KINCOPPAL-ROSE BAY

Cor Unum

2003

HIGHER SCHOOL CERTIFICATE
TRIAL EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown on every question

Total marks – 84

- Attempt Questions 1-7
- All questions are of equal value
- Start each question in new writing booklet

Examiners KH,MN June 2003

Question 1 (12 marks)

(a) Evaluate $\lim_{x \to 0} \dfrac{\tan 4x}{x}$. **1**

(b) Find $\dfrac{d}{dx}\left(2x^3 e^{3x}\right)$. **2**

(c) Solve $\dfrac{1}{2-x} > 3$. **3**

(d) State the domain and range of the function **2**

$$f(x) = 2\cos^{-1}\left(\frac{x}{3}\right).$$

(e) Find the acute angle between the lines **2**

$$y = 3x - 5$$
$$2x + y - 7 = 0$$

to the nearest degree.

(f) Evaluate $\displaystyle\int \frac{\cos x}{1 + 2\sin x}\, dx$ **2**

Question 2 (12 marks)

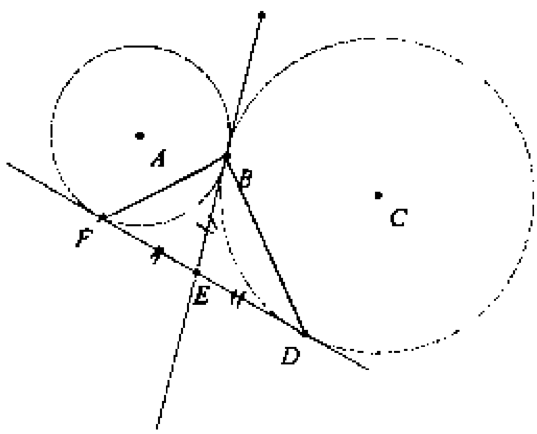
(a) Evaluate $\displaystyle\int_0^2 \frac{8x}{\sqrt{1+2x^2}}\,dx$, using the substitution $u = 1 + 2x^2$. **3**

(b) Find the general solution to $\sqrt{3}\tan x - 1 = 0$.
Express your answer in terms of π. **2**

(c) Prove that $(x\text{-}2)$ is a factor of $2x^4 - 4x^3 + 4x^2 - 15x + 14$ **1**

(d) Evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \sin^2 2x \; dx$ **3**

(e)



(e) (i) Explain why $BE =$ ~~BF~~ $EF = DE$ **1**

(ii) Let $\angle BFE = \alpha$ and $\angle BDE = \beta$.
Prove that $\angle FBD = 90°$ **2**

Question 3 (12 marks)

(a) Six people are seated in a straight line.

(i) How many seating arrangements are possible? **1**

(ii) How many arrangements are possible if Tarzan and Jane occupy the seats at either end? **2**

(b) (i) Show that $x^3 + 2x - 17 = 0$ has a root between $x=2$ and $x=3$ **1**

(ii) Using an approximation of $x = 2 \cdot 4$, use one application of Newton's method to find a better approximation for this root. Give your answer to two decimal places. **3**

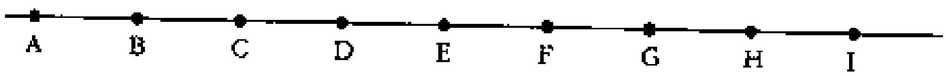
(c) Use a table of standard integral to evaluate **2**

$$\int \frac{1}{\sqrt{x^2+9}}\ dx$$

(d) $\displaystyle\int_0^{\frac{3}{4}} \frac{1}{9+16x^2}\,dx$ **3**

Question 4 (12 marks)

(a) (i) In what ratio does I divide AG? **1**



(ii) W(2,3) divides XY internally in the ratio $k:l$ where X(-1,1) and Y(7,9). Find the ratio $k:l$. **2**

(b) The polynomial $P(x)=x^3-3x^2+kx-2$ has roots α, β, γ .

(i) Find the value of $\alpha+\beta+\gamma$. **1**

(ii) Find the value of $\alpha\beta\gamma$. **1**

(iii) It is known that two roots are the reciprocal of each other. Find the value of the third root and hence find the value of k. **2**

(c) Marvin the Martian has a body temperature of 100°C. When Marvin sleeps his body temperature obeys Newton's Law of Cooling according tho the law $\frac{dT}{dt}=k(T-A)$, where T is Marvin's body temperature and A is the temperature of the surrounding air.

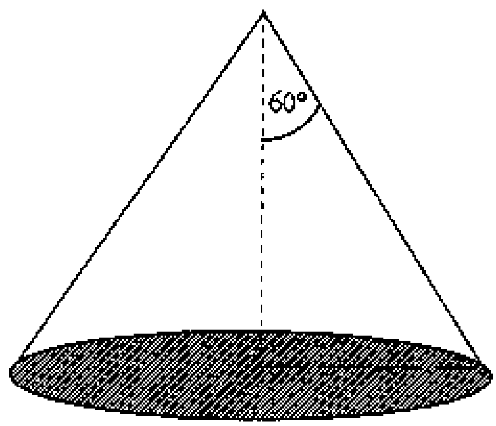
(i) Show that $T = A + Ce^{kt}$, where C and k are constants, satisfies Newton's Law of Cooling. **2**

(ii) Marvin goes to sleep at 10 pm. His temperature at midnight is $95\,°C$. Marvin's bedroom is air conditioned with the temperature set at $20\,°C$. Assuming Marvin continues to sleep what will be his body temperature at 8am? **3**

Question 5 (12 marks)

(a) Use the principle of Mathematical Induction to show that $7^n + 13^n$ is divisible by 10 for n odd integers. **3**

(b) Sand pours onto the ground and forms a cone where the semi-vertical angle is $60°$. The height of the cone at time t seconds is h cm and the radius of the base is r cm. Sand is being poured onto the pile at a rate of $12cm^3/s$.



(i) Show that $r = \sqrt{3}h$ **1**

(ii) Find the rate at which the height is increasing at the instant when the height is 12 cm. **3**

[Volume of a cone = $\frac{1}{3}\pi r^2 h$]

(c) Consider the function

$$f(x) = \tan^{-1} x + \tan^{-1}\left(\frac{1}{x}\right)$$

(i) State any values of x for which $f(x)$ is undefined. **1**

(ii) Show that $f(1) = \frac{\pi}{2}$ **1**

(iii) Show that $f'(x) = 0$ **2**

(iv) Sketch the graph of $y = f(x)$ **1**

Question 6 (12 marks)

A particle moves in Simple Harmonic Motion with amplitude a, in the form $\ddot{x} = -4x$ where x is the displacement, in metres, from the origin O and t is the time in seconds.

(i) Prove that $v^2 = 4(a^2 - x^2)$ **3**

(ii) The particle moves so that $x = 2$, $v = 4$ find the value of a. **1**

(iii) Find an expression for v in terms of displacement. **2**

(iv) By setting $v = \frac{dx}{dt}$ and taking the reciprocal, prove that $x = 2\sqrt{2}\sin 2t$ if when $t = \frac{\pi}{4}, x = 2\sqrt{2}$. **3**

(v) Where would you expect the maximum speed to occur? **1**

(vi) Hence, or otherwise, find the maximum speed of the particle. **2**

Question 7 (12 marks)

(a) A particle moves according to the equation $x = 2e^{-t}(\cos t + \sin t)$.

It moves in the interval $0 \le t \le 2\pi$.

(i)	Show that $\dot{x} = -4e^{-t}\sin t$ and find the acceleration function $\ddot{x}$.	2
(ii)	Discuss the displacement as $t \to \infty$.	1
(iii)	Find the times when the particle is at the origin.	2
(iv)	When is the particle moving in the positive direction.	1
(v)	Find the times when the particle will be stationary.	2
(vi)	Find the displacement at the times when the particle is stationary, (give your answers correct to three decimal places).	1
(vii)	Draw a neat, **full-page** sketch of $x = 2e^{-t}(\cos t + \sin t)$, giving endpoints, stationary points and intercepts	3

Kincoppal - Rose bay (2003)

Trial Extension 1 solutions 2003

Question 1:

(a) $4\lim_{x\to 0}\dfrac{\tan 4x}{4x}=4$ ✓

(b) $2x^3 3e^{3x}+6x^2e^{3x}=6x^2e^{3x}(x+1)$ ✓ ✓

(c) $(2-x)^2\times\dfrac{1}{2-x}\ge 3(2-x)^2 \quad x\ne 2$ ✓

$2-x\ge 3(2-x)^2$

$2-x-3(2-x)^2\ge 0$

$(2-x)(1-3(2-x))\ge 0$

$(2-x)(3x-5)\ge 0$

$2<x\le\dfrac{5}{3}$ never ever write like this

✓ ✓

(d) $-1\le\dfrac{x}{3}\le 1$ ✓ $\quad 0\le f(x)\le 2\pi$

$-3\le x\le 3$

✓

(e)

$m_1=3$ ✓

$m_2=-2$

For an acute angle

$\tan\theta=\left|\dfrac{m_1-m_2}{1+m_1m_2}\right|$

$\tan\theta=\left|\dfrac{3--2}{1-6}\right|$

$\tan\theta=\left|\dfrac{5}{-5}\right|=1$

$\theta=45°$ ✓

(f) $\dfrac{1}{2}\log(1+2\sin x)+C$

✓ ✓

Question 2

(a) $\dfrac{du}{dx}=4x \qquad x=2 \quad u=9$ ✓ $\qquad x=0 \quad u=1$

$dx=\dfrac{du}{4x}$

$\displaystyle\int_0^2\frac{8x}{\sqrt{1+2x^2}}\,dx=\int_1^9\frac{8x}{\sqrt{u}}\,\frac{du}{4x}=\int_1^9 2u^{-\frac{1}{2}}\,du=$ ✓

$\left[4u^{\frac{1}{2}}\right]_1^9=12-4=8$ ✓

(b) $\tan x=\dfrac{1}{\sqrt{3}}$ ✓

$x=n\pi+\tan^{-1}\dfrac{1}{\sqrt{3}}$

$x=n\pi+\dfrac{\pi}{6}$ ✓

(c) ✓

$P(2)=2(2)^4-4(2)^3+4(2)^2-15(2)+14$

$=0$

$\therefore (x-2)$ is a factor via the factor theorem

(d)

$\cos 2x=2\sin^2 x-1$

so $\cos 4x=2\sin^2 2x-1$

$\sin^2 2x=\dfrac{1}{2}(\cos 4x+1)$ ✓

$\displaystyle\int_0^{\frac{\pi}{4}}\sin^2 2x\,dx=\frac{1}{2}\int_0^{\frac{\pi}{4}}(\cos 4x+1)\,dx$

$=\dfrac{1}{2}\left[\dfrac{\sin 4x}{4}+x\right]_0^{\frac{\pi}{4}}$ ✓

$=\dfrac{1}{2}\left[\left(\dfrac{\sin\pi}{4}+\dfrac{\pi}{4}\right)-0\right]$

$=\dfrac{\pi}{8}$ ✓

(e) (i) Two tangents from an exterior point to a circle are equal. ✓

(ii)

Since $\angle BFE = \alpha$, then $\angle FBE = \alpha$ (isos $\triangle$) ✓
Since $\angle BDE = \beta$, then $\angle DBE = \beta$ (isos $\triangle$)
In $\triangle BFD$ $\quad \alpha + \alpha + \beta + \beta = 180$ ($\angle$ sum of $\triangle$)

$$2\alpha + 2\beta = 180$$
$$\alpha + \beta = 90 \quad ✓$$
$$\therefore \angle FBD = 90°$$

Question 3.

(a) (i) $6! = 720$ ✓

(ii) $4! \times 2! = 48$ ✓ ✓

(b) (i)

$$f(2) = (2)^3 + 2(2) - 17$$
$$= 8 + 4 - 17$$
$$= -5$$
✓
$$f(3) = (3)^3 + 2(3) - 17$$
$$= 16$$

Since $f(x)$ changes sign between $x=2$ and $x=3$ and since $f(x)$ is continuous for all x, $f(x)$ must be zero somewhere between $x=2$ and $x=3$.

(ii)

$$f'(x) = 3x^2 + 2$$
$$f'(2\cdot4) = 3(2\cdot4)^2 + 2 = 19\cdot24 \quad ✓$$
$$f(2\cdot4) = (2\cdot4)^3 + 2(2\cdot4) - 17 = 1\cdot624$$
$$x_2 = x_1 - \frac{f(x)}{f'(x)}$$
$$= 2\cdot4 - \frac{1\cdot624}{19\cdot24} \quad ✓$$
$$= 2\cdot3155925.....$$
$$= 2\cdot32 \quad (2d.p) \quad ✓$$

(c) $\ln(x + \sqrt{x^2 + 9}) + C$ ✓ ✓

(d)

$$\left[\frac{1}{16} \times \frac{4}{3} \tan^{-1} \frac{4x}{3}\right]_0^{\frac{3}{4}} \quad ✓ \quad ✓$$
$$= \frac{1}{16} \times \frac{4}{3} \tan^{-1} 1 - \frac{1}{16} \times \frac{4}{3} \tan^{-1} 0$$
$$= \frac{1}{16} \times \frac{4}{3} \times \frac{\pi}{4}$$
$$= \frac{\pi}{48} \quad ✓$$

Question 4.

(a) (i) -4:1 ✓

(ii)
$$x = \frac{kx_2 + lx_1}{k + l}$$
$$2 = \frac{7k - l}{k + l} \quad ✓$$
$$2k + 2l = 7k - l$$
$$-5k = -3l$$
$$\frac{k}{l} = \frac{3}{5}$$
$$k:l = 3:5 \quad ✓$$

bad question. use $y = \frac{ky_2 + ly_1}{k+l}$ and you get different answer. This suggests that... XWY is not a line

(b) (i) $\alpha + \beta + \gamma = 3$ ✓

(ii) $\alpha\beta\gamma = 2$ ✓

(iii)
$$\alpha \times \frac{1}{\alpha} \times \gamma = 2$$
$$\gamma = 2 \quad ✓$$
$$\alpha + \beta + 2 = 3$$
$$\alpha + \beta = 1$$

$$\alpha\beta + \beta\gamma + \alpha\gamma = -k$$
$$1 + 2\beta + 2\alpha = -k$$
$$1 + 2(\beta + \alpha) = -k$$
$$1 + 2 \times 1 = -k$$
$$k = -3 \quad ✓$$

(c) (i)

$T = A + Ce^{kt}$ ✓

$\frac{dT}{dt} = kCe^{kt}$

$\frac{dT}{dt} = k(T - A)$ as $Ce^{kt} = T - A$ ✓

(ii)

When $t = 0$ $T = 100$

$100 = 20 + Ae^0$

$A = 80$ ✓

$T = 20 + 80e^{kt}$

when $t = 2$ $T = 95$

$95 = 20 + 80e^{k2}$

$e^{2k} = \frac{15}{16}$

$2k = \ln\frac{15}{16}$

$k = \frac{1}{2}\ln\frac{15}{16}$ ✓

when $t = 10$

$T = 20 + 80e^{\frac{1}{2}\ln\frac{15}{16}.10}$

$T = 77.93571472$

$T = 78°$ ✓

Question 5:

(a) Let $7^n + 13^n = 10M$ where M is any integer.

For $n=1$ $7^1 + 13^1 = 20 = 10 \times 2$ which is divisible by 10. ✓

Assume $7^k + 13^k = 10M$ is true for $n=k$

Prove true for $n=k+2$

$13^k = 10M - 7^k$

$7^{k+2} + 13^{k+2} =$

$7^2 7^k + 13^2 13^k = 7^2 7^k + 13^2(10M - 7^k)$ ✓

$= 49.7^k + 1690M - 169.7^k$

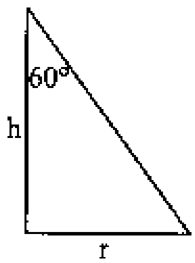
$= 1690M - 120.7^k$

$= 10(169 - 12.7^k)$ ✓

which is a multiple of 10, therefore true for $n=k+2$.

Since it is true for n=1, it is true for n=1+2 And so on, so it is true for all positive odd integers.

(b)



(i) $\tan 60° = \frac{r}{h}$

$h \tan 60° = r$ ✓

$r = \sqrt{3}h$

(ii) $\frac{dV}{dt} = 12$

$V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi(3h^2)h$

$V = \pi h^3$ ✓

$\frac{dV}{dh} = 3\pi h^2$

$\frac{dV}{dh} = 432\pi$ when h=12 ✓

$\frac{dh}{dt} = \frac{dh}{dV}.\frac{dV}{dt}$

$= \frac{1}{432\pi}.12$

$= \frac{1}{36\pi} cm/s$ ✓

(c) (i) $x \neq 0$ ✓

$$f(1) = \tan^{-1} 1 + \tan^{-1}\left(\frac{1}{1}\right)$$

(ii)

$$= \frac{\pi}{4} + \frac{\pi}{4} \quad ✓$$

$$= \frac{\pi}{2}$$

(iii)

$$f'(x) = \frac{1}{1+x^2} + \frac{1}{1+\left(\frac{1}{x}\right)^2} \cdot \frac{d}{dx}(x^{-1})$$

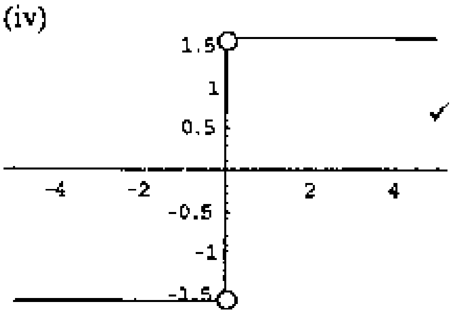
$$= \frac{1}{1+x^2} + \frac{1}{\frac{x^2+1}{x^2}} \cdot -x^{-2} \quad ✓$$

$$= \frac{1}{1+x^2} + \frac{x^2}{x^2+1} \cdot -\frac{1}{x^2}$$

$$= \frac{1}{1+x^2} - \frac{1}{x^2+1}$$

$$= 0 \quad ✓$$

(iv)



✓

Question 6 (12 marks) (Start a new booklet)

(i) $\ddot{x} = -4x$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -4x$ ☑

$\frac{1}{2}v^2 = -\frac{4x^2}{2} + C \qquad (v = 0, x = a)$

$0 = -2a^2 + C$

$C = 2a^2$ ☑

$\frac{1}{2}v^2 = -2x^2 + 2a^2$

$v^2 = 4\left(a^2 - x^2\right)$ ☑

(ii) $v^2 = 4\left(a^2 - x^2\right) \qquad x = 2, v = 4$

$16 = 4\left(a^2 - 4\right)$

$a^2 - 4 = 4$

$a^2 = 8$

$a = 2\sqrt{2} \qquad (a > 0)$ ☑

(iii) $v^2 = 4\left(8 - x^2\right)$

$v = \pm 2\sqrt{8 - x^2}$ ☑

but if $x = 2, v = 4 > 0$

$v = 2\sqrt{8 - x^2}$ ☑

↳ nope.
SHM so it's ±

(iv) $v = 2\sqrt{8 - x^2}$

$\frac{dx}{dt} = 2\sqrt{8 - x^2}$

$\frac{dt}{dx} = \frac{1}{2\sqrt{8 - x^2}}$ ☑

$t = \frac{1}{2}\sin^{-1}\left(\frac{x}{2\sqrt{2}}\right) + C, \left(t = \frac{\pi}{4}, x = 2\sqrt{2}\right)$ ☑

$\frac{\pi}{4} = \frac{1}{2}\sin^{-1}\left(\frac{2\sqrt{2}}{2\sqrt{2}}\right) + C$

$\frac{\pi}{4} = \frac{1}{2}\sin^{-1}(1) + C$

$\frac{\pi}{4} = \frac{\pi}{4} + C$

$C = 0$

$t = \frac{1}{2}\sin^{-1}\left(\frac{x}{2\sqrt{2}}\right)$ ☑

$\sin(2t) = \frac{x}{2\sqrt{2}}$

$x = 2\sqrt{2}\sin(2t)$

(v) at the origin (the centre of the motion. ☑

(vi) $v = 4\sqrt{2}\cos(2t)$ ☑

max speed $= 4\sqrt{2} \times 1$

$= 4\sqrt{2}\ m/s$ ☑

Question 7 (12 marks) (Start a new booklet)

(i) $x = 2e^{-t}(\cos t + \sin t)$

$\dot{x} = (\cos t + \sin t) \times -2e^{-t} + 2e^{-t}(-\sin t + \cos t)$

$= -2e^{-t} \times 2\sin t$ ☑

$= -4e^{-t}\sin t$

$\ddot{x} = \sin t \times 4e^{-t} + -4e^{-t}\cos t$

$= 4e^{-t}(\sin t - \cos t)$ ☑

(ii) As $t \to \infty$, $x \to 0$ since $e^{-t} \to 0$. ☑

(iii) $0 = 2e^{-t}(\cos t + \sin t)$ but $e^{-t} \neq 0$

$\cos t + \sin t = 0$

$\sin t = -\cos t$ ☑

$\tan t = -1$

$t = \frac{3\pi}{4}, \frac{7\pi}{4}$ ☑

(iv) $\dot{x} = -4e^{-t}\sin t$ moving in a positive direction $\dot{x} > 0$, ☑

$-4e^{-t}\sin t > 0$

$\sin t < 0$

$\pi < t < 2\pi$

(v) Stationary when $\dot{x} = 0 \to t = 0, \pi, 2\pi$ ☑☑

(vi) $x = 2e^{-t}(\cos t + \sin t)$ $t = 0, \pi, 2\pi$

$x = 2(\cos 0 + \sin 0) = 2$

$x = 2e^{-\pi}(\cos \pi + \sin \pi) = -0.086$

$x = 2e^{-2\pi}(\cos 2\pi + \sin 2\pi) = 0.004$ ☑☑

(vii)