Question 1 (12 marks) **Marks**

(a) Find the point P which divides the interval joining (2,-3) and (3,4) externally in the ratio 3:2. **2**

(b) Solve $\dfrac{3x}{x-2} \geq 1$. **3**

(c) Evaluate $\displaystyle\lim_{x \to 0} \frac{\tan 4x}{3x}$ **2**

(d) A curve has parametric equations $x = \sin\theta, y = 2\operatorname{cosec}^2\theta$. Find the cartesian equation. **1**

(e) Use the substitution $x = u^2 - 1$ to show that $\displaystyle\int \frac{3x}{\sqrt{x+1}}\,dx = 2\sqrt{x+1}\,(x-2) + C$. **3**

Question 2 (12 marks) Use a SEPARATE writing booklet.

(a) Sketch $y = \dfrac{1}{2}\cos^{-1}\dfrac{x}{3}$, clearly showing the endpoints **2**

(b) Find $\dfrac{d}{dx}\left(\tan^{-1}\sqrt{x}\right)$ **2**

(c) Evaluate $\displaystyle\int_0^2 \frac{1}{\sqrt{8-x^2}}\,dx$ **2**

(d) Find the coefficient of x^9 in the expansion of $\left(3 - x^3\right)^4$. **2**

(e) (i) Express $\cos x + \sqrt{3}\sin x$ in the form $A\cos(x - \alpha)$ where α is in radians. **2**

(ii) Hence solve $\cos x + \sqrt{3}\sin x = 1$ for $0 \leq x \leq 2\pi$. **2**

Question 3 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) How many nine letter arrangements can be made from the letters of the name NEEDLEMAN? **2**

(b) Find the exact value of $\cos\left(\tan^{-1}\frac{1}{5}\right)$ **2**

(c) Find the size of the acute angle (to the nearest minute) between the tangents drawn to $y = \log x$ at $x = 1$ and $x = 3$. **2**

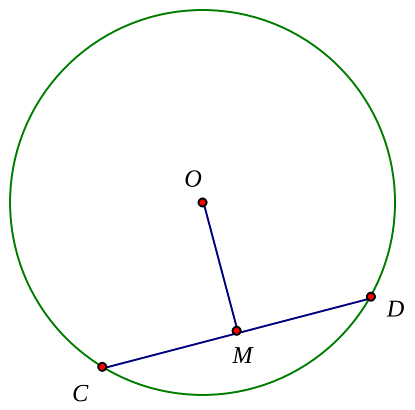
(d) (i) In a family of 3 children explain why the probability of 3 girls is $\frac{1}{8}$. **1**

(ii) If there are 7 families with 3 children, find the probability that four families have 3 girls. **2**

(e) Prove by mathematical induction that $7^n + 3^n$ is divisible by 10 when n is an odd integer. **3**

Question 4 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) O is the centre of the circle and M is the midpoint of the chord CD. **3**
Prove that $\angle OMC = 90°$



(b) (i) The points A and B are the points of intersection of the circle $x^2 + y^2 = 1$ and the line $y = x$. Find the coordinates of A and B. **2**

(ii) Draw a diagram that illustrates your answer in (i). Draw six different circles that pass through A and B. **1**

(iii) The centres of the circles you have drawn are collinear. Using part (a) of this question, prove that all the centres of the circles you have drawn must lie on the line $y = -x$. **2**

(iv) Let $H\ (u, -u)$ be the centre of one of these circles which passes through A and B. Show that the equation of this circle is $(x-u)^2 + (y+u)^2 = 2(u^2+1)$ **2**

(v) Find the equation of the circle passing through A, B and the point (4,-1) **2**

Question 5 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) A leadership committee of 2 boys and 3 girls is to be chosen from 5 boys and 4 girls.

(i) How many different leadership committees can be chosen? **2**

(ii) What is the probability that 2 particular boys and 1 particular girl will be chosen? **2**

(b) An arc AB of a circle subtends an angle of θ radians at the centre.
The length of the arc is ℓ and the length of the chord is d.

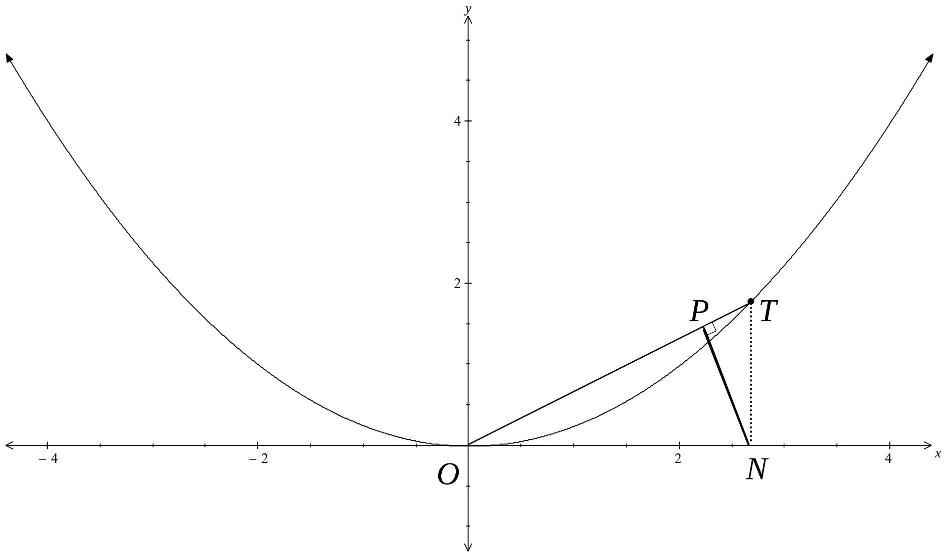
(i) If $\ell : d = 4 : 3$, show that $3\theta - 8\sin\frac{\theta}{2} = 0$. **3**

(ii) Using $\theta = 2{\cdot}5$ as a first approximation, use Newton's Method once to find a second approximation (correct to 3 decimal places) to the solution of the equation in (i). **3**

(iii) Is the second approximation a better solution than the first approximation? Use a numerical argument to justify your answer. **2**

Question 6 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) $T\left(2t, t^2\right)$ is a point on the parabola $x^2 = 4y$ whose vertex is O. N is the foot of the perpendicular from T to the x-axis. The perpendicular from N to OT meets OT at P.



(i) Copy the diagram into your answer booklet.

(ii) Write down the coordinates of N. **1**

(iii) Determine the equation of OT. **2**

(iv) Determine the equation of PN. **2**

(v) Determine the coordinates of P. **2**

(vi) Find the locus of P is satisfied by $x^2 + y^2 - 4y = 0$ and give a geometrical description of the locus. **3**

(vii) Sketch the locus of P on your diagram. **1**

(viii) Which point will be excluded from the locus? **1**

(b) The roots of $x^3 + ax^2 + bx + c = 0$ are in arithmetic sequence. By denoting the roots as $\alpha - d, \alpha$ and $\alpha + d$ prove that:

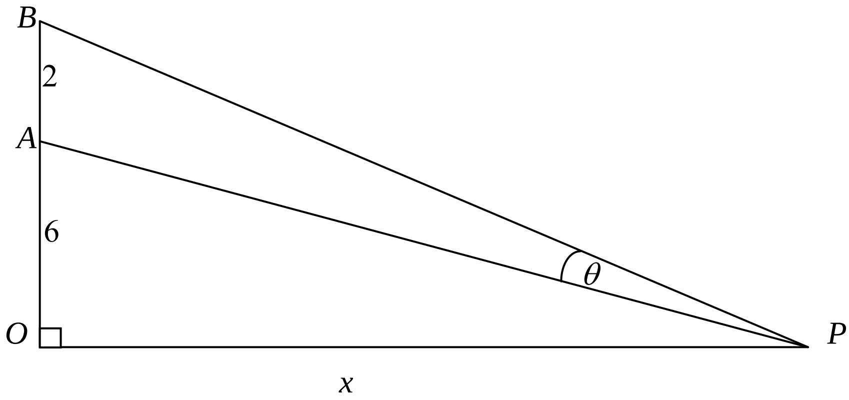
(i) one of the roots is $-\dfrac{a}{3}$ **1**

(ii) $2a^3 - 9ab + 27c = 0$ **2**

Question Seven (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find the exact value of x if $\log_e(2\log_e x) = 1$. **2**

(b)



In the diagram above ΔBOP has a right angle at *O*, *OA* is 6 units, *AB* is 2 units and $\angle BPA$ is θ.

(i) Show that $\theta = \tan^{-1}\frac{8}{x} - \tan^{-1}\frac{6}{x}$. **2**

(ii) Show that if $\frac{d\theta}{dx} = 0$, then $x = 4\sqrt{3}$. **4**

(iii) Deduce that when $x = 4\sqrt{3}$ then $\theta = \frac{\pi}{6}$ **2**

(iv) Without using calculus, do you think $\theta = \frac{\pi}{6}$ would be the minimum or maximum value of θ? Justify your answer. **2**

Question One

(a) $x=\dfrac{kx_2+lx_1}{k+l}$ $\quad y=\dfrac{ky_2+ly_1}{k+l}$

$x=\dfrac{9-4}{3-2}$ $\quad y=\dfrac{12+6}{3-2}$

$P=(5,18)$ $\boxed{\checkmark}\boxed{\checkmark}$

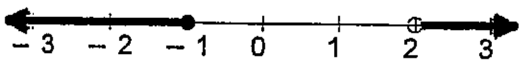
(b) $\dfrac{3x}{x-2}\geq 1 \qquad x\neq 2$

Let $\dfrac{3x}{x-2}=1$ $\boxed{\checkmark}$ $\qquad Test\; x=-3\checkmark$

$3x=x-2 \qquad Test\; x=0\checkmark$

$x=-1 \qquad Test\; x=3\checkmark$

$x\leq -1, x>2$ $\boxed{\checkmark}\boxed{\checkmark}$



(c) $\displaystyle\lim_{x\to 0}\frac{\tan 4x}{3x}=\lim_{x\to 0}\frac{\tan 4x}{4x}\times\frac{4}{3}$ $\boxed{\checkmark}$

$=1\times\dfrac{4}{3}$

$=\dfrac{4}{3}$ $\boxed{\checkmark}$

(d) $x=\sin\theta, y=2\operatorname{cosec}^2\theta$

$y=\dfrac{2}{\sin^2\theta}$

$y=\dfrac{2}{x^2}$ $\boxed{\checkmark}$

(e) $x=u^2-1\to dx=2du$ $\boxed{\checkmark}$

$\displaystyle\int\frac{3x}{\sqrt{x+1}}dx=\int\frac{3(u^2-1)}{\sqrt{u^2-1+1}}2udu$ $\boxed{\checkmark}$

$\displaystyle=6\int\frac{(u^2-1)}{\cancel{u}}\cancel{u}du$

$=2u^3-6u+C$

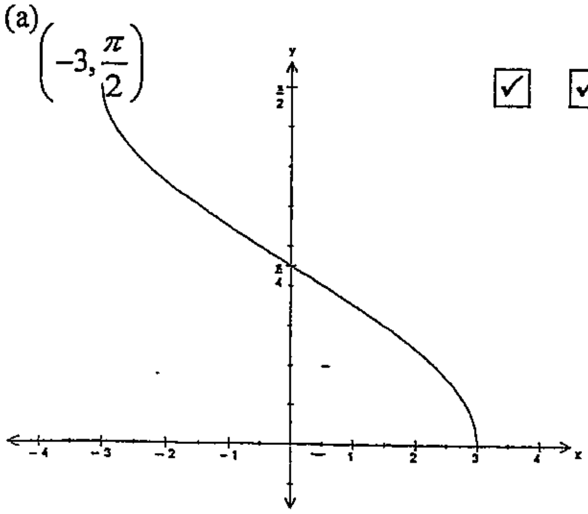
$=2(x+1)^{3/2}-6(x+1)^{1/2}+C$ $\boxed{\checkmark}$

$=2(x+1)^{1/2}\left[(x+1)-3\right]+C$

$=2\sqrt{x+1}(x-2)+C$ $\boxed{\checkmark}$

Question Two

(a)



$\boxed{\checkmark}$ $\boxed{\checkmark}$

(b) $\dfrac{d}{dx}\left(\tan^{-1}\sqrt{x}\right)=\dfrac{1}{1+\left(\sqrt{x}\right)^2}\times\dfrac{1}{2\sqrt{x}}$ $\boxed{\checkmark}$

$=\dfrac{1}{2\sqrt{x}(1+x)}$

$\boxed{\checkmark}$

(c) $\displaystyle\int_0^2\frac{1}{\sqrt{8-x^2}}dx=\left[\sin^{-1}\frac{x}{2\sqrt{2}}\right]_0^2$ $\boxed{\checkmark}$

$=\sin^{-1}\dfrac{\cancel{2}}{\cancel{2}\sqrt{2}}-\sin^{-1}0$

$=\dfrac{\pi}{4}$ $\boxed{\checkmark}$

(d) $\left(3-x^3\right)^4$

coefficient of $x^9={}^4C_3 3^1(-1)^3$ $\boxed{\checkmark}$.

$=-12$ $\boxed{\checkmark}$

(e) (i)

$\cos x+\sqrt{3}\sin x=A\cos x\cos\alpha+A\sin x\sin\alpha$

$A\cos\alpha=1, A\sin\alpha=\sqrt{3}\to\tan\alpha=\dfrac{1}{\sqrt{3}},\alpha=\dfrac{\pi}{3}$ $\boxed{\checkmark}$

$A\sin\dfrac{\pi}{3}=\sqrt{3}\to A=2\therefore\cos x+\sqrt{3}\sin x=2\cos\left(x-\dfrac{\pi}{3}\right)$

(ii) $2\cos\left(x-\dfrac{\pi}{3}\right)=1$

$x=0,\dfrac{2\pi}{3},2\pi$ $\boxed{\checkmark}\boxed{\checkmark}$

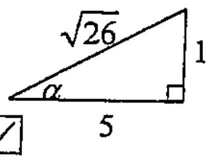
Question Three

(a) Number of arrangements $= \dfrac{9!}{2!\times 3!}$ ☑

$=30240$ ☑

(b) $\cos\left(\tan^{-1}\dfrac{1}{5}\right)$

let $\alpha = \tan^{-1}\dfrac{1}{5}$



$\tan\alpha = \dfrac{1}{5}$ ☑

$\cos\alpha = \dfrac{5}{\sqrt{26}}$ ☑

(c) $y = \log x$

$\dfrac{dy}{dx} = \dfrac{1}{x}$

$m_1 = \dfrac{1}{1} = 1, m_2 = \dfrac{1}{3}$ ☑

$\tan\theta = \dfrac{1-\frac{1}{3}}{1+\frac{1}{3}}$

$\tan\theta = \dfrac{1}{2}$

$\theta = 26°34'$ ☑

(d) (i) Because the probability of a girl is ½. The probability of 3 girls is given by ½ × ½ × ½ = ⅛ ☑

(ii) $\left(\dfrac{7}{8}+\dfrac{1}{8}\right)^7$

P(4 fam3girls) $= {}^7C_4\left(\dfrac{7}{8}\right)^3\left(\dfrac{1}{8}\right)^4$ ☑

$= 0.006$ (3 dec. pl) ☑

(e) When $n = 1$ $7^n + 3^n = 10$ which is divisible by 10 ☑

(ii)

Assume true for $n = k$

$\dfrac{7^k+3^k}{10} = M$ (M is a pos. int.)

$7^k = 10M - 3^k$

Prove true for $n = k+2$

$7^{k+2}+3^{k+2} = 49\times 7^k + 9\times 3^k$

$= 49\times\left(10-3^k\right)+9\times 3^k$

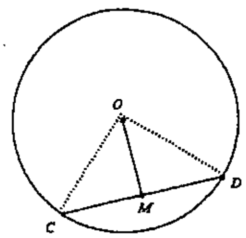
$= 10(49-4\times 3^k)$

(divisible by 10) ☑☑

Since it is true for $n = 1$ it will be true for $n = 3, 5, 7 \ldots$ and all n.

Question Four

(a) Prove that $\angle OMC = 90°$



In $\Delta OMC, \Delta OMC$;

OM is common.

$CM = MD$ (given)

$OC = OD$ (radii)

In $\Delta OMC \equiv \Delta OMD$ $(S.S.S)$

$\therefore \angle OMC = \angle OMD$ (corr. angles, cong. Δ's)

but $\angle OMC + \angle OMD = 180°$ (*straight angle*)

$\therefore \angle OMC = \angle OMD = 90°$

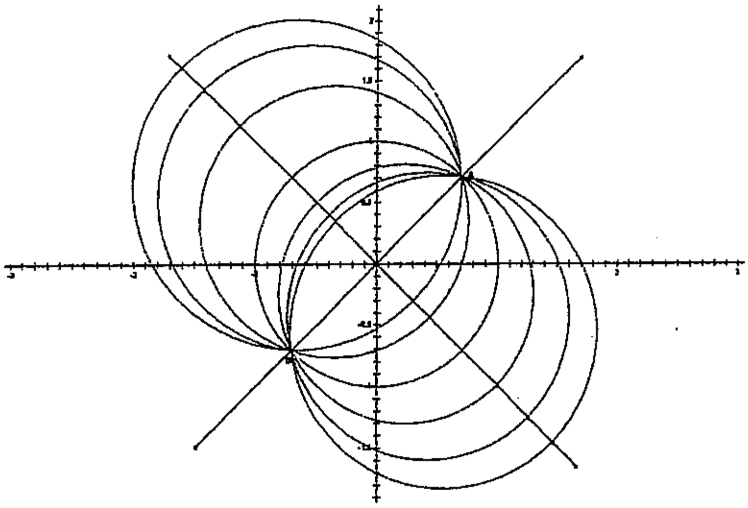
(b) $x^2+y^2 = 1, y = x$

$x^2 + x^2 = 1$

$2x^2 = 1$

$x = \pm\dfrac{1}{\sqrt{2}}, y = \pm\dfrac{1}{\sqrt{2}}$

$A = \left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right), B\left(-\dfrac{1}{\sqrt{2}}, -\dfrac{1}{\sqrt{2}}\right)$



(iii) The centre of all those circles lie on the perpendicular bisector of AB. This line passes through (0,0) and is perpendicular to $y = x$. It has $m = -1$ so is $y = -x$. ☑☑

(iv) Since the centre H lies on $y = -x$, it can be $(u,-u)$. Now

$$HO^2 + OB^2 = HB^2$$ ☑

$$u^2 + u^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = r^2$$

$$2u^2 + 1 = r^2$$ ☑

$$\therefore (x-u)^2 + (y+u)^2 = 2u^2 + 1$$

(v) $$(4-u)^2 + (-1+u)^2 = 2u^2 + 1$$

$$u^2 - 8u + 16 + u^2 - 2u + 1 = 2u^2 + 1$$

$$-10u = -16$$

$$u = \frac{8}{5}$$ ☑

$$\left(x-\frac{8}{5}\right)^2 + \left(y+\frac{8}{5}\right)^2 = 2\left(\frac{8}{5}\right)^2 + 1$$

$$\left(x-\frac{8}{5}\right)^2 + \left(y+\frac{8}{5}\right)^2 = 6\frac{3}{25}$$ ☑

Question Five

(a) (i) ${}^5C_2 \times {}^4C_3 = 10 \times 4 = 40$ ✓✓

(ii) No. of committees $= \dfrac{1 \times 1 \times {}^3C_2}{40} = \dfrac{3}{40}$ ✓✓

(b) (i) $l = r\theta$ and $\sin\dfrac{\theta}{2} = \dfrac{\frac{d}{2}}{r} \to d = 2r\sin\dfrac{\theta}{2}$ ✓

$l : d = 4 : 3$

$\cancel{r}\theta : 2\cancel{r}\sin\dfrac{\theta}{2} = 4 : 3$ ✓

$3\theta = 8\sin\dfrac{\theta}{2}$

$3\theta - 8\sin\dfrac{\theta}{2} = 0$ ✓

(ii) $f(\theta) = 3\theta - 8\sin\dfrac{\theta}{2}$ $\quad f(2.5) = -0.09188$

$f'(\theta) = 3 - 4\cos\dfrac{\theta}{2}$ $\quad f'(2.5) = 1.73871$ ✓

$$\theta_2 = \theta_1 - \frac{f(\theta_1)}{f'(\theta_1)}$$

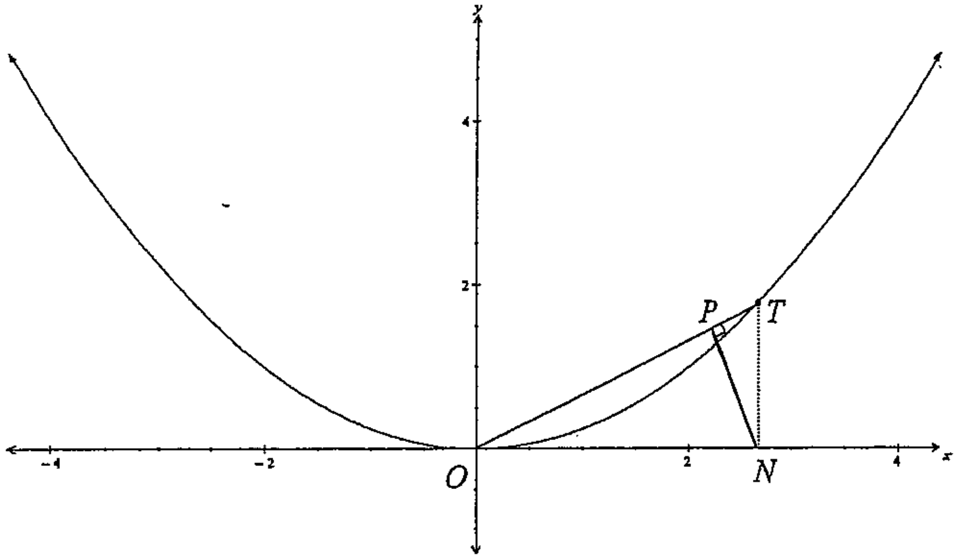
$$= 2.5 - \frac{-0.09188}{1.73871}$$ ✓

$$= 2.552 \text{ (3 d.p.)}$$ ✓

(iii) Yes since $f(2.552) = 0.002$ which is closer to 0 than $f(2.55)$ ✓✓

Question Six

(a) $T(2t, t^2)$ is a point on the parabola $x^2 = 4y$ whose vertex is O. N is the foot of the perpendicular from T to the x-axis. The perpendicular from N to OT meets OT at P.



(ii) $N = (2t, 0)$ ☑

(iii) $m(OT) = \frac{t^2}{2t} = \frac{t}{2}$ ☑

$y = \frac{t}{2}x$ ☑

(iv) $m(PN) = -\frac{2}{t}$ ☑

$y - 0 = -\frac{2}{t}(x - 2t)$

$ty = -2x + 4t$

$2x + ty - 4t = 0$ ☑

(v) To find the coordinates of P solve $2x + ty - 4t = 0$ and $y = \frac{t}{2}x$ simultaneously:

$2x + t\left(\frac{t}{2}x\right) - 4t = 0$

$x(4 + t^2) = 8t$

$x = \frac{8t}{4 + t^2}$ ☑

$y = \frac{t}{2} \times \frac{8t}{4 + t^2} = \frac{4t^2}{4 + t^2}$

$P = \left(\frac{8t}{4 + t^2}, \frac{4t^2}{4 + t^2}\right)$ ☑

(vi) $x=\dfrac{8t}{4+t^2}, y=\dfrac{4t^2}{4+t^2}$

$$LHS=\left(\frac{8t}{4+t^2}\right)^2+\left(\frac{4t^2}{4+t^2}\right)^2-4\left(\frac{4t^2}{4+t^2}\right)$$

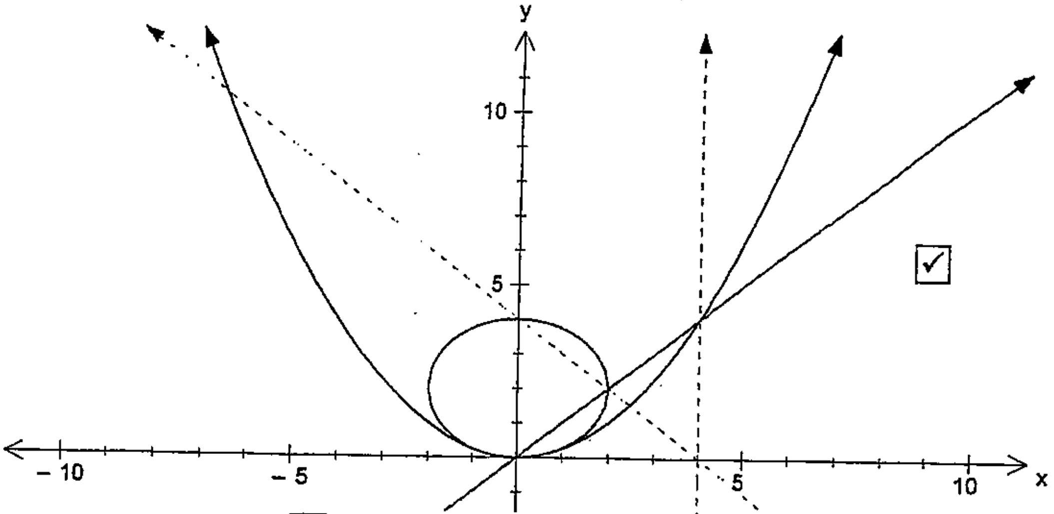
$$=\frac{64t^2}{\left(4+t^2\right)^2}+\frac{16t^4}{\left(4+t^2\right)^2}-16t^2\left(\frac{4+t^2}{\left(4+t^2\right)^2}\right)$$

$$=\frac{64t^2+16t^4-64t^2-16t^4}{\left(4+t^2\right)^2}=0 \ \boxed{\checkmark}$$

$$x^2+y^2-4y=0\to x^2+\left(y-2\right)^2=4 \ \boxed{\checkmark}$$

This is a circle centre (0,2), $r=2$.

(vii) Sketch the locus of P on your diagram.



$\boxed{\checkmark}$

(viii) (0,4) $\boxed{\checkmark}$

(b) The roots of $x^3+ax^2+bx+c=0$ are in arithmetic sequence. By denoting the roots as $\alpha-d, \alpha$ and $\alpha+d$

(i) $\alpha-d+\alpha+\alpha+d=-a$

$$3\alpha=-a$$

$$\alpha=-\frac{a}{3} \quad \boxed{\checkmark}$$

(ii) Since $\alpha=-\dfrac{a}{3}$ is a root of $2a^3-9ab+27c=0$ then :

$$\left(-\frac{a}{3}\right)^3+a\left(-\frac{a}{3}\right)^2+b\left(-\frac{a}{3}\right)+c=0$$

$$-\frac{a^3}{27}+\frac{a^3}{9}-\frac{ab}{3}+c=0 \quad \boxed{\checkmark}$$

$$-a^3+3a^3-9ab+27c=0$$

$$2a^3-9ab+27c=0 \quad \boxed{\checkmark}$$

Question Seven

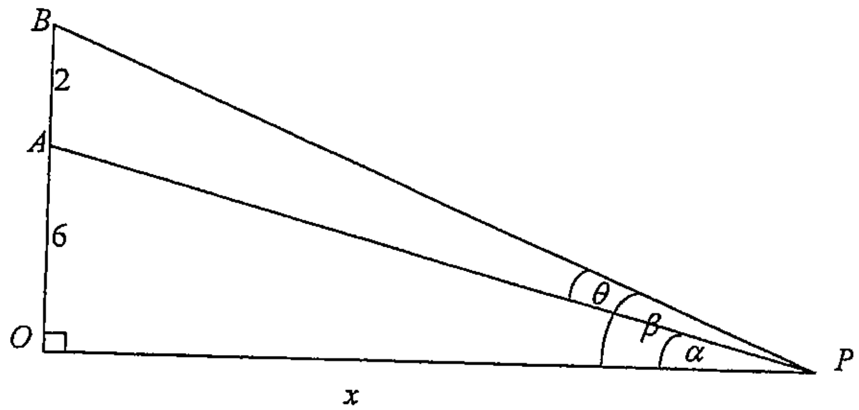
(a) $\log_e(2\log_e x) = 1$

$2\log_e x = e$ ☑

$\log_e x = \frac{e}{2}$

$x = e^{\frac{e}{2}}$ ☑

(b)



In the diagram above ΔBOP has a right angle at O, OA is 6 units, AB is 2 units and $\angle BPA$ is θ.

(i) $\tan\alpha = \frac{6}{x} \to \alpha = \tan^{-1}\left(\frac{6}{x}\right)$

$\tan\beta = \frac{8}{x} \to \beta = \tan^{-1}\left(\frac{8}{x}\right)$ ☑

$\theta = \beta - \alpha$ ☑

$\theta = \tan^{-1}\frac{8}{x} - \tan^{-1}\frac{6}{x}$

(ii) $\theta = \tan^{-1}\frac{8}{x} - \tan^{-1}\frac{6}{x}$

$$\frac{d\theta}{dx} = \frac{1}{1+\left(\frac{8}{x}\right)^2} \times -\frac{8}{x^2} - \frac{1}{1+\left(\frac{6}{x}\right)^2} \times -\frac{6}{x^2}$$

$$= \frac{-8}{64+x^2} + \frac{6}{36+x^2}$$

$$\frac{d\theta}{dx} = 0$$

$$\frac{-8}{64+x^2}+\frac{6}{36+x^2}=0$$

$$\frac{8}{64+x^2}=\frac{6}{36+x^2}$$

$$288+8x^2=384+6x^2$$

$$2x^2=96$$

$$x^2=48$$

$$x=4\sqrt{3}$$

(iii) $\theta=\tan^{-1}\frac{8}{4\sqrt{3}}-\tan^{-1}\frac{6}{4\sqrt{3}}$

$$\tan\theta=\tan\left(\tan^{-1}\frac{8}{4\sqrt{3}}-\tan^{-1}\frac{6}{4\sqrt{3}}\right) \quad \boxed{\checkmark}$$

$$=\frac{\frac{8}{4\sqrt{3}}-\frac{6}{4\sqrt{3}}}{1+\frac{8}{4\sqrt{3}}\times\frac{6}{4\sqrt{3}}}$$

$$=\frac{\frac{1}{2\sqrt{3}}}{2}$$

$$=\frac{1}{\sqrt{3}}$$

$$\theta=\frac{\pi}{6} \quad \boxed{\checkmark}$$

(iv) As $x\to\infty,\theta\to 0$ so this cannot be the minimum value of θ.