KINCOPPAL-ROSE BAY
SCHOOL OF THE SACRED HEART

2001

EXTENSION 1 MATHEMATICS

Friday 22nd June,2001

Time Allowed : Two hours
(plus five minutes reading time)

DIRECTIONS TO CANDIDATES:

- All questions may be attempted
- Start each question in a new booklet.
- All questions are of equal value.
- All necessary working should be shown in every question.
- Marks may be deducted for careless or badly arranged work.
- Only Board-approved calculators are to be used.
- Standard integrals are printed on a separate page.

Question 1 (12 marks)

(a) Solve for x: $\dfrac{x^2-5x}{x-4} \le 3$ **3**

(b) Sketch the graph of $y = 3\sin^{-1}\dfrac{x}{2}$, showing essential features. **3**

(c) Find the acute angle at which the tangents to the curve $y=x(x-4)$ at $x=-1$ and $x=3$ meet to the nearest minute. **2**

(d) The polynomial $P(x) = 2x^4 - 3x^3 - 4x^2 + ax + b$, where a and b are constants, is divisible by both $(x-2)$ and $(x+1)$. **4**

(i) Find the values of a and b

(ii) Hence find all real roots of $P(x) = 0$ for these values of a and b

(a) Solve the equation $\cos 2\theta - 1 = 2\sin\theta$ for $0 \le \theta \le 2\pi$ **3**

(b) It is known that the equation $e^{-x^2} - 5x^2 - 0.99 = 0$ has a positive root close to the origin. Attempt to find this root by using Newton's method of approximation, starting with a value of $x = 0$. Explain why this attempt fails. **3**

(c) Evaluate $\int 2\cos^2 \frac{x}{2}\, dx$ **3**

(d) The region which lies between the x axis and the curve **3**

$$f(x) = \frac{1}{\sqrt{x^2+9}}$$

from $x = 0$ to $x = 3$ is rotated about the x axis to form a solid. Find the volume of the solid, leaving the answer in exact form.

Question 3 (12 marks)

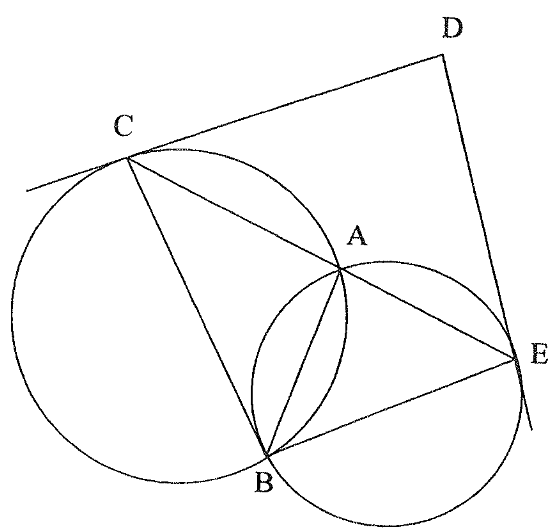
(a) Solve $\int_0^{\sqrt{3}} \frac{1}{(1+t^2)^{3/2}}\,dt$ using the substitution $t = \tan x$ **4**

(b) A spherical balloon is being inflated at the rate of 1000 cm^3s^{-1}.

(i) Show that $\frac{250}{\pi r^2}$ is an expression for the instantaneous rate of change of the radius **2**

(ii) Find the rate of change of the surface area of the balloon when the radius is 15 cm. **2**

(c)



Two circles intersect at A and B. CAE is a straight line where C is a point on the first circle and E is a point on the second circle. The tangent at C to the first circle and the tangent at E to the second circle meet at D.

(i) Copy the diagram.

(ii) Prove that BCDE is a cyclic quadrilateral. **4**

Question 4 (12 marks)

(a) Using the function

$$f(x) = \frac{1}{x^2 - 1}$$

(i) Draw a sketch of $f(x)$. **2**

(ii) Explain why the function does not have an inverse function. **1**

(iii) Suggest a suitable restriction to the domain so that the function will have an inverse. **1**

(iv) Draw a sketch of $f^{-1}(x)$. **2**

(b) Given the function $f(x) = 3\sin^{-1}x + 3\cos^{-1}x$

(i) Find the value of $f'(x)$. **2**

(ii) From part (i) what can be deduced from the graph of $y = f(x)$? **1**

(iii) Explain why $f(x) = \frac{3\pi}{2}$. **1**

(iv) Evaluate $\int_{-\frac{1}{2}}^{1} 3\sin^{-1}x + 3\cos^{-1}x \; dx$ **2**

Question 5 (12 marks)

(a) P is the point $(2at, at^2)$ on the parabola $x^2 = 4ay$ and l is the tangent at P. S is the focus. **5**

(i) Prove that the equation of l is $y = tx - at^2$.

(ii) If l cuts the y axis at A, show that A is the point $(0, -at^2)$.

(iii) Show that l makes equal angles with the y axis and with the interval PS.

(b) In a raffle the chance of winning a prize is one out of ten. If I purchase twelve tickets, what is the probability of winning exactly two prizes? (Leave your answer in index form) **2**

(c) At time t the temperature $T°$ of a body in a room of constant temperature $20°$ is decreasing according to the equation **5**

$$\frac{dT}{dt} = -k(T-20) \text{ for some constant } k > 0$$

(i) Verify that $T = 20 + Ae^{-kt}$, where A is a constant, is a solution of the above equation.

(ii) The initial temperature of the body is $90°$ and it falls to $70°$ after 10 minutes. Find the temperature of the body after a further 5 minutes.

Question 6 (12 marks)

(a) Find the term independent of x in the expansion of $\left(x + \frac{2}{x^2} \right)^{12}$ **3**

(b) (i) Write out the binomial expansion of $x(1+x)^n$ **1**

(ii) Hence, show that **3**

$$\sum_{r=0}^{n} (r+1)\,{}^nC_r = (n+2)2^{n-1}$$

(c) (i) Show that $(n+1)! = (n+1) \times n!$ **1**

(ii) Use the method of mathematical induction to show that **4**

$1 \times 1! + 2 \times 2! + 3 \times 3! + \ldots + n \times n! = (n+1)! - 1$ for all positive integers n.

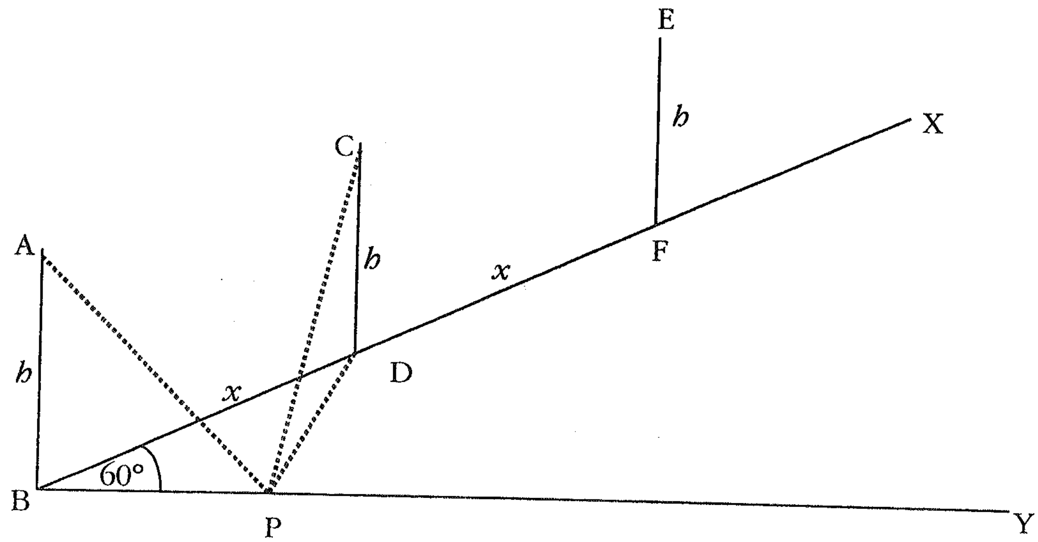
Question 7 (12 marks)

(a) An employer wishes to choose two people for a job. There are eight applicants, three are women and five are men.

(i) If each applicant is interviewed separately and all of the women are interviewed before any of the men, find how many ways there are of carrying out the interviews. **1**

(ii) If the employer chooses two of the applicants at random, find the probability that at least one of those chosen is a woman. **3**

(b)



In the above diagram, BX and BY represent two roads intersecting at an angle of 60°. On the road BX are situated three telegraph poles AB, CD and EF all of equal height, the same distance, x metres apart. (ie BD = DF = x)

P is a point on the road BY and the angles of elevation of A and C from P are 45° and 30° respectively.

(i) Show that BP = h and DP = $h\sqrt{3}$. **2**

(ii) By the use of the sine rule in triangle BDP, show that angle BDP = 30° and hence that triangle BDP is right-angled at P. **2**

(iii) Prove that $x = 2h$. **1**

(iv) By the use of the cosine rule in triangle PDF, show that PF = $h\sqrt{13}$ and hence show that the angle of elevation of E **3**

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$