Killara HIGH SCHOOL
MATHEMATICS DEPARTMENT

Centre Number

Student Number

2020 | Mathematics Extension 1

Trial HSC Examination

Date: 05/08/20

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using blue or black pen
- NESA approved calculators may be used
- Show relevant mathematical reasoning and/or calculations

Total Marks: 70

Section I – 10 marks

- Allow about 15 minutes for this section

Section II – 60 marks

- Allow about 1 hour and 45 minutes for this section

Section I (10 marks)	Multiple Choice	/10
Section II (60 marks)	Question 11	/13
	Question 12	/17
	Question 13	/15
	Question 14	/15
	Total	**/70**

This question paper must not be removed from the examination room.
This assessment task constitutes 30% of the course.

Section I

10 marks
Allow about 15 minutes for this section

Use the multiple-choice sheet for Question 1–10

---

1. Quentin, Elliott and five friends arrange themselves at random in a circle. What is the probability that Quentin and Elliott are not together?

(A) $\frac{1}{2}$

(B) $\frac{2}{3}$

(C) $\frac{5}{6}$

(D) $\frac{20}{21}$

2. An examination consists of 30 multiple-choice questions, each question having five possible answers. A student guesses the answer to every question. Let X be the number of correct answers. What is $E(X)$?

(A) 5

(B) 6

(C) 9

(D) 15

3. $\sin x + \sqrt{3}\cos x$ written in the form $R\sin(x+\alpha)$ is:

(A) $2\sin\left(x+\frac{\pi}{4}\right)$

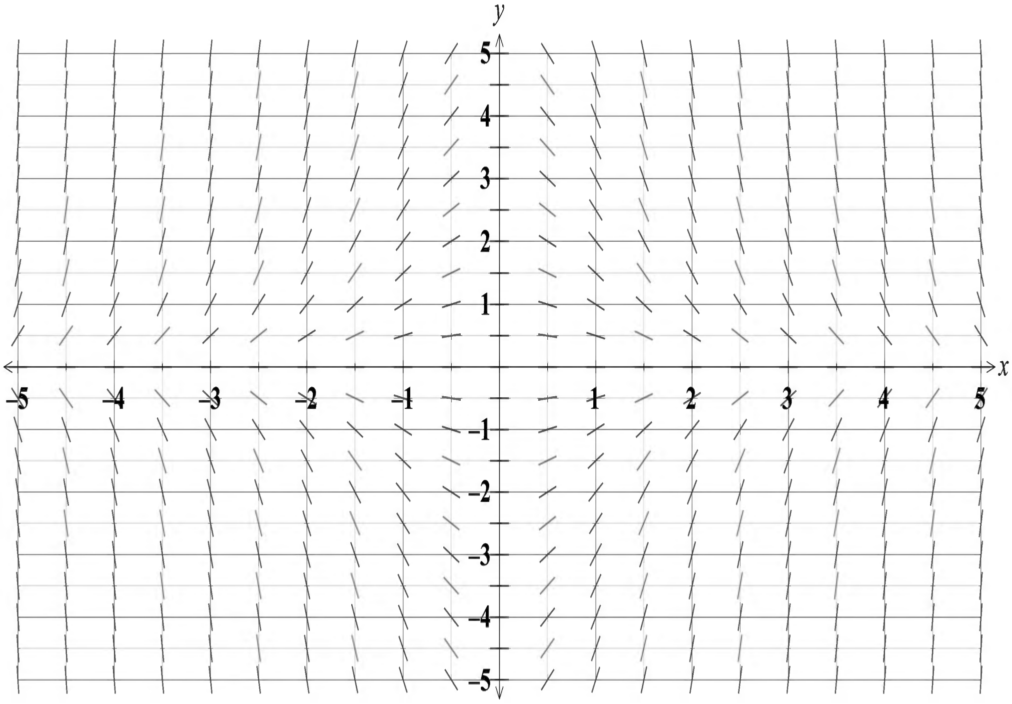
(B) $\sin\left(x+\frac{\pi}{3}\right)$

(C) $2\sin\left(x+\frac{\pi}{3}\right)$

(D) $2\sin\left(x-\frac{\pi}{3}\right)$

Section I continues on next page

4. Which of the following differential equations could be represented by the slope field diagram below?



(A) $y' = -xy$

(B) $y' = xy$

(C) $y' = -x^2y$

(D) $y' = x^2y$

5. If $y = \sin^{-1}\dfrac{a}{x}$, then $\dfrac{dy}{dx}$ equals:

(A) $\dfrac{-a}{x^2\sqrt{x^2 - a^2}}$

(B) $\dfrac{x}{\sqrt{x^2 - a^2}}$

(C) $\dfrac{-x}{\sqrt{x^2 - a^2}}$

(D) $\dfrac{-a}{x\sqrt{x^2 - a^2}}$

Section I continues on next page

6. P, Q and R are three collinear points with position vectors $\underset{\sim}{p}, \underset{\sim}{q}$ and $\underset{\sim}{r}$ respectively, where Q lies between P and R.

If $\left|\overrightarrow{QR}\right| = \frac{1}{2}\left|\overrightarrow{PQ}\right|$, then $\underset{\sim}{r}$ is equal to:

(A) $\frac{3}{2}\underset{\sim}{q} - \frac{1}{2}\underset{\sim}{p}$

(B) $\frac{3}{2}\underset{\sim}{p} + \frac{1}{2}\underset{\sim}{q}$

(C) $\frac{3}{2}\underset{\sim}{q} - \frac{3}{2}\underset{\sim}{p}$

(D) $\frac{1}{2}\underset{\sim}{p} - \frac{3}{2}\underset{\sim}{q}$

7. Given that $f(x) = e^x - 1$, and $y = f^{-1}(x)$, find an expression for $\frac{dy}{dx}$.

(A) $\frac{1}{e^x - 1}$

(B) $\frac{1}{x + 1}$

(C) $\ln x$

(D) $\ln (x + 1)$

Section I continues on next page

8. Which of the following expressions represents the area of the region bounded by the curve $y = \sin^3 x$ and the x-axis from $x = -\pi$ to $x = 2\pi$? Use the substitution $u = \cos x$.

(A) $-\int_{-\pi}^{2\pi} (1 - u^2) du$

(B) $-3\int_{0}^{\pi} (1 - u^2) du$

(C) $-\int_{-1}^{1} (1 - u^2) du$

(D) $3\int_{-1}^{1} (1 - u^2) du$

9. A body of still water has suffered an oil spill and a circular oil slick is floating on the surface of the water.

The area of the oil slick is increasing by $0.01 \text{ m}^2/\text{minute}$.

At what rate is the radius increasing when the area is 0.03 m^2?

(A) 0.006 m/minute

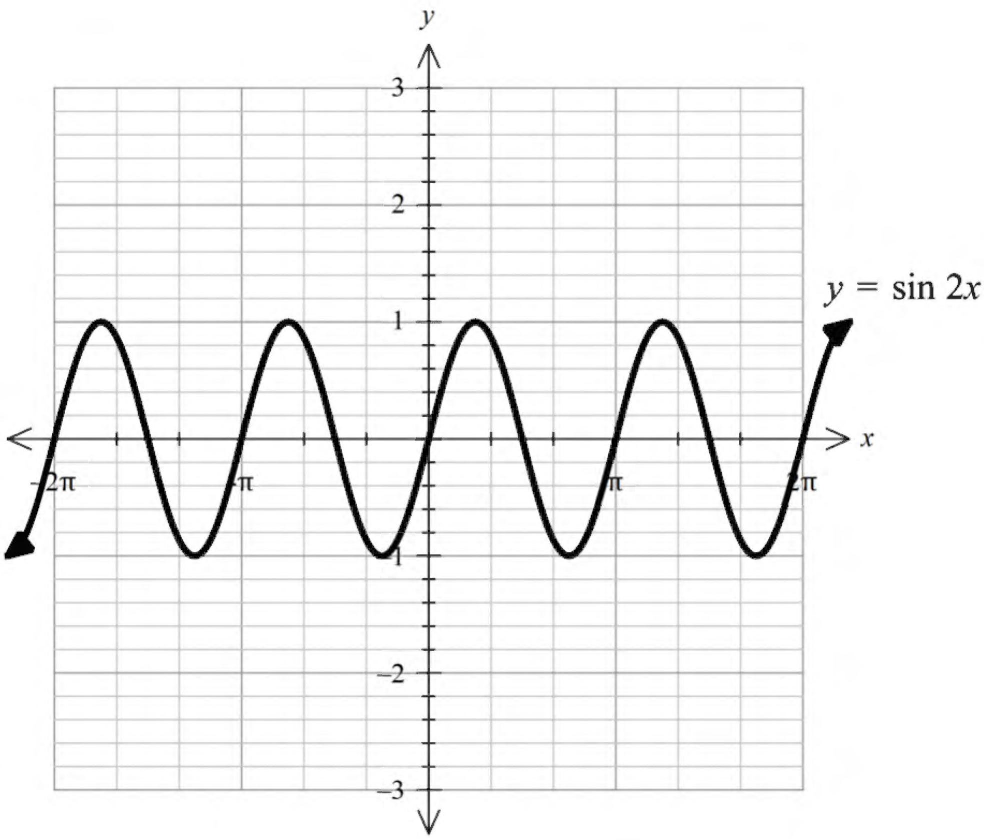
(B) 0.03 m/minute

(C) 0.0163 m/minute

(D) 0.0017 m/minute

Section I continues on next page

10. The function $y = \sin 2x$ is shown in the diagram.



If this function is transformed using steps I, II and III as below:

I.	Reflected about the x −axis
II.	Vertically translated 1 unit down
III.	Dilated horizontally by a scale factor of 2.

Which equation would represent the transformed function?

(A) $y = -(\sin x + 1)$

(B) $x = 2\,(\sin(2y) - 1\,)$

(C) $y = -(\sin(4x)) + 2$

(D) $y = -2\sin(2x) - 1$

End of Section I

Section II

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

In Questions 11 – 14, your response should include relevant mathematical reasoning and/or calculations.

Question 11 (13 marks) Use the Question 11 Writing Booklet.

(a) Solve the inequality **2**

$$\frac{2}{x-1} \leq 1$$

(b) Find: **2**

$$\frac{d}{dx}\left(e^x \tan^{-1} x\right)$$

(c) Consider vectors $\underset{\sim}{a} = 4\underset{\sim}{i} - 5\underset{\sim}{j}$ and $\underset{\sim}{b} = -2\underset{\sim}{i} + 4\underset{\sim}{j}$.

(i) Find the magnitude and direction of $\underset{\sim}{a} + \underset{\sim}{b}$. **2**

(ii) Calculate the dot product $\underset{\sim}{a} \cdot \underset{\sim}{b}$. **1**

(iii) Find the projection of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$. **2**

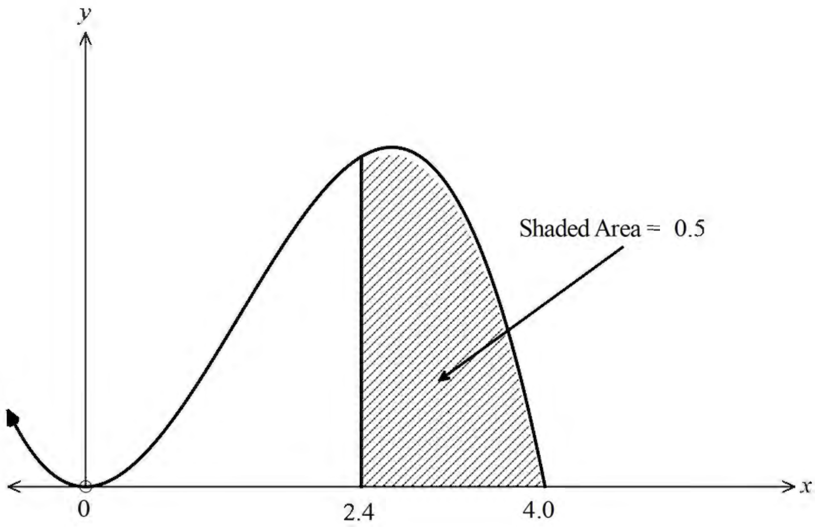
(e) The continuous random variable X has the following probability density function:

$$f(x) = \begin{cases} ax^2(4-x) & 1 \leq x \leq 4 \\ 0 & x < 1 \text{ or } x > 4 \end{cases}$$

(i) Find the value of a. **2**

(ii) Write an expression that could be used to correctly calculate $P(3 < x < 4)$. **1**
(Do not evaluate your expression)

(iii) The graph of the probability density function $f(x)$ is shown below.
The line $x = 2.4$ creates an area of 0.5 square units to the right of the line and under the curve, as shown.



Explain what measure the value of $x = 2.4$ represents in relation to $f(x)$. **1**

End of Question 11

Question 12 (17 marks) Use the Question 12 Writing Booklet.

(a) Find the constant term in the expansion: **3**

$$\left(4x^4 - \frac{2}{x^2}\right)^9$$

(b) (i) Write an expression for $\sin 5x \sin x$ in terms of $\cos 4x$ and $\cos 6x$. **1**

(ii) Hence, find **2**

$$\int_0^{\frac{\pi}{4}} \sin 5x \sin x \; dx$$

(c) Evaluate $\int_{-8}^{0} \frac{x}{\sqrt{1-x}} dx$ using the substitution $u = 1 - x$. **3**

(d) (i) Use the substitution $t = \tan\frac{x}{2}$ to show that $\operatorname{cosec} x + \cot x = \cot\frac{x}{2}$. **2**

(ii) Hence, evaluate $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (\operatorname{cosec} x + \cot x)\, dx$. **3**

(Answer in simplest form)

(e) A restaurant knows from past experience that 27% of customers will order a dessert after the main course. For next week, the restaurant has taken 288 customer bookings. **3**

Determine the probability that less than 100 will order dessert.

A normal distribution table is included overleaf.

	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0	0	0.004	0.008	0.012	0.016	0.02	0.024	0.028	0.032	0.036
0.1	0.04	0.044	0.048	0.052	0.056	0.06	0.064	0.068	0.071	0.075
0.2	0.079	0.083	0.087	0.091	0.095	0.099	0.103	0.106	0.11	0.114
0.3	0.118	0.122	0.126	0.129	0.133	0.137	0.141	0.144	0.148	0.152
0.4	0.155	0.159	0.163	0.166	0.17	0.174	0.177	0.181	0.184	0.188
0.5	0.192	0.195	0.199	0.202	0.205	0.209	0.212	0.216	0.219	0.222
0.6	0.226	0.229	0.232	0.236	0.239	0.242	0.245	0.249	0.252	0.255
0.7	0.258	0.261	0.264	0.267	0.27	0.273	0.276	0.279	0.282	0.285
0.8	0.288	0.291	0.294	0.297	0.3	0.302	0.305	0.308	0.311	0.313
0.9	0.316	0.319	0.321	0.324	0.326	0.329	0.332	0.334	0.337	0.339
1	0.341	0.344	0.346	0.349	0.351	0.353	0.355	0.358	0.36	0.362
1.1	0.364	0.367	0.369	0.371	0.373	0.375	0.377	0.379	0.381	0.383
1.2	0.385	0.387	0.389	0.391	0.393	0.394	0.396	0.398	0.4	0.402
1.3	0.403	0.405	0.407	0.408	0.41	0.412	0.413	0.415	0.416	0.418
1.4	0.419	0.421	0.422	0.424	0.425	0.427	0.428	0.429	0.431	0.432
1.5	0.433	0.435	0.436	0.437	0.438	0.439	0.441	0.442	0.443	0.444
1.6	0.445	0.446	0.447	0.448	0.45	0.451	0.452	0.453	0.454	0.455
1.7	0.455	0.456	0.457	0.458	0.459	0.46	0.461	0.462	0.463	0.463
1.8	0.464	0.465	0.466	0.466	0.467	0.468	0.469	0.469	0.47	0.471
1.9	0.471	0.472	0.473	0.473	0.474	0.474	0.475	0.476	0.476	0.477
2	0.477	0.478	0.478	0.479	0.479	0.48	0.48	0.481	0.481	0.482
2.1	0.482	0.483	0.483	0.483	0.484	0.484	0.485	0.485	0.485	0.486
2.2	0.486	0.486	0.487	0.487	0.488	0.488	0.488	0.488	0.489	0.489
2.3	0.489	0.49	0.49	0.49	0.49	0.491	0.491	0.491	0.491	0.492
2.4	0.492	0.492	0.492	0.493	0.493	0.493	0.493	0.493	0.493	0.494
2.5	0.494	0.494	0.494	0.494	0.495	0.495	0.495	0.495	0.495	0.495
2.6	0.495	0.496	0.496	0.496	0.496	0.496	0.496	0.496	0.496	0.496
2.7	0.497	0.497	0.497	0.497	0.497	0.497	0.497	0.497	0.497	0.497
2.8	0.497	0.498	0.498	0.498	0.498	0.498	0.498	0.498	0.498	0.498
2.9	0.498	0.498	0.498	0.498	0.498	0.498	0.499	0.499	0.499	0.499
3	0.499	0.499	0.499	0.499	0.499	0.499	0.499	0.499	0.499	0.499

End of Question 12

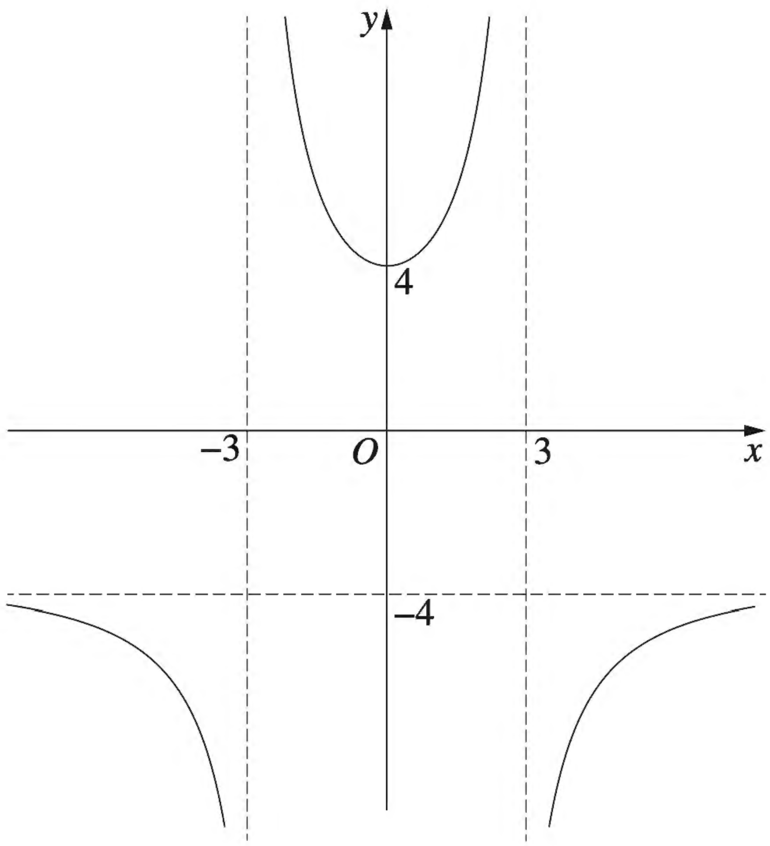
Question 13 (15 marks) Use the Question 13 Writing Booklet.

(a) Use mathematical induction to prove that **3**

$$4 + 7 + 10 + \cdots + (3n + 1) = \frac{n(3n + 5)}{2}$$

for all integers $n \geq 1$.

(b) The diagram shows the graph of $y = f(x)$.



Draw a half-page diagram for the following function, showing all asymptotes and intercepts. **2**

$$y = \frac{1}{f(x)}$$

Question 13 continues on next page

(c) Sienna intends to row her boat from the south bank of a river to meet with her friends on the north bank. The river is 100 metres wide. Sienna's rowing speed is 5 metres per second when the water is still. The river is flowing east at a rate of 4 metres per second. Sienna's boat is also being impacted by a wind blowing from the south-west, which pushed the boat at 8 metres per second. She starts rowing across the river by steering the boat such that it is perpendicular to the south bank.

(i) Show that the velocity of Sienna's boat can be expressed as the component vector: **2**

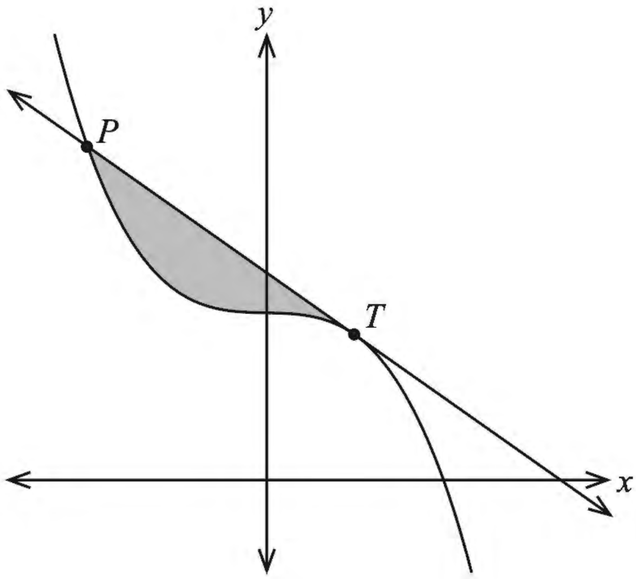
$$\left(4 + 4\sqrt{2}\right)\underset{\sim}{i} + \left(5 + 4\sqrt{2}\right)\underset{\sim}{j}$$

(ii) Calculate the speed of the boat, correct to 2 decimal places. **1**

(iii) Determine the distance rowed from Sienna's starting point to her landing point and how long it will take her to reach the north bank. **3**

Question 13 continues on next page

(d) Part of the graph of $x^3 + 8y = 64$ is shown below. A tangent is drawn to the curve, at the point $T(2, 7)$, intersecting the curve again at the point P.



The equation of the line PT is

$$y = -\frac{3}{2}x + 10.$$

(i) Find the coordinates of P. **1**

(ii) The shaded region shown in the diagram above is rotated about the x-axis. Calculate the volume of the resulting solid. **3**

End of Question 13

Question 14 (15 marks) Use the Question 14 Writing Booklet.

(a) Two chemicals A and B are poured into a mixing machine. Initially the machine has 40 L of chemical A. Then chemical A is poured in at a rate of 2 L/min and at the same time chemical B is poured in at 6 L/min. The mixing machine constantly mixes the chemicals and the mixture flows out at 4 L/min.

(i) Show that the expression for the rate of change of the volume of chemical B in the mixing machine t minutes after the pouring commences is given by: **1**

$$\frac{dB}{dt} = 6 - \frac{B}{10 + t}$$

(ii) Find values of m and n, real numbers, such that **2**

$$B(t) = \frac{mt^2 + nt}{10 + t}$$

is a solution to the differential equation in part (i).

(iii) A useable mixture is roughly two parts chemical A to three parts chemical B. How many minutes should elapse before the outflow is the required mix? **2**

(b) If the sum of two unit vectors is a unit vector, ($\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{a} + \underset{\sim}{b}$ are all unit vectors) prove using vector properties that the magnitude of their difference **3**

$$\left|\underset{\sim}{a} - \underset{\sim}{b}\right| = \sqrt{3}$$

Question 14 continues on the next page

(c) A particle is projected from O, with speed V ms^{-1} at an angle of elevation of α to the horizontal on a flat plane.
You may assume the following equations of motion of the projectile:

$$a = -g\underset{\sim}{j}$$

$$v = (V\cos\alpha)\,\underset{\sim}{i} + (-gt + V\sin\alpha)\underset{\sim}{j}$$

$$r = (V\cos\alpha t)\,\underset{\sim}{i} + (-\frac{gt^2}{2} + V\sin\alpha t)\underset{\sim}{j}$$

(i) Prove that the horizontal range of the projectile is given by **2**

$$\frac{V^2\sin2\alpha}{g}\text{ metres}$$

A garden sprinkler sprays water symmetrically about a vertical axis at a constant speed $V\ ms^{-1}$. The initial direction of spray varies continuously between angles of $15°$ and $60°$ to the horizontal.

(ii) Prove that, from the fixed position O on the level ground, the sprinkler will wet the surface of an annular region with internal and external radii $\frac{V^2}{2g}$ metres and $\frac{V^2}{g}$ metres respectively. **2**

(iii) Deduce that by locating the sprinkler at O, a rectangular garden bed of size 6m by 3m can be completely watered only if **3**

$$\frac{V^2}{2g} \geq 1 + \sqrt{7}$$

End of Examination

Q1 Seven people so $(7-1)!$ options.

2 sit together: 6 blocks of people $2!\,5!$

$P(\,) = \frac{2!\,5!}{6!} = \frac{1}{3}$ so $1 - \frac{1}{3}$

$= \frac{2}{3}$

(B)

Q2 $p = \frac{1}{5}$ $q = \frac{4}{5}$

$E(X) = 30 \times \frac{1}{5} = 6$

(B) change question lose $y =$

Q3 $y = \sin x + \sqrt{3}\cos x$ $R\sin(x+\alpha)$

$\sin(x+\alpha) = \sin x \cos\alpha + \cos x \sin\alpha$

$R\cos\alpha = 1$ $R\sin\alpha = \sqrt{3}$

$\alpha = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right)$

$(\sqrt{3})^2 + (1)^2 = 4$ $2\sin(x + \pi/3)$

(C)

Q4 (A) by wolframalpha

Q5 $y = \sin^{-1}\frac{a}{x}$

$f(x) = \frac{a}{x} = a x^{-1}$

$f'(x) = -a x^{-2}$

$= -\frac{a}{x^2}$

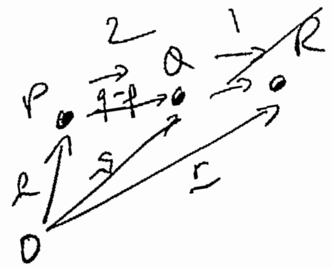
$\frac{dy}{dx} = \frac{f'(x)}{\sqrt{1-(f(x))^2}}$

$= \frac{-\frac{a}{x^2}}{\sqrt{1-\left(\frac{a}{x}\right)^2}} = \frac{-\frac{a}{x^2}}{\sqrt{\frac{x^2-a^2}{x^2}}}$

$= \frac{-a}{x\sqrt{x^2-a^2}}$

(D)

Q6



$\frac{1}{2}(q-p)$

$q + \frac{1}{2}(q-p)$

$\frac{3}{2}q - \frac{1}{2}p$

(A)

Q 7 $f(x) = e^{x} - 1$ $y = f^{-1}(x)$

$x = e^{y} - 1$

$x+1 = e^{y}$

$\ln(x+1) = y$

$y = \ln(x+1)$

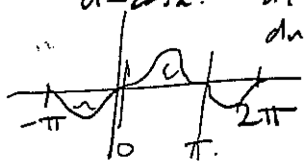
$\frac{dy}{dx} = \frac{1}{x+1}$

(B)

Q8 $y = \sin^{3} x$ $u = \cos x$. $\frac{du}{dx} = -\sin x$

$du = -\sin x$

$\int_{-\pi}^{2\pi}$



$\cos 2\pi = 1$

$\cos -\pi = -1$

$y = (1 - \cos^2 x) \sin x$

$\int_{-1}^{1} (1 - u^2)\, du$

(D)

Q 9 $\frac{dA}{dt} = 0.01$ $\frac{dr}{dA} = \frac{dr}{dt} \frac{dt}{dA}$ 100

$A = \pi r^2$ When $A = 0$ $r = \sqrt{\frac{0.03}{\pi}}$ $\frac{dA}{dr} = \frac{dt}{dr} \times \frac{dA}{dt}$.

$\frac{dA}{dr} = 2\pi r$

$2\pi r = \frac{dt}{dr} \times 0.01$

$\frac{dr}{dA} = \frac{dr}{dt} \frac{dt}{dA}$ $\frac{dr}{dt} = \frac{0.01}{2 \times \pi \times \sqrt{\frac{0.03}{\pi}}}$

$= 0.01628675\ldots$

Q 10 $y = \sin 2x$

$y = -\sin 2x$

$= -\sin 2x - 1$

~~(D)~~ A

Question 11:

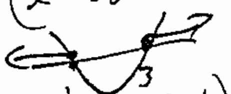
a) $\frac{2}{x-1} \leq 1$

$2(x-1) \leq (x-1)^2$

$2x-2$ ~~$2x-1$~~ $\leq x^2 - 2x + 1$

$x^2 - 4x + 3 \geq 0$ ~~$x^2 - 4x + 2 \geq 0$~~ (1)

$(x-3)(x-1) \geq 0$



~~$x \leq 1$~~, $x \geq 3$

$x < 1$

$x^2 - 4x = -2$

$x^2 - 4x + 4 = 2$

$(x-2)^2 = 2$

$x - 2 = \pm\sqrt{2}$

$x = 2 \pm \sqrt{2}$

$x \geq 2 + \sqrt{2}$

$x \leq 2 - \sqrt{2}$ (1)

(1) for quadratic

(1) for correct answers

b) $\frac{d}{dx}(e^x \tan^{-1} x) = e^x \tan^{-1} x + \frac{e^x}{1+x^2}$

(1) for u' & v'

(1) for substitution

or (1) for each term

c) $\underset{\sim}{a} = 4\underset{\sim}{i} - 5\underset{\sim}{j}$ & $\underset{\sim}{b} = -2\underset{\sim}{i} + 4\underset{\sim}{j}$

i) $\underset{\sim}{a} + \underset{\sim}{b} = 2\underset{\sim}{i} - \underset{\sim}{j}$ $|\underset{\sim}{b}| = \sqrt{20}$

magnitude: $\sqrt{5}$, direction $\tan^{-1}(\frac{-1}{2})$ (2) 1 for each direction & magnitude

ii) $\underset{\sim}{a} \cdot \underset{\sim}{b} = -8 + -20$

$= -28$ (1)

iii) $\text{proj}_b\, \underset{\sim}{a} = \frac{(\underset{\sim}{a} \cdot \underset{\sim}{b})\underset{\sim}{b}}{\ldots} = \frac{-28}{\sqrt{20}}(-2\underset{\sim}{i} + 4\underset{\sim}{j})$

$|\underset{\sim}{b}| \times |\underset{\sim}{b}|$ (1) for correct formula

$= \frac{14}{\sqrt{5}}\underset{\sim}{i} - \frac{28}{\sqrt{5}}\underset{\sim}{j}$ (1) for correct answer in correct form

$|b|^2 = 20$

$\frac{14}{5}\underset{\sim}{i} - \frac{28}{5}\underset{\sim}{j}$

OR

$\begin{pmatrix} \frac{14}{5} \\ \frac{-28}{5} \end{pmatrix}$

~~d) $y = \tan^{-1} x + \tan^{-1}\frac{1}{x}$ Domain: $x \in \mathbb{R} : x \neq 0$~~

d) $f(x) = \begin{cases} ax^2(4-x) & 1 \leq x \leq 4 \\ 0 & x < 1 \text{ or } x > 4 \end{cases}$

i) ~~$\int_1^4 ax^2(4-x)\,dx = 1$~~

~~$\int_1^4 4ax^2 - ax^3\,dx = 1$~~

~~$\left[\frac{4ax^3}{3} - \frac{ax^4}{4}\right]_1^4 = 1$~~

~~$\frac{256}{3}a - \frac{256}{4}a - \left(\frac{-4}{3}a - \frac{a}{4}\right) = 1$~~

~~$a = \frac{12}{275}$~~

(1) for forming expression

(1) for correct answer from correct working

$a\int_1^4 4x^2 - x^3\,dx =$

$\left[\frac{4x^3}{3} - \frac{x^4}{4}\right]_1^4 =$

$\left(\frac{4^4}{3} - \frac{4^4}{4}\right) - \left(\frac{4}{3} - \ldots\right)$

$\frac{64}{3} - \frac{13}{12} =$

$\frac{81}{4} = \frac{1}{a}$

$a = \frac{4}{81}$

ii) $P(3 < x < 4) = \int_3^4 ax^2(4-x)\,dx$

(1) for expression or with $a = \frac{4}{81}$

iii) $x = 2.4$ represents the ~~expected value~~ of $f(x)$ (1) for expected value or median value?

MEDIAN

– Not a normal distribution so median ≠ mean

Question 12:

a) $\left(4x^4 - \frac{2}{x^2}\right)^9 = {}^9C_0 (4x^4)^9 + {}^9C_1 (4x^4)^8\left(-\frac{2}{x^2}\right)^1$

${}^9C_6 (4x^4)^3\left(-\frac{2}{x^2}\right)^6$

$84 \times 64 \times 64$ $= 344\,064$

① for identifying correct term
① for expression
① for correct final answer

③

b) i) $\sin 5x \sin x$ ... $\cos 4x$ $\cos 6x$

$\sin 5x \sin x = \frac{1}{2}\left[\cos 4x - \cos 6x\right]$ ①

by formula

①

ii) hence $\int_0^{\pi/4} \sin 5x \sin x \, dx$

$= \frac{1}{2}\int_0^{\pi/4} \cos 4x - \cos 6x \, dx$

$= \frac{1}{2}\left[\frac{\sin 4x}{4} - \frac{\sin 6x}{6}\right]_0^{\pi/4}$

① for substitution & integration

$= \frac{1}{2}\left[\left(\frac{\sin \pi}{4} - \frac{\sin 3\pi/2}{6}\right) - 0\right]$

① for substitution and correct final answer

$= + \frac{1}{12}$

②

c) $\int_{-8}^{0} \frac{x}{\sqrt{1-x}} \, dx$

$u = 1 - x$

$\frac{du}{dx} = -1$

$du = -dx$

$u = 1 - 0 = 1$
$u = 1 - -8 = 9$

$\int_9^1 \frac{1-u-1}{\sqrt{u}} \, du$

$\int_9^1 \frac{u-1}{\sqrt{u}} \, du = \int_9^1 u^{1/2} - u^{-1/2} \, du$

① change boundaries and use u sub.

① split and integrate

$= \left[\frac{u^{3/2}}{3/2} - \frac{u^{1/2}}{1/2}\right]_9^1$

① for final correct answer

$= \frac{1^{3/2}}{3/2} - \frac{1^{1/2}}{1/2} - \left(\frac{9^{3/2}}{3/2} - \frac{9^{1/2}}{1/2}\right)$

$= -\frac{64}{3}$ $-\frac{40}{3}$

③

$-13\frac{1}{3}$

d) i) $t = \tan\frac{x}{2}$ show $\operatorname{cosec} x + \cot x = \cot\frac{x}{2}$

LHS $= \frac{1}{\sin x} + \frac{1}{\tan x}$

RHS $= \frac{1}{\tan x/2}$

$= \frac{1+t^2}{2t} + \frac{1-t^2}{2t}$

$= \frac{1}{t}$

$= \frac{1+t^2+1-t^2}{2t}$

① for correct t substitution

$= \frac{2}{2t} = \frac{1}{t} =$ RHS

① for simplifying to get $\frac{1}{t}$

②

ii) $\int_{\pi/3}^{\pi/2} (\operatorname{cosec} x + \cot x)\,dx$

$= \int_{\pi/3}^{\pi/2} \cot \frac{x}{2}\, dx$

$= 2\int_{\pi/3}^{\pi/2} \frac{\frac{1}{2}\cos \frac{x}{2}}{\sin \frac{x}{2}}\, dx$ ① for resolving to $\frac{\cos x/2}{\sin x/2}$

$= 2\left[\ln\left(\sin\left(\frac{x}{2}\right)\right)\right]_{\pi/3}^{\pi/2}$ ① for integrating with log

$= 2\left\{\ln(\sin(\pi/4)) - \ln(\sin(\pi/6))\right\}$ ① for simplified final answer

$= 2\left[\ln\left(\frac{1}{\sqrt{2}}\right) - \ln\left(\frac{1}{2}\right)\right]$

$= 2\ln\left(\frac{\frac{1}{\sqrt{2}}}{\frac{1}{2}}\right) = 2\ln\left(\frac{2}{\sqrt{2}}\right) = 2\ln\sqrt{2}$

$= \ln 2$

(3)

c) $p = 0.27$ $\quad P(X < 100)$

$q = 0.73$ $\quad 99.5?$

$np > 5$ ✓

$nq > 5$ ✓

$\mu = np = 77.76$ ① for μ

$Var(X) = np(1-p)$

$= 56.7648$ ① for σ

$\sqrt{Var(X)} = \sigma = \sqrt{56.7648}$

$= 7.5342\ldots$

z-score $2.95182\ldots$

0.498 ① for z-score & correct Prob.

$P(X < 100) \doteqdot 0.998$

(3)

49.8% ②/3

Question 13:

a) $4 + 7 + 10 + \ldots + (3n+1) = \frac{n(3n+5)}{2}$

$\forall x \in \mathbb{Z}, x \geq 1$

for $n = 1$

LHS $= 4$ $\quad$ RHS $= \frac{1(3+5)}{2} = 4$ ✓

① for $n=1$

$\therefore$ true for $n = 1$

Assume true for $n = k$

that is: $4 + 7 + 10 + \ldots + (3k+1) = \frac{k(3k+5)}{2}$

for $n = k+1$ $\quad$ $(k+1)(3k+8)$

LHS $= 4 + 7 + 10 + \ldots + (3k+1) + (3(k+1)+1)$

$= \frac{k(3k+5)}{2} + 3k + 4$ by assumption ① for use of assumption

$= \frac{3k^2 + 5k + 6k + 8}{2}$

$= \frac{3k^2 + 11k + 8}{2}$

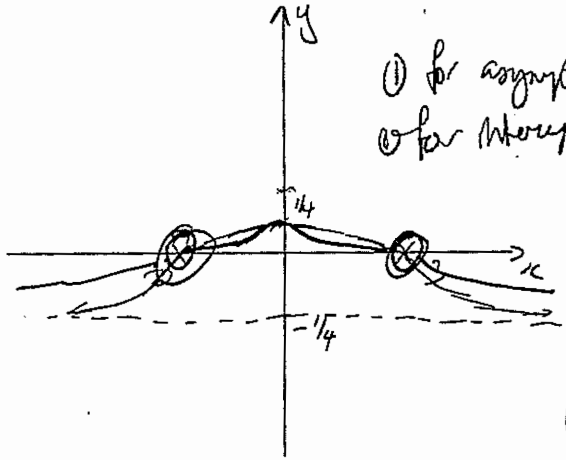
$= \frac{(k+1)(3k+8)}{2} = \frac{(k+1)(3(k+1)+5)}{2}$

$=$ RHS as required □

$\therefore$ Statement true by the principle of Mathematical Induction ① for correct working with correct conclusion

(3)

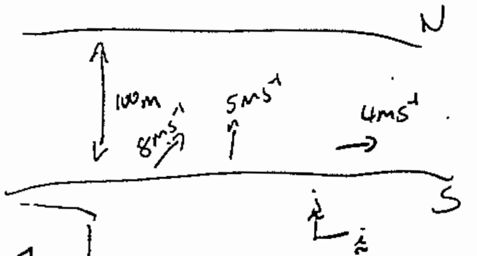
b)

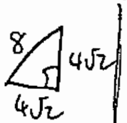


① for asymptotes
0 for intercepts: being open

②

c)





i) $(4+4\sqrt{2})\underset{\sim}{i} + (5+4\sqrt{2})\underset{\sim}{j}$

② : some working

ii)

$\sqrt{(4+4\sqrt{2})^2+(5+4\sqrt{2})^2}$

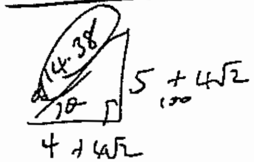
$= \sqrt{16+16\sqrt{2}+32+25+40\sqrt{2}+32}$

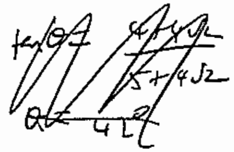
$= \sqrt{105+56\sqrt{2}}$

$= 14.38 \ (2dp)$

①

iii)





$\frac{d}{14.38} = \frac{100}{5+4\sqrt{2}}$

$d = 134.94$ distance

$\frac{d}{14.38} = 9.38$ seconds

$t = 9.38$ seconds

②/3 ③

d) i) $x^3+8y=64$ $\quad y=\frac{64-x^3}{8}$

$y=-\frac{3}{2}x+10$

$x^3+8\left(-\frac{3}{2}x+10\right)=64$

$P(-4,16)$ ① $(2,7)$

ii) $\pi\int_{-4}^{2}\left(-\frac{3}{2}x+10\right)^2-\left(\frac{64-x^3}{8}\right)^2 dx$

$-\left(8-\frac{x^3}{8}\right)^2$

$$\left(64 - \frac{2x^3 + x^6}{64}\right)$$

$$\pi \int_{-4}^{2} \frac{9}{4}x^2 - 30x + 100 - ~~64 + \frac{x^6}{64}~~ \, dx$$

~~$\pi \left[\frac{x^7}{64 \cdot 7} - 30x^2 - 36x \right]_{-4}^{2}$~~

~~$\pi \left[\frac{2^7}{488} - 15 \cdot 2^2 - 36 \cdot 2 - \left(\frac{(-4)^7}{488} - 15(-4)^2 - 36(-4) \right) \right]$~~

$$= \pi \int_{-4}^{2} \frac{9}{4}x^2 - 30x + 36 + 2x^3 - \frac{x^6}{64} \, dx$$

$$= \pi \left[-\frac{x^7}{64 \times 7} + \frac{x^4}{2} + \frac{3x^3}{4} - 15x^2 + 36x \right]_{-4}^{2}$$

~~$= \pi \left(\frac{72}{7} + \frac{4176}{7} \right)$~~ $\frac{2052}{7}\pi$

$= \frac{4248\pi}{7}$

$\approx$ ~~1906.~~50 u^3

(3)

920.9354465

Question 14:

a) i) +24, +6a, 40A, −4, $60t+4$

+6 in

out $\left(\frac{B(t)}{40+4t}\right) \times 4$

$= \frac{B(t)}{10+t}$

$$\frac{dB}{dt} = 6 - \frac{B(t)}{10+t}$$

ii) max $\Rightarrow \frac{B(t)}{4+4t} = \frac{3}{5}$ need $B(t)$.

$$\frac{dB}{dt} = 6 - \frac{B(t)}{10+t} \qquad B(t) = \frac{3t^2 + 60t}{10+t}$$

$B = \frac{mt^2 + nt}{10+t}$. where $m, n \in \mathbb{R}$ ~~positive~~ real

$$\frac{dB}{dt} = \frac{(10+t)(2mt+n) - (mt^2+nt)(1)}{(10+t)^2}$$

$$= 6\frac{\left(\frac{2m}{6}t + \frac{n}{6}\right)(10+t)}{(10+t)^2} - \frac{\frac{mt^2+nt}{10+t}}{10+t} \quad \frac{B}{10+t}$$

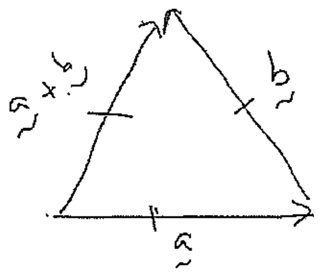
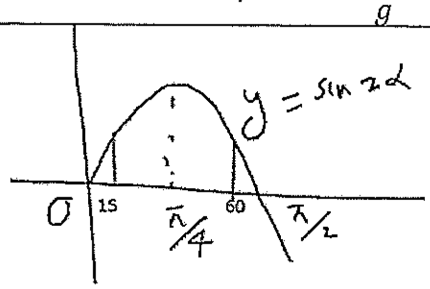
$= \frac{2m}{6} = 1 \qquad \frac{n}{6} = 10$ Equating coefficients

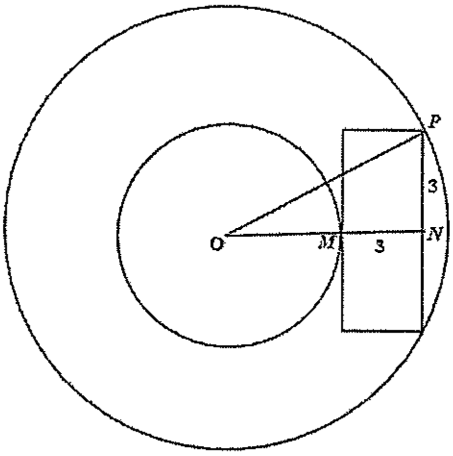
$2m = 6 \qquad n = 60$

$m = 3$

Question 14 Solution Marker's Feedback

a) (i)	Total Volume of liquid in the tank at time t = $(40+2t+6t)-(4t)=40+4t$ Litres Proportion of B in the container = $\frac{B}{40+4t}$ Amount of B flowing out per minute = $\frac{B}{40+4t}\times 4 = \frac{B}{10+t}\ L/min$ $\frac{dB}{dt}$ = Rate of inflow of B − rate of outflow of B $= 6\ L/\min - \frac{B}{10+t}\ L/min$ $\frac{dB}{dt}=6-\frac{B}{10+t}$	1 mark: Proves the expression for rate of outflow and subtract from 6 L/Min	Must have this expression to get the mark for this question Outflow= $\frac{B}{40+4t}\times 4$ Many students are adding rate with time. Can't add L/min + minutes). Parity of units must be maintained
(ii)	$B=\frac{mt^2+nt}{10+t}$ $\frac{dB}{dt}=\frac{(10+t)(2mt+n)-(mt^2+nt)}{(10+t)^2}$ $=\frac{2mt+n}{10+t}-\frac{mt^2+nt}{(10+t)^2}$ Equating, $\frac{2mt+n}{10+t}-\frac{mt^2+nt}{(10+t)^2}=6-\frac{B}{10+t}$ $\frac{2mt+n}{10+t}=6$ $2mt+n=60+6t$ $n=60,\quad 2m=6,\ \text{ie. } m=3$ $B=\frac{3t^2+60t}{10+t}$	2 marks: Correctly differentiates and evaluates m and n 1 mark: Differentiates and equates coefficients	Very poorly done. Students did not make the connection between the given $\frac{dB}{dt}$ expression from (i) and the derivative of $B=\frac{mt^2+nt}{10+t}$
(iii)	$\frac{B}{40+4t}=\frac{3}{5}$ $\frac{3t^2+60t}{4(10+t)^2}=\frac{3}{5}$ $\frac{t^2+20t}{4(10+t)^2}=\frac{1}{5}$ $5t^2+100t=400+80t+4t^2$ $t^2+20t-400=0$ $t=10\sqrt{5}-10\approx 12.36\quad t>0$	1 mark: Equates the ratio of B in the liquid in the container to $\frac{3}{5}$ 1 mark: Uses the expression for B from (ii) and solves the quadratic to give the time.	Very few students attempted this question

b)	 $\underset{\sim}{a}.\underset{\sim}{b} = \|a\|\|b\|\cos\frac{2\pi}{3}$ $= 1 \times 1 \times \frac{-1}{2} = -\frac{1}{2}$ Consider $(\underset{\sim}{a} - \underset{\sim}{b}).(\underset{\sim}{a} - \underset{\sim}{b}) = \|(\underset{\sim}{a} - \underset{\sim}{b})\|^2$ $\|(\underset{\sim}{a} - \underset{\sim}{b})\|^2 = \underset{\sim}{a}.\underset{\sim}{a} - \underset{\sim}{a}.\underset{\sim}{b} - \underset{\sim}{b}.\underset{\sim}{a} + \underset{\sim}{b}.\underset{\sim}{b}$ $= \|\underset{\sim}{a}\|^2 + \|\underset{\sim}{b}\|^2 - 2\underset{\sim}{a}.\underset{\sim}{b}$ $= 1 + 1 - 2 \times -\frac{1}{2} = 3$ $\|\underset{\sim}{a} - \underset{\sim}{b}\| = \sqrt{3}$	1 mark: correctly finds $\underset{\sim}{a}.\underset{\sim}{b}$ 1 mark: expresses $\|(\underset{\sim}{a} - \underset{\sim}{b})\|^2$ as a dot product 1 mark: correctly expands the expression for $(\underset{\sim}{a} - \underset{\sim}{b}).(\underset{\sim}{a} - \underset{\sim}{b})$ and evaluates the result	Question clearly tells you to use the vector properties. **Addition and Scalar Multiplication** 1. $\vec{a} + \vec{b} = \vec{b} + \vec{a}$ 2. $\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$ 3. $\vec{a} + \vec{0} = \vec{a}$ 4. $\vec{a} + (-\vec{a}) = \vec{0}$ 5. $c(\vec{a} + \vec{b}) = c\vec{a} + c\vec{b}$ 6. $(c + d)\vec{a} = c\vec{a} + d\vec{a}$ 7. $(cd)\vec{a} = c(d\vec{a})$ 8. $1\vec{a} = \vec{a}$ **Dot Product** 1. $\vec{a} \cdot \vec{a} = \|\vec{a}\|^2$ 2. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$ 3. $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$ 4. $(c\vec{a}) \cdot \vec{b} = c(\vec{a} \cdot \vec{b})$ 5. $\vec{0} \cdot \vec{a} = 0$ 6. $\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta$ 7. $\vec{a} \cdot \vec{b} = 0 \iff$ $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\vec{a} \perp \vec{b}$
c) (i)	$x = V\,cos\alpha t \rightarrow t = \frac{x}{V\,cos\alpha}$ $y = -\frac{gt^2}{2} + V\,sin\alpha t$ $y = -\frac{g}{2}\left(\frac{x}{V\,cos\alpha}\right)^2 + V\,sin\alpha\,\frac{x}{V\,cos\alpha}$ Hence, the Cartesian equation is $y = x tan\alpha - \frac{gx^2}{2V^2\cos^2\alpha}$ But when $y = 0, x = R$ $R\,tan\alpha = \frac{gR^2}{2V^2\cos^2\alpha}$ $2V^2 R sin\alpha cos\alpha = gR^2$ $R \neq 0, \quad R = \frac{V^2 sin2\alpha}{g}$	1 mark: Derives the Cartesian equation 1 mark: Derives the result for the range	Well done
(ii)	 The maximum range is attained when $\alpha = 45°$ The sprinkler water annual region for the angles between 15° and 45°	2 marks: correctly explains and reasons the result	You need to explain why $\alpha = 15°$ gives the minimum range and $\alpha = 45°$ gives the maximum range

	When $\alpha = 15°$, the range $R = \frac{V^2 sin30}{g} = \frac{V^2}{2g}$ When $\alpha = 45°$, the range $R = \frac{V^2 sin90}{g} = \frac{V^2}{g}$	1 mark: Substitutes into the range expression and attempts to explain why the maximum range is $\frac{V^2}{g}$	
(iii)	 In the right-angled triangle ONP, $ON^2 = OP^2 - NP^2$ $= \left(\frac{V^2}{g}\right)^2 - 9$ **1 mark** The entire garden bed will be wet when $ON \geq \frac{V^2}{2g} + 3$ Thus $\left(\frac{V^2}{g}\right)^2 - 9 \geq \left(\frac{V^2}{2g} + 3\right)^2$ 1 *Mark* $\frac{V^4}{g^2} - 9 \geq \frac{V^4}{4g^2} + \frac{3V^2}{g} + 9$ $\frac{3V^4}{4g^2} - \frac{3V^2}{g} - 18 \geq 0$ $\frac{V^4}{4g^2} - \frac{V^2}{g} \geq 6$ $\left(\frac{V^2}{2g} - 1\right)^2 \geq 7$ $\frac{V^2}{2g} \geq 1 \pm \sqrt{7}$ $\frac{V^2}{2g} > 0$ Hence, $\frac{V^2}{2g} > 1 \pm \sqrt{7}$ 1 mark	1 mark: Represents the annular sprinkler and the garden bed with dimensions, showing clearly the internal and external range 1 mark: forms the inequality 1 mark: solves the quadratic to give the result	Very few students attempted this question.