KINCOPPAL-ROSE BAY
Cor Unum

2002
HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 120 minutes
- Write using black or blue pen
- Board-approved scientific calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 84

- Attempt all questions 1-7
- All questions are of equal value
- Start a new writing booklet for each question

Total marks (84)
Attempt Questions 1-7
All questions are of equal value.

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (12 marks) Use a SEPARATE Writing Booklet. **Marks**

(a) Evaluate $\cos\left[\tan^{-1}\left(-\sqrt{3}\right)\right]$. Show all working. **2**

(b) Solve the inequality $\frac{x}{2x-1} > 0$. **3**

(c) If α, β, γ are the roots of the equation $2x^3 - 6x^2 + x + 2 = 0$, **3**
find the value of $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$.

(d) Find the exact value of $\int_{\sqrt{2}}^{\sqrt{6}} \frac{dx}{4+2x^2}$. **4**

Question 2 (12 marks) Use a SEPARATE Writing Booklet. **Marks**

(a) Evaluate $\int_0^{\frac{\pi}{4}} \sec^2 x \tan^2 x \, dx$ using the substitution $u = \tan x$. **3**

(b) Taking $\frac{\pi}{2}$ as your first approximation, apply Newton's method once to find a better approximation to the root of the equation $3\sin x - 2x = 0$. **3**

(c) Differentiate $y = \tan^{-1}(\log_e x)$. **2**

(d) A committee of six people is to be chosen from 8 men and 6 women.
What is the probability that at least 3 men and 2 women will be chosen for this committee ? **4**

Question 3 (12 marks) Use a SEPARATE Writing Booklet. **Marks**

(a) $P(x) = x^3 + x^2$

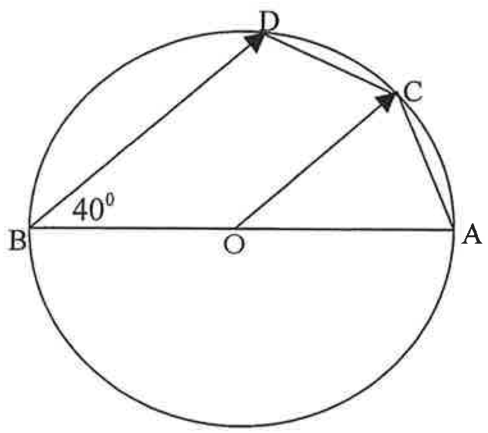
i) Sketch the polynomial $P(x)$ indicating all x and y intercepts. **2**

ii) The tangent at $A(1, 2)$ to the curve $y = x^3 + x^2$ meets the curve again at B. **2**
Draw this on your diagram in part (i).
Use calculus to show that AB has equation $y = 5x - 3$.

iii) Show that the tangent AB and the curve $y = x^3 + x^2$ meet where $x^3 + x^2 - 5x + 3 = 0$. **1**

iv) Hence or otherwise, find the coordinates of B. **3**

(b) **4**



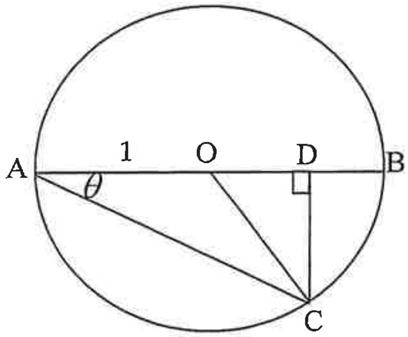
ABDC is a cyclic quadrilateral. *AB* is a diameter of the circle with centre *O*. *BD* and *OC* are parallel, and $\angle OBD = 40^0$. *C* and *D* are joined.

i) Copy this diagram into your writing booklet.

ii) Prove that *OC* bisects $\angle ACD$, giving geometric reasons for your answer.

Question 4 (12 marks) Use a SEPARATE Writing Booklet. **Marks**

a)



AB is a diameter of a circle centre O, radius 1 unit. CD is perpendicular to AB. $\angle OAC = \theta$.

i) Explain why $\angle DOC = 2\theta$ giving geometrical reasons. **2**

ii) Show that the perimeter P of $\triangle DOC$ is given by $P = 1 + \sin 2\theta + \cos 2\theta$. **2**

iii) Find the value of R and α, such that $\sin 2\theta + \cos 2\theta = R\sin(2\theta + \alpha)$. **3**

iv) Hence or otherwise, find the maximum value of the perimeter P of $\triangle DOC$, and the corresponding value of θ. **2**

(b) A machine manufacturing clone soldiers for the Republic is known to produce on average, one defective clone soldier per hundred. **3**
What is the probability that just one defective clone soldier will be made in the manufacture of 10 such clone soldiers ?
Give your answer correct to 3 decimal places.

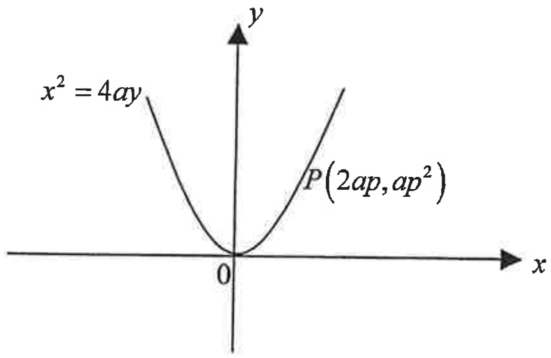
Question 5 (12 marks) Use a SEPARATE Writing Booklet. **Marks**

(a) Consider the function $f(x) = x^2 - 2x$.

i) Find the domain over which $f(x) = x^2 - 2x$ is monotonic increasing. **1**

ii) Find the inverse function $f^{-1}(x)$ over this restricted domain. **3**

(b) $P(2ap, ap^2)$ is a point on the parabola $x^2 = 4ay$.



i) Show that the equation of the normal to the curve of the parabola at the point P is $x + py = 2ap + ap^3$. **2**

ii) Find the coordinates of the point Q where the normal at P meets the y-axis. **1**

iii) Show that the coordinates of the point R, which divides PQ externally in the ratio 2:1 are given by $(-2ap, 4a + ap^2)$. **2**

iv) Find the cartesian equation of the locus of R and describe the locus in geometrical terms. **3**

Question 6 (12 marks) Use a SEPARATE Writing Booklet. **Marks**

(a) Let $f(x) = \dfrac{x}{x^2 - 1}$.

i) For what values of x is $f(x)$ undefined ? **1**

ii) Show that $y = f(x)$ is an odd function. **2**

iii) Show that $f'(x) < 0$ at all values of x for which the function is defined. **2**

iv) Hence sketch $f(x)$ showing all important features. **2**

(b) Find the term independent of x in the expansion $\left(x^3 + \dfrac{2}{x}\right)^8$. **3**

(c) A boat seats 8 people and has 4 seats on the stroke side and 4 seats on the bow side. In how many ways can 8 oarsmen be seated if 3 can row only on the stroke side and 2 can only row on the bow side ? **2**

Question 7 (12 marks) Use a SEPARATE Writing Booklet. **Marks**

(a) By considering the expansion $(1+x)^n$, show that : **4**

$${}^nC_1+2.{}^nC_2+3.{}^nC_3+4.{}^nC_4+....+n.{}^nC_n=n.2^{n-1}$$

(b) The region bounded by the curve $y=\cos 2x$ and the x-axis between $x=0$ and $x=\dfrac{\pi}{4}$ is rotated about the x-axis. Find the exact volume of the solid of revolution formed. **4**

(c) Use mathematical induction to prove that for all positive integers n : **4**

$$\sum_{r=1}^{n} r(r!)=(n+1)!-1.$$

Question 1.

a) $\cos\left[\tan^{-1}\left(-\sqrt{3}\right)\right]=\cos\left[-\tan^{-1}\sqrt{3}\right]$

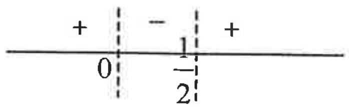
$=\cos\left(-\frac{\pi}{3}\right)$ ☑

$=\cos\frac{\pi}{3}$

$=\frac{1}{2}$ ☑

b) $\frac{x}{2x-1}>0 \quad x\neq\frac{1}{2}$

$x<0$ or $x>\frac{1}{2}$ ☑☑☑



c) $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{\beta\gamma+\alpha\gamma+\alpha\beta}{\alpha\beta\gamma}$ ☑

$=\frac{\frac{1}{2}}{-1}$ ☑

$=-\frac{1}{2}$ ☑

d) $\int_{\sqrt{2}}^{\sqrt{6}}\frac{dx}{4+2x^2}=\frac{1}{2}\int_{\sqrt{2}}^{\sqrt{6}}\frac{dx}{2+x^2}$

$=\frac{1}{2}\times\frac{1}{\sqrt{2}}\left[\tan^{-1}\frac{x}{\sqrt{2}}\right]_{\sqrt{2}}^{\sqrt{6}}$ ☑

$=\frac{1}{2\sqrt{2}}\left[\tan^{-1}\sqrt{3}-\tan^{-1}1\right]_{\sqrt{2}}^{\sqrt{6}}$ ☑

$=\frac{1}{2\sqrt{2}}\left[\frac{\pi}{3}-\frac{\pi}{4}\right]$ ☑

$=\frac{\pi}{24\sqrt{2}}$ or $\frac{\pi\sqrt{2}}{48}$ ☑

Question 2.

a) $\int_0^{\frac{\pi}{4}} \sec^2 x \tan^2 x dx$ $\quad u = \tan x,\ du = \sec^2 x dx$

$x = 0,\ u = 0$ ☑

$x = \frac{\pi}{4},\ u = 1$

$\int_0^1 u^2 du = \left[\frac{u^3}{3}\right]_0^1$ ☑

$= \frac{1}{3}$ ☑

b) $f\left(\frac{\pi}{2}\right) = 3 - \pi$ ☑

$f'\left(\frac{\pi}{2}\right) = 3\cos\frac{\pi}{2} - 2 = -2$ ☑

$x_1 = \frac{\pi}{2} - \frac{3-\pi}{-2}$

$= \frac{3}{2}$ ☑

c) $\frac{dy}{dx} = \frac{1}{1+(\log_e x)^2} \times \frac{1}{x}$ ☑☑

$= \frac{1}{x(1+2\log_e x)}$ *or* $\frac{1}{x(1+\log_e x^2)}$

d) $P(\text{at least } 3M2W) = P(3M3W)$ *or* $P(4M2W)$

$P(3M3W) = \frac{{}^8C_3 \times {}^6C_3}{{}^{14}C_6}$ ☑☑

$P(4M2W) = \frac{{}^8C_4 \times {}^6C_2}{{}^{14}C_6}$ ☑

$\therefore P(\text{at least } 3M2W) = \frac{1120}{3003} + \frac{1050}{3003}$

$= \frac{2170}{3003}$

$= \frac{310}{429}$ ☑

Question 3

a)(i)

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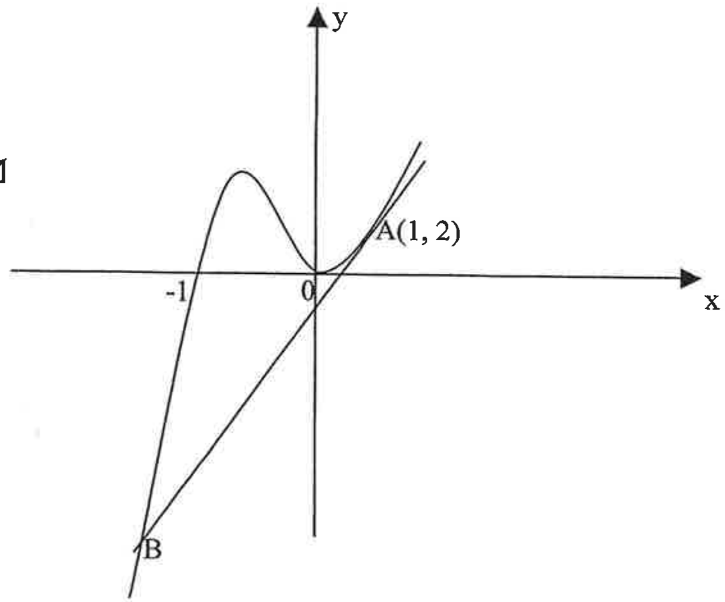

(ii) $P(x)=x^2(x+1)$

$P'(x)=3x^2+2x$

$P'(1)=5$ ☑

$y-2=5(x-1)$

$y=5x-3$ ☑

(iii) $y=5x-3$ and $y=x^3+x^2$

$5x-3=x^3+x^2$ ☑

$\therefore x^3+x^2-5x+3=0$

(iv) $x=1$ is a solution, $(x-1)$ is a factor

$$\begin{array}{r} x^2+2x-3 \\ x-1\overline{)x^3+x^2-5x+3} \\ \underline{x^3-x^2} \\ 2x^2-5x \\ \underline{2x^2-2x} \\ -3x+3 \end{array}$$ ☑

$x^3+x^2-5x+3=(x-1)^2(x+3)$ ☑

$\therefore x=-3 \qquad y=-18$

$B(-3,-18)$ ☑

b) $\angle COA=40^0$ (*corresponding* $\angle$*'s; BD* $\parallel$ *OC*) ☑

$\angle OCA=70^0$ (*angle sum of isos.* ΔOCA*; OC = OA equal radii*) ☑

$\angle DCA=140^0$ (*opp.* $\angle$*'s of cyclic quad. BDCA*) ☑

$\angle DCO=140^0-70^0$ (*adjacent* $\angle s$)

$\therefore \angle DCO=\angle OCA=70^0$ ☑

and hence OC bisects $\angle ACD$.

Question 4

a) (i) $\angle OCA = \theta$ (equal $\angle$'s of isos. ΔOAC; $OA = OC$ equal radii) ☑

$\angle DOC = 2\theta$ (exterior $\angle$ of ΔOAC=sum of opposite 2 interior $\angle$'s) ☑

or

angle at circumference is half the angle at the centre subtended by the same arc BC.

(ii) $OC = 1(radius)$

$\frac{DC}{1} = \sin 2\theta \qquad DC = \sin 2\theta$ ☑

$\frac{OD}{1} = \cos 2\theta \qquad OD = \cos 2\theta$

$P = 1 + \sin 2\theta + \cos 2\theta$ ☑

(iii) $R\sin(2\theta + \alpha) = R\sin 2\theta \cos\alpha + R\cos 2\theta \sin\alpha$ ☑

$R = \sqrt{1^2 + 1^2} = \sqrt{2}$ ☑

$\tan\alpha = 1, \ \alpha = \frac{\pi}{4}$ ☑

$\therefore \sin 2\theta + \cos 2\theta = \sqrt{2}\sin\left(2\theta + \frac{\pi}{4}\right)$

(iv) $\sin 2\theta + \cos 2\theta = \sqrt{2}\sin\left(2\theta + \frac{\pi}{4}\right)$ has a max value of $\sqrt{2}$ when:

$\sin\left(2\theta + \frac{\pi}{4}\right) = 1$

$2\theta + \frac{\pi}{4} = \frac{\pi}{2}$

$\therefore \theta = \frac{\pi}{8}$

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and maximum value of P is $1 + \sqrt{2}$ ☑

b) $P(defective) = p = 0.01$

$P(not\ defective) = q = 0.99$

$P(1\ def.\ 9\ not\ def.) = {}^{10}C_1 pq^9$ ☑

$= 10 \times 0.01 \times 0.99^9$ ☑

$= 0.091(3d.p.)$ ☑

Question 5

(a)

i) $x > 1$ ☑

ii) $x = y^2 - 2y$

$x + 1 = (y-1)^2$ ☑

$y = 1 \pm \sqrt{x+1}$ ☑

$choose\ y > 1; \therefore\ y = 1 + \sqrt{x+1}$ ☑

(b)

i) $\frac{dy}{dx} = \frac{x}{2a}; at\ x = 2ap, \frac{dy}{dx} = p$

$m_N = \frac{-1}{p} at\left(2ap, ap^2\right)$ ☑

$y - ap^2 = \frac{-1}{p}(x - 2ap)$

$x + py = 2ap + ap^3$ ☑

ii) $x = 0$ $\qquad py = 2ap + ap^3$

$y = 2a + ap^2$

$Q\left(0, 2a + ap^2\right)$ ☑

iii) $k : l = 2 : -1$

$$\left(\frac{2(0) - 2ap}{2-1}, \frac{2\left(2a + ap^2\right) - ap^2}{2-1}\right)$$ ☑☑

$\left(-2ap, 4a + ap^2\right)$

iv) $x = -2ap; \quad p = \frac{-x}{2a}$

$y = 4a + ap^2$

$y = 4a + a\left(\frac{-x}{2a}\right)^2$ ☑

$y = 4a + \frac{x^2}{4a}$

$4a(y - 4a) = x^2$ ☑

parabola vertex $(0, 4a)$, focal length a i.e. focus $(0, 5a)$ ☑

Question 6

(a)

i) $x = \pm 1$ ☑

ii) $f(-x) = \dfrac{-x}{(-x)^2 - 1}$

$= \dfrac{-x}{x^2 - 1}$ ☑

$= -f(x)$

$f(-x) = f(x)$ hence $f(x)$ is odd. ☑

iii) $f'(x) = \dfrac{(x^2-1).1 - x(2x)}{(x^2-1)^2}$

$= -\dfrac{(x^2+1)}{(x^2-1)^2}$ ☑

Now $x^2 + 1 > 0$ and $(x^2-1)^2 > 0; x \neq \pm 1$ ☑

Hence $-\dfrac{x^2+1}{(x^2-1)^2} < 0$

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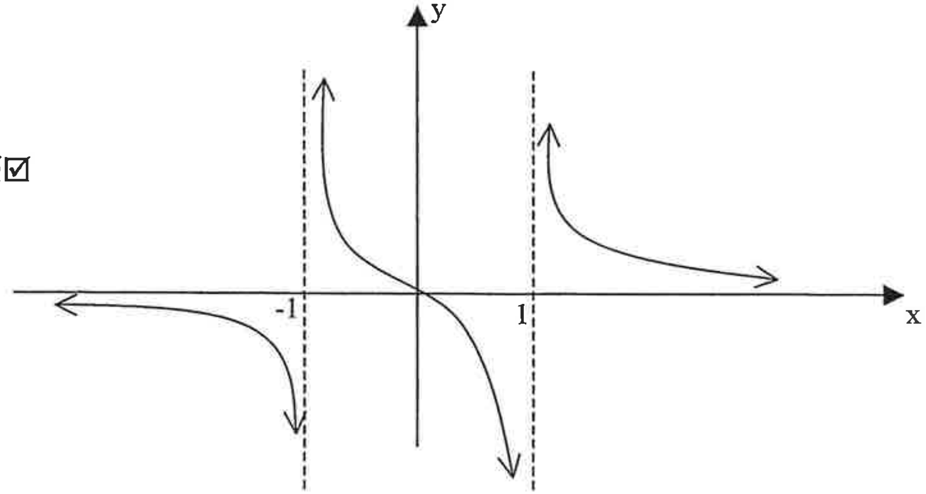

(b)

$T_{k+1} = {}^8C_k \left(x^3\right)^{8-k} \left(\dfrac{2}{x}\right)^k$

$= {}^8C_k x^{24-3k} (2)^k x^{-k}$

$= {}^8C_k 2^k x^{24-4k}$ ☑

$24 - 4k = 0$

$k = 6$ ☑

$T_7 = {}^8C_6 2^6$

$= 1792$ ☑

(c)

${}^4P_3 \times {}^4P_2 \times {}^3P_3$ ☑

$= 1728\, ways$ ☑

Question 7

(a)

$$(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + {}^nC_3x^3 + \ldots\ldots + {}^nC_rx^r + \ldots + {}^nC_nx^n$$ ☑

Differentiate both sides with respect to x

$$n(1+x)^{n-1} = {}^nC_1 + 2\,{}^nC_2x + 3\,{}^nC_3x^2 + \ldots\ldots + r\,{}^nC_rx^{r-1} + \ldots + n\,{}^nC_nx^{n-1}$$ ☑☑

let $x=1$

$$n(2)^{n-1} = {}^nC_1 + 2\,{}^nC_2 + 3\,{}^nC_3 + \ldots\ldots + r\,{}^nC_r + \ldots + n\,{}^nC_n$$ ☑

(b)

$$V = \pi\int_0^{\frac{\pi}{4}}(\cos 2x)^2\,dx$$ ☑

$$\cos^2 2x = \frac{1}{2}(\cos 4x + 1)$$

$$V = \frac{\pi}{2}\int_0^{\frac{\pi}{4}}\cos 4x + 1\,dx$$ ☑

$$= \frac{\pi}{2}\left[\frac{\sin 4x}{4} + x\right]_0^{\frac{\pi}{4}}$$ ☑

$$= \frac{\pi}{2}\left[\left(\frac{\sin\pi}{4} + \frac{\pi}{4}\right) - \left(\frac{\sin 0}{4} + 0\right)\right]$$

$$= \frac{\pi}{2}\left[\frac{\pi}{4}\right]$$

$$= \frac{\pi^2}{8}$$ ☑

(c)

Test for $n=1$

LHS $=1(1)!$ $=1$ RHS $=2!-1$ $=1$

LHS=RHS and it is true for $n=1$. ☑

Assume it is true for $n=k$

i.e. $1(1)!+2(2)!+3(3)!+\ldots+k(k)!=(k+1)!-1$

Try to prove true for $n=k+1$

$$1(1)!+2(2)!+3(3)!+\ldots+k(k)!+(k+1)(k+1)! = (k+1)!-1+(k+1)(k+1)!$$
$$= (k+1)!\{1+k+1\}-1$$
$$= (k+1)!(k+2)-1$$
$$= (k+2)!-1$$
$$= S_{k+1}$$ ☑☑

Hence if it is true for $n=k$, then it is true for $n=k+1$

Now it is true for $n=1$; hence it must be true for $n=1+1$, i.e. $n=2$.

If it is true for $n=2$, then it is true for $n=3$ and so on for all positive integers n ☑