2006
HIGHER SCHOOL CERTIFICATE
TRIAL EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown on every question

Total marks – 84

- Attempt Questions 1-7
- All questions are of equal value
- Start each question in a new writing booklet

Examiners:BW,SE

Question 1 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) Solve $\dfrac{5}{x-1} \leq 2$ **3**

(b) A is the point (–2, -1), and B is the point (1, 5). Find the coordinates of the point P which divides AB externally in the ratio 5 : 2. **2**

(c) Let $P(x) = (x+2)\,(x-5)Q(x) + a(x+2) + b$, where $Q(x)$ is a polynomial and a and b are real numbers.

When $P(x)$ is divided by $(x+2)$ the remainder is 5.
When $P(x)$ is divided by $(x-5)$ the remainder is 19.

(i) What is the value of b? **2**

(ii) What is the remainder when $P(x)$ is divided by $(x+2)(x-5)$? **2**

(d) Evaluate $\displaystyle\int_{1}^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\,dx$ in terms of π **3**

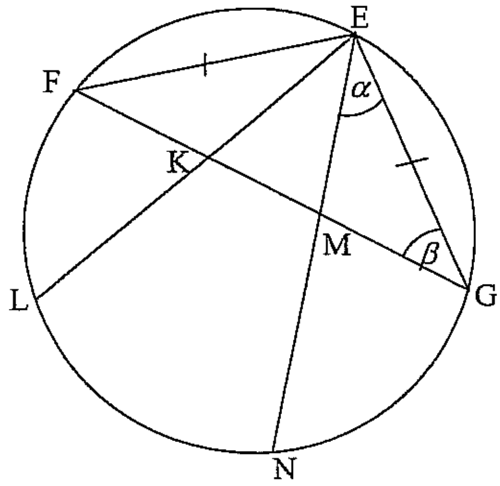
Examiners:BW,SE

Question 2 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) (i) Show that $f(x)=x^4-5x-8$ has a root between $x=2$ and $x=3$ **1**

(ii) Using an approximation of $x=2.1$, use one application of Newton's method to find a better approximation for this root. Give your answer to two decimal places. **1**

(b)



Let $EGNLF$ be a circle such that $EF=EG$, EL meets FG at K and EN meets FG at M, as in the diagram. Let $\angle GEN=\alpha$ and $\angle EGF=\beta$.

(i) Copy this diagram into your Writing Booklet

(ii) Prove that $\angle GLN=\alpha$ **1**

(iii) Prove that $\angle GLE=\beta$ **1**

(iv) Prove that LKMN is a cyclic quadrilateral. **2**

(c) Use the binomial theorem to find the term independent of x in the expansion of **3**

$$\left(x^2+\frac{4}{x}\right)^8$$

(d) Use the substitution $u=1+x$ to find $\int \frac{x}{\sqrt{1+x}}\,dx$ **3**

Examiners:BW,SE

Question 3 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) (i) By expanding the left hand side, show that

$\sin(7x-4x)+\sin(7x+4x)=2\sin 7x\cos 4x$ **2**

(ii) Hence find $\int \sin 7x\cos 4x dx$ **2**

(b) The equation $x^3-kx+2=0$ has roots $\alpha, \alpha,$ and β.

(i) Show that $2\alpha+\beta=0$ and $\alpha^2\beta=-2$ **2**

(ii) Hence find the values of α, β and k. **2**

(c) A geometric series is given by $1-\tan^2 x+\tan^4 x-\tan^6 x+....$for $0<x<\frac{\pi}{4}$

(i) Show that the limiting sum exists and is given by $Z=\cos^2 x$ **2**

(ii) Find the set of possible values of Z **2**

Question 4 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) Differentiate $e^{5x}(\log(x^2-3x+1))$ **3**

(b) Use the substitution $t=\tan\frac{\theta}{2}$ to solve the equation **3**

$$\sin\theta+\cos\theta=1 \text{ for } 0\le\theta\le 2\pi$$

(c) Two points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2=4ay$

(i) Derive the equation of the tangent to the parabola at P **2**

(ii) Find the coordinates of the point of intersection T of the tangents to the parabola at P and Q. **2**

(iii) You are given that the tangents at P and Q in (ii) intersect at an angle of $\frac{\pi}{4}$ **2**
Show that $p-q=1+pq$

Examiners:BW,SE

Question 5 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) Prove by mathematical induction that 3

$1\times 2^0 + 2\times 2^1 + 3\times 2^2 + ... + n\times 2^{n-1} = 1 + (n-1)2^n$ for all integers $n \geq 1$

(b) For the function $y = x + e^{-x}$

(i) Find the coordinates and nature of any stationary points on the graph of $y = f(x)$ and show that the graph is concave upwards for all values of x. 2

(ii) Sketch the graph of $y = f(x)$ showing clearly the coordinates of any turning points and the equations of any asymptotes. 2

(c) A group consisting of 3 men and 6 women attends a prizegiving ceremony.

(i) If the members of the group sit down at random in a straight line, find the probability that the 3 men sit next to each other. 1

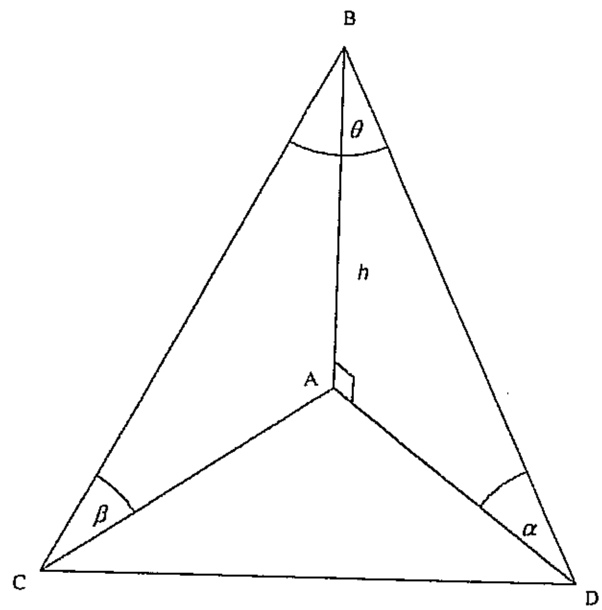
(ii) If 5 prizes are awarded at random to members of the group, find the probability that exactly 3 of the prizes are awarded to women if

(α) there is a restriction of at most one prize per person 2

(β) there is no restriction on the number of prizes per person 2

Question 6 (12 marks) Use a SEPARATE writing booklet **Marks**

(a)



The above diagram represents a balloonist B, being sighted simultaneously by two different observers, C and D on level ground. C is due south of the balloon and D is due east of it. Let A be the foot of the perpendicular from the balloon to the ground. Then,

Let $AD = x$, $AC = y$, $AB = h$, $\angle CBD = \theta$, $\angle BDA = \alpha$, and $\angle ACB = \beta$,
As indicated on the diagram.

(i) Show that $x = h\cot\alpha$ and obtain a similar expression for y. **1**

(ii) Show that $\cos\theta = \dfrac{h^2}{\sqrt{\left(x^2+h^2\right)\left(y^2+h^2\right)}}$ **2**

(iii) Hence show that $\sin\alpha\sin\beta = \cos\theta$ **2**

Question 6 continued on Next Page

Question 6 (12 marks) Use a SEPARATE writing booklet **Marks**

(b) Consider the function $f(x)=\dfrac{x-4}{x-2}$ *for* $x>2$

(i) Show that $f(x)$ is an increasing function for all values of x in its domain **2**

(ii) Explain briefly why the inverse function $f^{-1}(x)$ exists **1**

(iii) State the domain and range of $f^{-1}(x)$ **2**

(iv) Find the gradient of the tangent to $y=f^{-1}(x)$ at the point $(0,4)$ on it. **2**

Examiners:BW,SE

Question 7 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) Find the value of x if $\log_{10}(x^2+x)-\log_{10}(x+1)=2$ **2**

(b) Christine sets up a prize fund with a single investment of \$2000 to provide her School with an annual prize valued at \$144. The fund accrues interest at a rate of 6% per annum, compounded annually. The first prize is awarded one year after the investment is set up

(i) Calculate the balance in the fund at the beginning of the second year. **1**

(ii) Let $\$B_n$ be the balance in the fund at the end of n years (and after the nth prize has been awarded). **2**

Show that $B_n = 2400 - 400 \times (1.06)^n$

(iii) At the end of the tenth year (and after the tenth prize has been awarded) it is decided to increase the prize value to \$200. **3**

For how many more years can the prize fund be used to award the prize?

(c) (i) Write out the binomial expansion of $x(1+x)^n$ **1**

(ii) Hence by differentiating $x(1+x)^n$, show **3**

$$\sum_{r=0}^{n}(r+1)\,{}^nC_r=(n+2)2^{n-1}$$

Examiners:BW,SE

SOLUTIONS:

Question 1 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) $\frac{5}{x-1} \le 2$ **3**

$5 \le 2x-2$

$2x-2 \ge 5$

$2x \ge 7$

$x \ge \frac{7}{2}$ [1]

$x-1 \ne 0 \quad \therefore x \ne 1$ [1]

$\therefore \ x<1 \quad or \quad x \ge \frac{7}{2}$ [1]

(b) $x = \frac{(-2\times 2+1\times -5)}{-5+2}$, $y = \frac{(-1\times 2+5\times -5)}{-5+2}$ [1] **2**

$\therefore (3, 9)$ [1]

(c) (i) $P(-2) = (-2+2)(-2-5)Q(x) + a(-2+2) + b = 5$ [1] **2**

$\therefore b = 5$ [1]

(ii) $P(5) = (5+2)(5-5)Q(x) + a(5+2) + b = 19$ **2**

$\therefore 7a + 5 = 19$

$a = 2$ [ ]

$\therefore$ *divide by* $(x+2)(x-5)$

$P(x) = (x+2)(x-5)Q(x) + 2(x+2) + 5$

$\therefore$ *remainder is* $2x + 9$ [1]

(d) $\int_1^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}} dx = \left[\sin^{-1}\left(\frac{x}{2}\right)\right]_1^{\sqrt{3}}$ [1] **3**

$= \left[\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1}\left(\frac{1}{2}\right)\right] = \frac{\pi}{6}$ [2]

Examiners:BW,SE

Question 2 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) (i) $f(2)=(2)^4-5(2)-8=-2$ **1**

$f(3)=(3)^4-5(3)-8=58$

$\therefore f(2)<0 \ \ and \ \ f(3)>0$

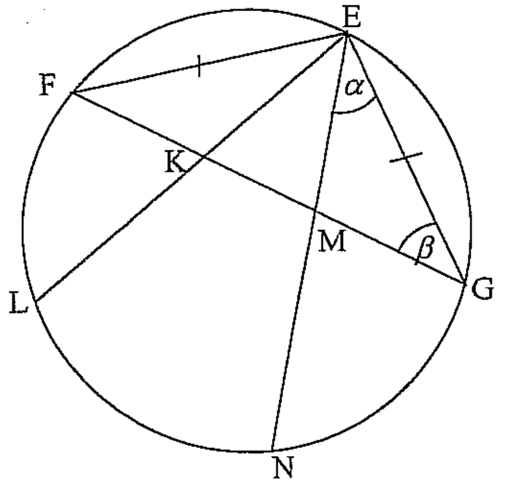
$\therefore roots \ between \ 2 \ and \ 3$ [1]

(ii) $f(2.1)=(2.1)^4-5(2.1)-8=0.9481$

$f'(2.1)=4(2.1)^3-5=32.044$ [1]

$\therefore x=a-\dfrac{f(a)}{f'(a)} = 2.1-\dfrac{0.9481}{32.044}=2.07$ [1]

(b)



(ii) Join GL such that $\angle GLN=\angle GEN$ (angles at the circumference standing on the same arc GN are equal. [1] **1**

Therefore $\angle GEN=\alpha$

(iii) Since EF=EG. Therefore EFG is an isosceles triangle. **1**

Therefore $\angle EGF=\angle EFG=\beta$

Since $\angle GLE=\angle EFG$. (angles at the circumference standing on the same arc EG are equal. [1]

$\therefore \angle GLE=\beta$

Examiners:BW,SE

(iv) $\angle NMK + \angle EMK = 180$ (*straight line*) **2**

$\therefore \angle NMK + (\alpha + \beta) = 180$ [1]

$\angle NMK + NLK = 180$ since $\angle NLK = \angle NLG + \angle GLK = \alpha + \beta$ [1]

$\therefore LKMN$ is a cyclic quadrilateral (opp angles of cyclic quad are supp)

(c) $\left(x^2 + \frac{4}{x}\right)^8 = {}^8C_r\,(x^2)^{8-r}\left(\frac{4}{x}\right)^8 = {}^8C_r\,(x^{16-2r})\left(4^8 x^{-8}\right) = {}^8C_r\left(4^8\right)(x^{8-2r})$ ☐ **3**

$\therefore\ (x^{8-2r}) = x^0$

$r = 4$ ☐

${}^8C_4\left(4^8\right)(x^{8-2(4)}) = 4587520$ ☐

(d) $\int \frac{u-1}{u^{\frac{1}{2}}}\,dx = \int \frac{u}{u^{\frac{1}{2}}} - \frac{1}{u^{\frac{1}{2}}}\,dx = \int u^{\frac{1}{2}} - u^{\frac{-1}{2}}\,dx$ [1] **3**

$= \frac{2u^{\frac{3}{2}}}{3} - 2u^{\frac{1}{2}} + C$ [1]

$= \frac{2(1+x)^{\frac{3}{2}}}{3} - 2(1+x)^{\frac{1}{2}} + C$ [1]

Question 3 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) (i) $\sin 7x \cos 4x - \sin 4x \cos 7x + \sin 7x \cos 4x + \sin 4x \cos 7x = 2\sin 7x \cos 4x$ [2] **2**

2

(ii) $\frac{1}{2}\int 2\sin 7x \cos 4x dx$ [1]

$\therefore \frac{1}{2}\left[\sin(7x-4x) + \sin(7x+4x)\right] + C$

$= \frac{1}{2}\int \sin 3x dx + \frac{1}{2}\int \sin 11x dx$

$= \frac{-1}{2} \times \frac{1}{3}\cos 3x + \frac{-1}{2} \times \frac{1}{11}\cos 11x$

$= \frac{-1}{6}\cos 3x - \frac{1}{22}\cos 11x + c$ [1]

(b) (i) $x^3 - kx + 2 = 0$ **2**

$\alpha + \beta + \gamma = \alpha + \beta + \alpha = \frac{-b}{a}$ $\therefore\ 2\alpha + \beta = 0$ [1]

$\alpha\beta\gamma = \alpha\alpha\beta = \frac{-d}{a}$ $\therefore\ \alpha^2\beta = -2$ [1]

(ii) $x^3 - kx + 2 = 0$ **2**

$\beta = -2\alpha \quad , \quad \alpha^2\beta = -2 \quad and \quad \alpha^2 + 2\alpha\beta = -k$

$\therefore\ \alpha^2(-2\alpha) = -2 \quad \Rightarrow \quad -2\alpha^3 = -2 \qquad \therefore \alpha = 1$ [1]

$\therefore\ \beta = -2\alpha \quad \Rightarrow \quad \beta = -2(1) \qquad \therefore \beta = -2$

$\therefore\ \alpha^2 + 2\alpha\beta = -k \quad \Rightarrow \quad (1)^2 + 2(1)(-2) = -k \qquad \therefore\ k = 3$ [1]

(c) (i) $1 - \tan^2 x + \tan^4 x - \tan^6 x + \ldots.$ **2**

$\therefore a = 1 \qquad r = -\tan^2 x$ [1]

$$\therefore S_\infty = \frac{a}{1-r} = \frac{1}{1--\tan^2 x} = \frac{1}{1+\tan^2 x}$$

$$= \frac{1}{\sec^2 x}$$

$Z = \cos^2 x$ [1]

(ii) $|r| < 1 \qquad \therefore -1 < r < 1 \quad \Rightarrow -1 < -\tan^2 x < 1$ [1] **2**

$\tan^2 x < 1 \ and\ \tan^2 x > -1$

$\therefore\ -\frac{\pi}{4} < x < \frac{\pi}{4} \quad and \quad$ no solution.

$\therefore\ -\frac{\pi}{4} < x < \frac{\pi}{4}$ [1]

Question 4 (12 marks) Use a SEPARATE writing booklet **Mark**

(a) $$\frac{d}{dx}\left[e^{5x}(\log(x^2 - 3x + 1))\right] = \left[vu' + +uv'\right] = \left[\log(x^2 - 3x + 1) \times 5e^{5x} + e^{5x} \times \frac{2x-3}{x^2 - 3x + 1}\right]$$ [1] **2**

$$= e^{5x}\left[5\log(x^2 - 3x + 1) + \frac{2x-3}{x^2 - 3x + 1}\right]$$ [1]

(b) $\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2} = 1 \quad \Rightarrow \quad 2t + 1 - t^2 = 1 + t^2 \qquad \therefore 2t^2 - 2t = 0$ **3**

$2t(t-1) = 0$

$\therefore t = 0 \quad and \quad t = 1$ [1]

$\therefore \tan\frac{\theta}{2} = 0 \quad and \quad \tan\frac{\theta}{2} = 1$

$\frac{\theta}{2} = 0, \pi, 2\pi, \frac{\pi}{4} \ \ and\ \ \frac{5\pi}{4}$ [1]

$\therefore \quad \theta = 0, 2\pi, 4\pi, \frac{\pi}{2} \ \ and\ \ \frac{5\pi}{2}$ [1]

(c)(i) $x^2 = 4ay \quad \Rightarrow \quad \frac{dy}{dx} = \frac{2x}{4a} = \frac{x}{2a} = \frac{2ap}{2a} = p$ **2**

$\therefore\ m_{\text{tangent at p}} = p$ [1]

$\therefore y - ap^2 = p(x - 2ap)$

$y = px - ap^2$ [1]

(c)(ii) $Tangent\ \ P: y = px - ap^2$ 2

$Tangent\ \ Q: y = qx - aq^2$

$\therefore$ Point of intersection = $px - ap^2 = qx - aq^2$ $\quad \therefore x = a(p+q)$ [1]

$y = ap(p+q) - ap^2$ $\quad \therefore y = apq$ [1]

$\therefore$ Point of intersection $(a(p+q)\ ,\ apq)$

(c)(iii) $\tan 45 = 1 = \left|\dfrac{p-q}{1+pq}\right|$ [1] 1

$\therefore\ 1 + pq = p - q$

Question 5 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) $1\times 2^0 + 2\times 2^1 + 3\times 2^2 + ... + n\times 2^{n-1} = 1 + (n-1)2^n$ 3

Step 1: Prove true for n=1

LHS= $1\times 2^{1-1} = 1\times 1 = 1$

RHS= $1 + (1-1)2^1 = 1 + 0 = 1$

$\therefore LHS = RHS \quad \rightarrow \quad true \ \ for \ \ n = 1$ [1]

Step 2: Assume true for n=k

$\therefore 1\times 2^0 + 2\times 2^1 + 3\times 2^2 + ... + k\times 2^{k-1} = 1 + (k-1)2^k$

Step 3: Prove true for n = k + 1

LHS = $1\times 2^0 + 2\times 2^1 + 3\times 2^2 + ... + k\times 2^{k-1} + (k+1)\times 2^{k+1-1}$

$= 1 + (k-1)2^k + (k+1)\times 2^k$

$= 1 + 2^k(k - 1 + k + 1)$

$= 1 + 2^k(2k)$

$= 1 + 2^{k+1}(k)$ [1]

RHS= $1 + (k+1-1)2^{k+1} = 1 + 2^{k+1}(k)$

$\therefore LHS = RHS \quad \rightarrow \quad true \ \ for \ \ n = k+1$ [1]

Step 4: Since true for n=k and n=k+1, then true for n=1, 2, 3, ... $\therefore$ true for $n \geq 1$

(b) (i) $\dfrac{dy}{dx} = 1 - e^{-x} = 0 \qquad \therefore -e^{-x} = -1 \qquad \Rightarrow \quad e^{-x} = 1$ 2

$\therefore\ \log 1 = -x$

$\therefore x = 0$

Stationary point (0, -1) [1]

$\dfrac{dy}{dx} = 1 - e^{-x} > 0 \qquad when\ x \to \infty\ then\ \dfrac{dy}{dx} > 0$

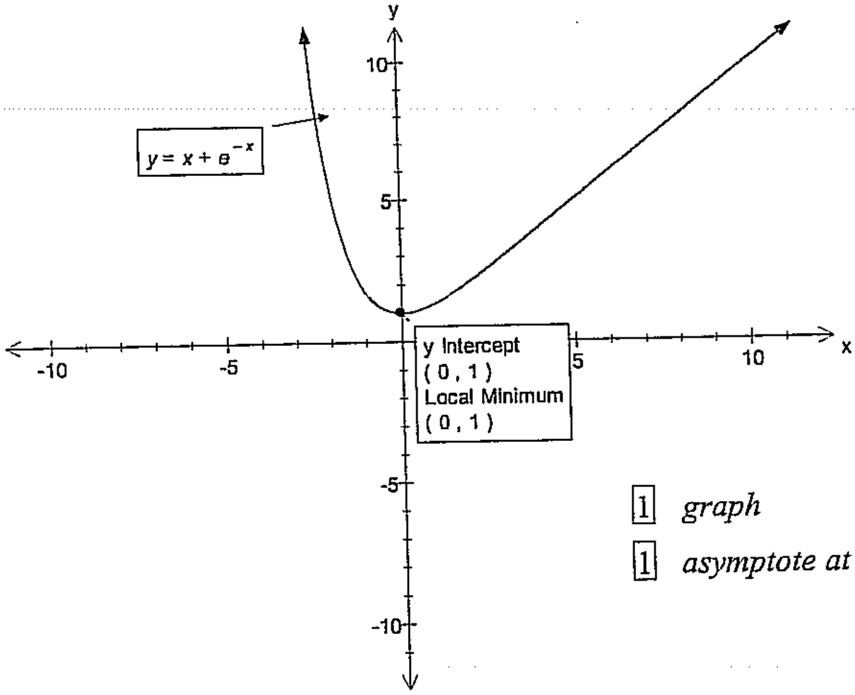
$when\ x \to -\infty\ then\ \dfrac{dy}{dx} > 0$

$\therefore y = x - e^{-x}$

is concave up for all x values [1]

Examiners:BW,SE

(ii) **2**



[1] *graph*

[1] *asymptote at* $y = x$

(c) A group consisting of 3 men and 6 women attends a prizegiving ceremony.

(i) **1**

F F F F F F M M M

F F F F F M M M F

F F F F M M M F F

F F F M M M F F F

F F M M M F F F F

F M M M F F F F F

M M M F F F F F F

$7 \times 3! \times 6!$

$$\therefore \text{P(3 men together)} = \frac{7 \times 3! \times 6!}{9!} = \frac{7 \times 3 \times 2 \times 6!}{9 \times 8 \times 7 \times 6!}$$

$$= \frac{6}{72} = \frac{1}{12}$$

(ii) (α) ${}^6C_3 \times {}^3C_2$ [1] **2**

$$\frac{{}^6C_3 \times {}^3C_2}{{}^9C_5} = \frac{60}{126} = \frac{10}{21}$$ [1]

$$\frac{10}{21}$$

(ii) (β) **2**

$${}^5C_3 \times \left(\frac{6}{9}\right)^3 \left(\frac{3}{9}\right)^2$$ [1]

$$10 \times \frac{8}{27} \times \frac{1}{9} = \frac{80}{243}$$ [1]

Question 6 (12 marks) Use a SEPARATE writing booklet **Marks**

(b) Consider the function $f(x)=\dfrac{x-4}{x-2}$ *for* $x>2$

(i) $f'(x)=\dfrac{vu'-uv'}{v^2}=\dfrac{(x-2)(1)-(x-4)(1)}{(x-2)^2}=\dfrac{2}{(x-2)^2}$ [1] **2**

$f'(x)=\dfrac{2}{(x-2)^2}$ *as* $x>2$ $f'(x)>0$ $\therefore$ increasing for $x>2$ [1]

(ii) $x>2$ $\dfrac{2}{(x-2)^2}>0$ $\therefore$ increasing **1**

$x<2$ $\dfrac{2}{(x-2)^2}>0$ $\therefore$ increasing

$\therefore$ When use the horizontal line test it will only cut it once.

Therefore it has an inverse [1]

(iii) $y=\dfrac{x-4}{x-2}$ **2**

$\therefore x=\dfrac{y-4}{y-2}$

$xy-2x=y-4$

$xy-y=2x-4$

$y(x-1)=2x-4$

$y=\dfrac{2x-4}{x-1}$

$\therefore f^{-1}(x)=\dfrac{2x-4}{x-1}$ *Domain*: $x\in\square$ *except* $x=1$

Range: $y\in\square$ *except* $y=2$

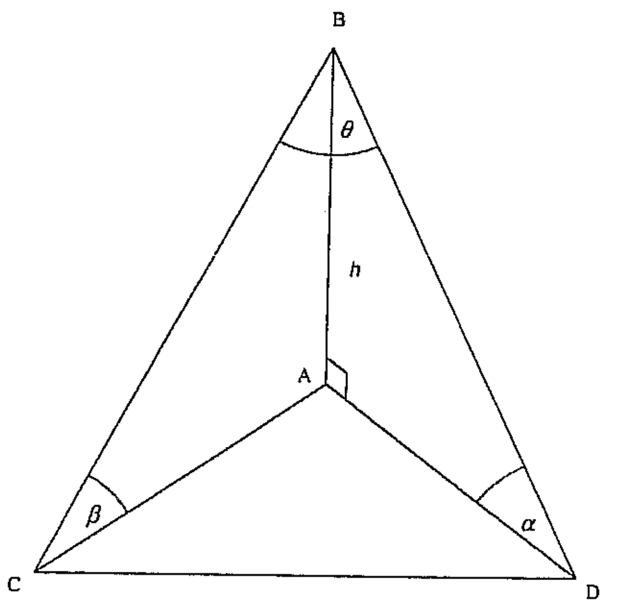
(iv) $\therefore f^{-1}(x)=\dfrac{2x-4}{x-1}$ **2**

$f'(x)=\dfrac{vu'-uv'}{v^2}=\dfrac{(x-1)(2)-(2x-4)(1)}{(x-1)^2}=\dfrac{-x+2}{(x-1)^2}$ [1]

$\therefore$ *gradient at* $(0,4)=\dfrac{-0+2}{(0-1)^2}=\dfrac{2}{1}=2$ [1]

Examiners:BW,SE

(a)



The above diagram represents a balloonist B, being sighted simultaneously by two different observers, C and D on level ground. C is due south of the balloon and D is due east of it. Let A be the foot of the perpendicular from the balloon to the ground. Then,

Let $AD = x, AC = y$, $AB = h$, $\angle CBD = \theta, \angle BDA = \alpha,$ and $\angle ACB = \beta,$
As indicated on the diagram.

(i) Show that $x = h\cot\alpha$ and obtain a similar expression for y. **1**

$\tan\alpha = \dfrac{h}{x}$

and $\cot\alpha = \dfrac{x}{h}$ $\boxed{1}$

$\therefore x = h\cot\alpha$

in $\square ABC$

$\cot\beta = \dfrac{y}{h}$

$\therefore y = h\cot\beta$

2

(ii) Show that $\cos\theta = \dfrac{h^2}{\sqrt{(x^2+h^2)(y^2+h^2)}}$

$$\cos\vartheta = \frac{BC^2+BD^2-CD^2}{2(BC)(BD)}$$

$$\cos\vartheta = \frac{h^2+y^2+x^2+h^2-(x^2+y^2)}{2\sqrt{(x^2+y^2)}\sqrt{x^2+y^2}}$$

$$\cos\vartheta = \frac{2h^2}{2\sqrt{(x^2+y^2)}\sqrt{x^2+y^2}} \quad \boxed{1}$$

$$\cos\vartheta = \frac{h^2}{\sqrt{(x^2+y^2)}\sqrt{x^2+y^2}}$$

$\boxed{1}$

2

(iii) Hence show that $\sin\alpha\sin\beta = \cos\theta$

RTP

$LHS: \sin\alpha.\sin\beta$

$$= \frac{h}{\sqrt{h^2+x^2}}.\frac{h}{\sqrt{h^2+y^2}}$$

$$= \frac{h^2}{\sqrt{(h^2+x^2)(h^2+y^2)}} \quad \boxed{1}$$

$= \cos\vartheta$

$= RHS\boxed{1}$

Examiners:BW,SE

Question 7 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) $\log_{10}\dfrac{x(x+1)}{x+1}=2$ [1] **2**

$\log_{10} x = 2$

$\therefore\ x = 10^2 = 100$ [1]

(b) (i) $B = 2000(1+0.06)^1 - 144 = \$1976$ [1] **1**

(ii) $2\text{nd year} = 2000(1.06)^1 - 144$ **2**

$3\text{rd year} = 2000(1.06)^2 - 144(1.06) - 144$

$4\text{th year} = 2000(1.06)^3 - 144(1.06)^2 - 144(1.06)^1 - 144$

$$nth\ year = 2000(1.06)^n - 144(1.06)^{n-1} - \ldots - 144(1.06)^1 - 144$$

$$= 2000(1.06)^n - 144\left[(1.06)^{n-1} + (1.06)^{n-1} + \ldots + 1\right]$$

$$= 2000(1.06)^n - 144\left[\frac{1(1.06^n - 1)}{1.06 - 1}\right]$$

$$= 2000(1.06)^n - \left[2400(1.06^n - 1)\right]$$

$$= 2000(1.06)^n - \left[2400(1.06^n) - 2400\right]$$

$$= 2400 + 2000(1.06)^n - 2400(1.06^n)$$

$$= 2400 - 400(1.06)^n$$

Examiners:BW,SE

(iii) At the end of the tenth year (and after the tenth prize has been awarded) it is decided to increase the prize value to \$120. **3**

For how many more years can the prize fund be used to award the prize?

$B_{10} = 2400 - 400 \times 1.06^{10}$

$B_{10} = 1683.66$ [1]

$A_n = 1683.66 \times 1.06^n - 200(1 + 1.06 + ... + 1.06^{n-1})$

When $A_n = 0$

$1683.66 \times 1.06^n = 200(1 + 1.06 + ... + 1.06^{n-1})$ [1]

$1683.66 \times 1.06^n = 3333.33(1.06^n - 1)$

$3333.33 = 3333.33(1.06^n) - 1683.66 \times 1.06^n$

$3333.33 = (3333.33 - 1683.66) \times 1.06^n$

$1.06^n = 2.02$

$n = \frac{\ln 2.02}{\ln 1.06} = 12.07$ [1]

Hence after $10 + 12.07 = 22.07$ *years the prize ends.*

(c) (i) **1**

$$x(1+x)^n = x({}^nC_0(1)^x(x)^0 + {}^nC_1(1)^{x-1}(x)^1 + {}^nC_2(1)^{x-2}(x)^2 + ... + {}^nC_n(1)^0(x)^n)$$
$$= {}^nC_0(1)^x(x)^1 + {}^nC_1(1)^{x-1}(x)^2 + {}^nC_2(1)^{x-2}(x)^3 + ... + {}^nC_n(1)^0(x)^{n+1}$$ [1]

(ii) Differentiate LHS $x(1+x)^n$

$$\frac{d}{dx}x(1+x)^n = (1+x)^n.1 + nx(1+x)^{n-1} = (1+x)^{n-1}(1+x+nx)$$ 1

Differentiate RHS ${}^nC_0(1)^x(x)^1 + {}^nC_1(1)^{x-1}(x)^2 + {}^nC_2(1)^{x-2}(x)^3 + \ldots + {}^nC_n(1)^0(x)^{n+1}$

$$\frac{d}{dx}{}^nC_0(1)^x(x)^1 + {}^nC_1(1)^{x-1}(x)^2 + {}^nC_2(1)^{x-2}(x)^3 + \ldots + {}^nC_n(1)^0(x)^{n+1}$$

$$= {}^nC_0 + 2\,{}^nC_1(x) + 3\,{}^nC_2(x)^2 + 4\,{}^nC_3(x)^3 + \ldots + (n+1)\,{}^nC_n(x)^n$$ 1

When $x = 1$

LHS= $(1+1)^{n-1}(1+1+n1) = (2)^{n-1}(n+2)$

RHS = ${}^nC_0 + 2\,{}^nC_1 + 3\,{}^nC_2 + 4\,{}^nC_3 + \ldots + (n+1)\,{}^nC_n = \sum_{r=0}^{n}(r+1)\,{}^nC_r$ 1

$$\therefore \sum_{r=0}^{n}(r+1)\,{}^nC_r = (n+2)2^{n-1}$$

Examiners:BW,SE