a reason for your answer. **2**

Question 14 is continued on the next page

(c) Two particles are fired simultaneously from the ground at time $t = 0$. Particle 1 is projected from the origin at an angle of θ, $0 < \theta < \frac{\pi}{2}$, with an initial velocity V.

Particle 2 is projected vertically upward from point A, a distance α to the right of the origin, also with an initial velocity V.



It can be shown that while both particles are in flight, Particle 1 has equations of motion: $x = Vt\cos\theta \quad y = Vt\sin\theta - \frac{1}{2}gt^2$

and Particle 2 has equations of motion: $x = \alpha \quad y = Vt - \frac{1}{2}gt^2$

(Do NOT prove these equations of motion).

Let L be the distance between the particles at time t.

(i) Show that while both particles are in flight, **2**

$L^2 = 2V^2t^2(1-\sin\theta) - 2\alpha Vt\cos\theta + \alpha^2$.

(ii) An observer notices that the distance between the particles in flight first decreases then increases. **3**

Show that the distance between the particles in flight is smallest when $t = \frac{\alpha\cos\theta}{2V(1-\sin\theta)}$ and that the smallest distance is $\alpha\sqrt{\frac{1-\sin\theta}{2}}$.

End of Paper

Killara HIGH SCHOOL

STUDENT NUMBER: | | | | | | | | | |

Multiple Choice Answer Sheet

Completely fill the response oval representing the most correct answer.

1. A ○ B ○ C ○ D ○
2. A ○ B ○ C ○ D ○
3. A ○ B ○ C ○ D ○
4. A ○ B ○ C ○ D ○
5. A ○ B ○ C ○ D ○
6. A ○ B ○ C ○ D ○
7. A ○ B ○ C ○ D ○
8. A ○ B ○ C ○ D ○
9. A ○ B ○ C ○ D ○
10. A ○ B ○ C ○ D ○

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

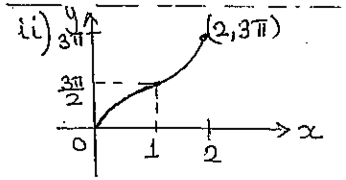
$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Mple Choice :

1. B, 2D, 3C, 4B, 5A, 6D, 7D, 8B, 9C, 10C

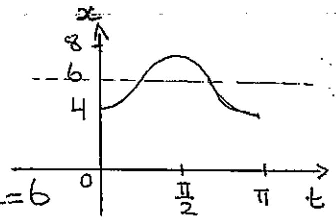
SOLUTIONS

	Award
11 a) $f^{-1}(x) = 3\cos^{-1}(1-x)$	
i) Domain $-1 \le 1-x \le 1$, Range $0 \le \cos^{-1}(1-x) \le \pi$ $-2 \le -x \le 0$ $0 \le 3\cos^{-1}(1-x) \le 3\pi$ $0 \le x \le 2$ or $0 \le y \le 3\pi$	2 marks correct domain and range. 1 mark correct domain 1 mark correct range
ii) 	2 marks correct shape correct boundaries. 1 mark some working towards solution
iii) $f^{-1}(x) = 3\cos^{-1}(1-x)$ $x = 3\cos^{-1}(1-y)$ $\frac{x}{3} = \cos^{-1}(1-y)$ $\cos\frac{x}{3} = 1-y$ $y = 1-\cos\frac{x}{3}$	2 marks correct equation of $f(x)$ 1 mark $\frac{x}{3} = \cos^{-1}(1-y)$
b) $\frac{d}{dx}(e^{2x}.\cos^{-1}x) = 2e^{2x}\cos^{-1}x - \frac{e^{2x}}{\sqrt{1-x^2}}$	2 marks correct answer 1 mark correct differention of e^{2x} or $\cos^{-1}x$
c) i) $\sin 2x - \cos 2x = R\sin(2x-\alpha)$ where $a=1$, $b=1$ $R = \sqrt{a^2+b^2} = \sqrt{2}$ $\tan\alpha = \frac{b}{a}$ $= 1$ $\alpha = \frac{\pi}{4}$ OR $\sin 2x - \cos 2x \equiv R\sin(2x-\alpha)$ $\equiv R\sin 2x\cos\alpha - R\cos 2x\sin\alpha$ $\therefore R\cos\alpha = 1$ ①, $R\sin\alpha = 1$ ② square and add $R^2\cos^2\alpha = 1$, $R^2\sin^2\alpha = 1$ $R^2(\sin^2\alpha + \cos^2\alpha) = 2$ $R = \sqrt{2}$ $\frac{②}{①}$ $\tan\alpha = 1$, $\alpha = \frac{\pi}{4}$ $\therefore \sin 2x - \cos 2x = \sqrt{2}\sin(2x - \frac{\pi}{4})$	2 marks correct R, α and expression 1 mark correct R or correct α. $1\sin 2x - 1\cos 2x$ $\sqrt{1+1}$ $\sqrt{2}\left[\frac{1}{\sqrt{2}}\sin 2x - \frac{1}{\sqrt{2}}\cos 2x\right]$ $\sqrt{2}\left[\cos\frac{\pi}{4}\sin 2x - \sin\frac{\pi}{4}\cos 2x\right]$ $\sqrt{2}\left[\sin(2x - \frac{\pi}{4})\right]$

Q11 c ii continue

Hence if the statement is true for $n=k$ then it is true for $n=k+1$.

Thus since $S(1)$ is true then $S(2)$ must be true.
If $S(2)$ is true, then $S(3)$ must be true and so on.

	Award
Q12 a) $\int\frac{1}{1+16x^2}dx = \frac{1}{16}\int\frac{1}{\frac{1}{16}+x^2}dx$ where $a^2 = \frac{1}{16}$ and $a = \frac{1}{4}$ $= \frac{1}{16}\int\frac{\frac{1}{4}}{\frac{1}{4}\left(\frac{1}{16}+x^2\right)}dx$ $= \frac{1}{4}\tan^{-1}4x + C$	2 marks correct integration 1 mark any attempt towards solution Do not penalise for not writing 'c'.
b) $x = 6 - 2\cos 2t \Rightarrow 2\cos 2t = 6 - x$ i) $\dot{x} = 4\sin 2t$ velocity $\ddot{x} = 8\cos 2t$ $\ddot{x} = 4(6-x)$ $\ddot{x} = -4(x-6)$ If $x-6 = y$ then $\dot{x} = \dot{y}$ and $\ddot{x} = \ddot{y}$ (*Important) then $\ddot{y} = -4y$ It is in $\ddot{y} = -n^2y$ format therefore motion is simple harmonic where $n^2 = 4$ and $n = 2$, $n > 0$.	2 marks Establishes that $\ddot{x} = -4x$ or $\ddot{y} = -4y$ 1 mark $\ddot{x} = -4(x-6)$
ii) Amplitude $= 2$, Period $\frac{2\pi}{2} = \pi$ Centre of the motion $x = 6$ 	2 marks for 3 properties 1 mark 2 properties established.

Q12 b continues

iii) $\ddot{x} = 8\cos 2t$ $\quad -1 \leq \cos 2t \leq 1$ $\quad -8 \leq 8\cos 2t \leq 8$

Acceleration is maximum when $\cos 2t = 1$.

$\therefore$ Maximum $\ddot{x} = 8$ cm/sec²

$8 = 8\cos 2t$

$\cos 2t = 1$

$2t = 2\pi$

$t = \pi$ sec

when $t = \pi$ sec

$x = 6 - 2\cos 2\pi$

$x = 4$ cm

Award: 1 mark establishes that $\ddot{x} = 8$ cm/sec². 1 mark for finding time and position.

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iv) Maximum speed when $\sin 2t = 1$, as $-1 \leq \sin 2t \leq 1$

$\therefore \dot{x} = 4\sin 2t = 4$, then half maximum speed $= 2$ cm/s

$2 = 4\sin 2t$

$\sin 2t = \frac{1}{2}$

$2t = \frac{\pi}{6}, \frac{5\pi}{6}$

$t = \frac{\pi}{12}, \frac{5\pi}{12}$ s

Award: 1 mark establishes that half of maximum speed is 2 cm/s. 1 mark finds $t = \frac{\pi}{12}, \frac{5\pi}{12}$ s

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c) $\sin^{-1}(\sin 2) \neq 2$, $\quad y = \sin^{-1} x$ is restricted $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

$2 > \frac{\pi}{2}$ therefore $\sin^{-1}\sin 2 \neq 2$

Award: 1 mark correct explanation

d) $\sin^{-1}\left(\sin \frac{4\pi}{3}\right) = \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$

Award: 1 mark must show working

e) $y = e^{-x} - 2x$, to show "y" has only one root, show that it does not have turning points, i.e. it is monotonic increasing or decreasing.

$y' = -e^{-x} - 2$ at turning point $y' = 0$

$e^{-x} = -2$, $\frac{1}{e^x} = -2$ not true as $\frac{1}{e^x} > 0$

Award: 1 mark shows that $y' = -e^{-x} - 2$ and $\frac{1}{e^x}$ is always positive. $\therefore \frac{1}{e^x} \neq -2$

To find an approximation: $\quad x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$x_1 = 0.35$ $\quad f(0.35) = \frac{1}{e^{0.35}} - 0.7$

$f'(0.35) = -\frac{1}{e^{0.35}} - 2$

$x_2 = 0.35 - \dfrac{\frac{1}{e^{0.35}} - 0.7}{-\frac{1}{e^{0.35}} - 2}$ $\quad x_2 = 0.351733\ldots$

$x_2 = 0.352$

Award: 1 mark substituting into formula correctly. 1 mark correct answer (3 sig. fig)

Question 13

(a) $\int \sin^2 5x \, dx$

$= \int \frac{1}{2}(1 - \cos 10x) \, dx$ ← correct conversion to double angle — 1 mark

$= \frac{1}{2}\left(x - \frac{1}{10}\sin(10x)\right) + C$ ← correct answer — 1 mark

(b) (i) $f(x) = x^2 - kx + 3$

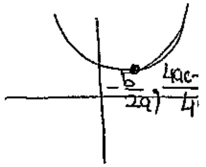
$f(x) = 3 \Rightarrow x = 0$ or $x = k$

$\therefore$ for a given value of $f(x)$, there are two distinct values of x ($k \neq 0$)

$\therefore f(x)$ doesn't have an inverse — 1 mark

or any other suitable explanation

(ii) $f(x) \geq -1 \Rightarrow \frac{12 - k^2}{4} = -1$



$\Rightarrow k = \pm 4$ — 1 mark

min. value of $f(x)$ occurs at $x = \frac{k}{2} < 0 \Rightarrow x \leq -2$

$\therefore k = -4$

$f(x) \geq -1$

$\therefore y = -1$ is turning pt.

$x = -\frac{b}{2a} = -\frac{-k}{2} = \frac{k}{2}$ $\quad \left(\frac{k}{2}, -1\right)$.

sub into $f(x)$

$-1 = \frac{k^2}{4} - \frac{k^2}{2} + 3$ $\quad \frac{-k^2}{4} + 4 = 0$

$k^2 = 16$

$k = \mp 4$

domain is restricted to negative real no's.

$k = -4$ $\quad x = \frac{-4}{2} = -2$ $\therefore x \leq -2$

(iii) when $k = -4$

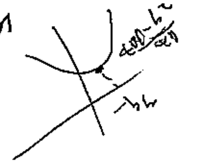
$f(x) = x^2 + 4x + 3$

$x = [f^{-1}(x)]^2 + 4f^{-1}(x) + 3$ — 1 mark

$\Rightarrow f^{-1}(x) = -2 \pm \sqrt{1 + x}$

since $f^{-1}(x) \leq -2$

$f^{-1}(x) = -2 - \sqrt{1 + x}$



(c) (i) $\frac{dN}{dt} = -kN$

$\int \frac{dN}{N} = \int -k\,dt$ —— 1 mark

$\ln N = -kt + C$ —— 1 mark

$t=0,\ N = N_0$

$\therefore c = \ln N_0$

$\ln\left(\frac{N}{N_0}\right) = -kt$

$\underline{N = N_0 e^{-kt}}$

(ii) $N = N_0 e^{-kt}$

$\frac{N_0}{2} = N_0 e^{-kt}$

$\Rightarrow e^{kt_{1/2}} = 2$

$\Rightarrow t_{\frac{1}{2}} = \frac{1}{k}\ln 2 = 5570$ —— correct answer 1 mark

$\Rightarrow k = \frac{\ln 2}{5570}$

(iii) $75 = 100e^{-kt}$

$t = \frac{1}{k}\ln\left(\frac{4}{3}\right)$

$= \frac{5570}{\ln 2}\ln\left(\frac{4}{3}\right)$

$= 2311.76 \sim 2312$ yrs. Mayan civilisation

(d) $32x^3 - 48x^2 + 22x + 24 = 0$

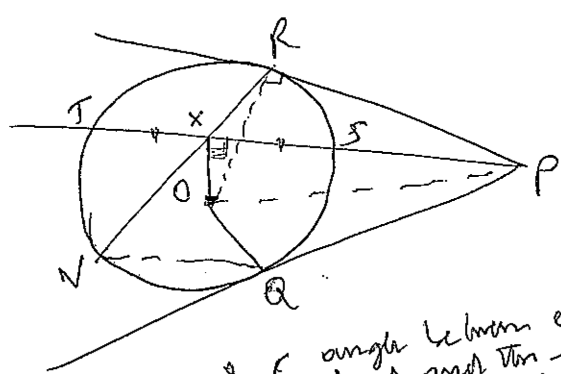
let the roots be $a-d, a, a+d$

sum of roots $= -\left(\frac{-48}{32}\right)$

$\Rightarrow 3a = \frac{48}{32}$ — 1 mark for correct working and correct answer

$\Rightarrow a = \frac{1}{2}$

(e)



(i) $\angle OXP = 90°$ (angle between a chord and the line joining the mid point of the chord to the centre of the circle) (or equivalent statement) 1 mark

$\angle ORP = 90°$ (PR is a tangent, O is the centre of the circle $\therefore OR \perp PR$)

Since $\angle ORP = \angle OXP = 90°$, OXRP is a cyclic quadrilateral with OP as a diameter (angles subtended by OP are equal) Angles in semicircle. 1 mark

(ii) let $\angle PXR = \theta$

$\therefore \angle POR = \theta$ ($\because$ OXRP is a cyclic quad)

But $\angle POR = \angle POQ$ ($\because$ PR and PQ are tangents)

$\Rightarrow \angle ROQ = 2\theta$

$\Rightarrow \angle RVQ = \theta$ (angle subtended by arc RSQ at the centre is double the angle subtended by the same arc on the circumference)

$\angle TXV = \angle PXR = \theta$ (vert opp ∠s)

$\Rightarrow \angle TXV = \angle RVQ = \theta$

$\Rightarrow$ TS ∥ VQ (alternate angles $\angle TXV = \angle RVQ$ are equal)

OR $\angle PXR = \theta$

$\angle RVQ = \theta$

$\angle PXR = \angle RVQ = \theta$

Corresponding ∠'s

$\therefore$ TS // VQ

Question 14

(a) (i) Dummy bomb reaches ground when $y_D = 0$

ie $0 = 9500 - 125\sqrt{3}t - 5t^2$

$t^2 + 25\sqrt{3}t - 1900 = 0$

$t = \frac{-25\sqrt{3} \pm \sqrt{625\times3+7600}}{2}$

$= \frac{-25\sqrt{3} + \sqrt{9475}}{2}$ ignore $(-t)$

$\doteqdot 27.61917074$

$t \doteqdot 27$ seconds

when $t = 27$

$x_D = 125 \times 27$

$= 3375$ metres from O.

1 for finding $t = 2$.
1 for $x = 3375$ m

(ii) for intersection to occur $x_D = x_M$

ie $125t = 325t\cos\theta$

$125 = 325\cos\theta$

$\cos\theta = \frac{125}{325} = \frac{5}{13}$

$\therefore \theta = \cos^{-1}\frac{5}{13}$

1 for answer (given in question) but working must be shown

Question 14

(a) (iii)

Collision occurs when $y_D = y_M$

ie $9500 - 125\sqrt{3}t - 5t^2 = 325t\sin\theta - 5t^2$

$325t\sin\theta + 125\sqrt{3}t = 9500$

$t(325\sin\theta + 125\sqrt{3}) = 9500$

$t = \frac{9500}{325\sin\theta + 125\sqrt{3}}$

$= \frac{9500}{325\sin(\cos^{-1}\frac{5}{13}) + 125\sqrt{3}}$

$= \frac{9500}{325 \times \frac{12}{13} + 125\sqrt{3}}$

$t \doteqdot 18.4$ secs

Height $= 325 \times t \times \sin\theta - 5t^2$

$= 325 \times (18.4) \times \frac{12}{13} - 5(18.4)^2$

$= 3827$ m

1 for showing $y_D = y_M$ (or the equation).

1 mark for $t = 18.4$ (or equivalent, may be exact value).

1 mark for 3827 m (if exact t used it is 3826 m).

Question 14

(b) (i) $\frac{d^2x}{dt^2} = 2x^3 + 4x$; $t=0, x=2, v=6$

$\frac{d(\frac{1}{2}v^2)}{dx} = 2x^3 + 4x$

$d(\frac{1}{2}v^2) = (2x^3+4x)\,dx$

$\int d(\frac{1}{2}v^2)\,dv = \int (2x^3+4x)\,dx$

$\frac{1}{2}v^2 = \frac{1}{2}x^4 + 2x^2 + C$

$\frac{1}{2}\times(6)^2 = \frac{1}{2}\times(2)^4 + 2\times(2)^2 + C$; $x=2, v=6$

$18 = 8 + 8 + C$

$C = 2$

$\therefore \frac{1}{2}v^2 = \frac{1}{2}x^4 + 2x^2 + 2$

$v^2 = x^4 + 4x^2 + 4$

1 for knowing to write $\frac{d^2x}{dt^2}$ as $\frac{d(\frac{1}{2}v^2)}{dx}$

ie $\frac{d(\frac{1}{2}v^2)}{dx} = 2x^3 + 4x$

1 for answer

(ii) $v^2 = x^4 + 4x^2 + 4$

$v^2 = (x^2+2)^2$

The minimum value for v^2 occurs where $x^2 = 0$

This equates to $v^2 = 4$

ie $v = \pm 2$

$\therefore$ the minimum speed is $2\,ms^{-1}$.

1 for explaining why $v^2 = 4$

1 for answer

Question 14

(c) (i)

Particle 1 : $A(Vt\cos\theta, Vt\sin\theta - \frac{gt^2}{2})$

Particle 2 : $B(\alpha, Vt - \frac{gt^2}{2})$

Let L be the distance between A and B

$L = \sqrt{(Vt\cos\theta - \alpha)^2 + (Vt\sin\theta - \frac{gt^2}{2} - Vt + \frac{gt^2}{2})^2}$

$= \sqrt{V^2t^2\cos^2\theta + \alpha^2 - 2\alpha Vt\cos\theta + (Vt\sin\theta - Vt)^2}$

$= \sqrt{V^2t^2\cos^2\theta + \alpha^2 - 2\alpha Vt\cos\theta + V^2t^2\sin^2\theta - 2V^2t^2\sin\theta + V^2t^2}$

$= \sqrt{2V^2t^2 - 2\alpha Vt\cos\theta - 2V^2t^2\sin\theta + \alpha^2}$

$L^2 = 2V^2t^2(1-\sin\theta) - 2\alpha Vt\cos\theta + \alpha^2$

1 for this line

1 for working out to arrive at final answer

(ii) L^2 is a quadratic in terms of t (concave up)

$\therefore$ Min value occurs at $t = \frac{-b}{2a}$

ie $t = \frac{2\alpha V\cos\theta}{2(2V^2)(1-\sin\theta)} = \frac{\alpha\cos\theta}{2V(1-\sin\theta)}$

sub into L^2 gives

$L^2 = 2V^2\left(\frac{\alpha^2\cos^2\theta}{4V^2(1-\sin\theta)^2}\right)(1-\sin\theta) - 2\alpha V\left(\frac{\alpha\cos\theta}{2V(1-\sin\theta)}\right)\frac{\cos\theta}{1} + \alpha^2$

$= \frac{\alpha^2\cos^2\theta}{2(1-\sin\theta)} - \frac{2\alpha^2\cos^2\theta}{2(1-\sin\theta)} + \alpha^2$

$= \frac{-\alpha^2\cos^2\theta}{2(1-\sin\theta)} + \alpha^2$

$= \alpha^2\left(1 - \frac{\cos^2\theta}{2(1-\sin\theta)}\right)$

1 for explaining and arriving at t value.

1 for sub into L^2 (this line)

1 for working to arrive at final answer.

Question 14

(c) (ii) continued

$$L = \alpha\sqrt{1 - \frac{(1-\sin\theta)(1+\sin\theta)}{2(1-\sin\theta)}}$$

$$= \alpha\sqrt{1 - \frac{1+\sin\theta}{2}}$$

$$= \alpha\sqrt{\frac{1-\sin\theta}{2}}$$

(or)

We can find $\frac{d(L^2)}{dt}$ and show that a stationary point exists at $t = \frac{\alpha\cos\theta}{2V(1-\sin\theta)}$ then substitute into L^2.

End.