THE KING'S SCHOOL

2003
Higher School Certificate
Trial Examination

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 84

- Attempt Questions 1-7
- All questions are of equal value

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Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (12 marks) Use a SEPARATE writing booklet.

(a) P(5,7) divides the interval AB externally in the ratio $m:n$

If $A=(-1,-5)$ and $B=(0,-3)$, find $m:n$. **2**

(b) Find the derivative of $\tan^{-1}(1+x^2)$ **2**

(c) In how many ways can 8 rowers be divided into two groups of 5 and 3 rowers? **2**

(d) Find the acute angle between the lines $y=2x+1$ and $y=7-3x$. **2**

(e) $x+1$ is a factor of the polynomial

$$P(x)=x^{2n+1}-x^{2n}+b,\ n \text{ a positive integer}$$

Find the value of b. **2**

(f) Evaluate $\displaystyle\sum_{n=1}^{9}\left(\frac{1}{n}-\frac{1}{n+1}\right)$ **2**

Marks

Question 2 (12 marks) Use a SEPARATE writing booklet.

(a) Show that the function $f(x) = 5x - \sin 4x - 12$

increases for all values of x **3**

(b) (i) Find $R > 0$ and α, $0 < \alpha < \dfrac{\pi}{2}$, so that

$R\sin(x - \alpha) = \sin x - \sqrt{3}\cos x$ **2**

(ii) Solve $\sin x - \sqrt{3}\cos x = \sqrt{2}$, $0 < x < 2\pi$, exactly **2**

(c) (i) Use the substitution $u = 4 - x^2$ to evaluate

$\displaystyle\int_0^{\sqrt{3}} \frac{x}{\sqrt{4 - x^2}}\, dx$ **3**

(ii) Evaluate $\displaystyle\int_0^{\sqrt{3}} \frac{4 - x}{\sqrt{4 - x^2}}\, dx$ **2**

Marks

Question 3 (12 marks) Use a SEPARATE writing booklet.

(a) The function $f(x) = e^{x^2} - x - 3$ has a zero near $x = 1.2$.

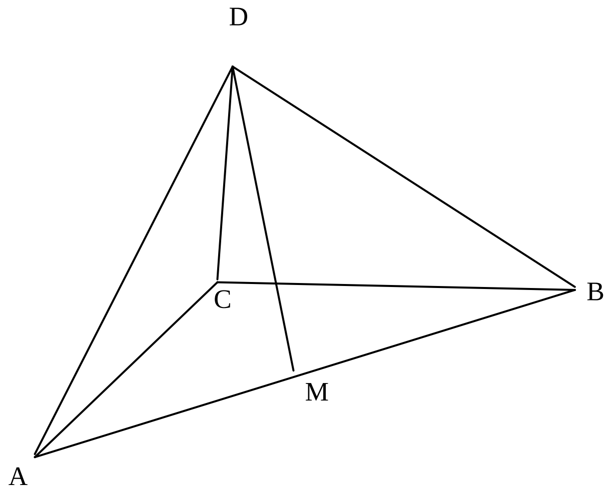
Use Newton's Method once to find a two decimal place approximation to this zero. **3**

(b) (i) Write $\sin 2A$ in terms of t, where $t = \tan A$ **1**

(ii) Prove the identity $\operatorname{cosec} 2A - 3\cot 2A = 2\tan A - \cot A$ **3**

(c) A, B, C are three points in a horizontal plane and M is the mid-point of AB. CD is at right-angles to the horizontal plane ABC.

The length of AB = 20 metres.
The angles of elevation from A, M, B to D are 30°, 45°, 30°, respectively.



(i) Copy the diagram into your booklet and include on it the information given. **1**

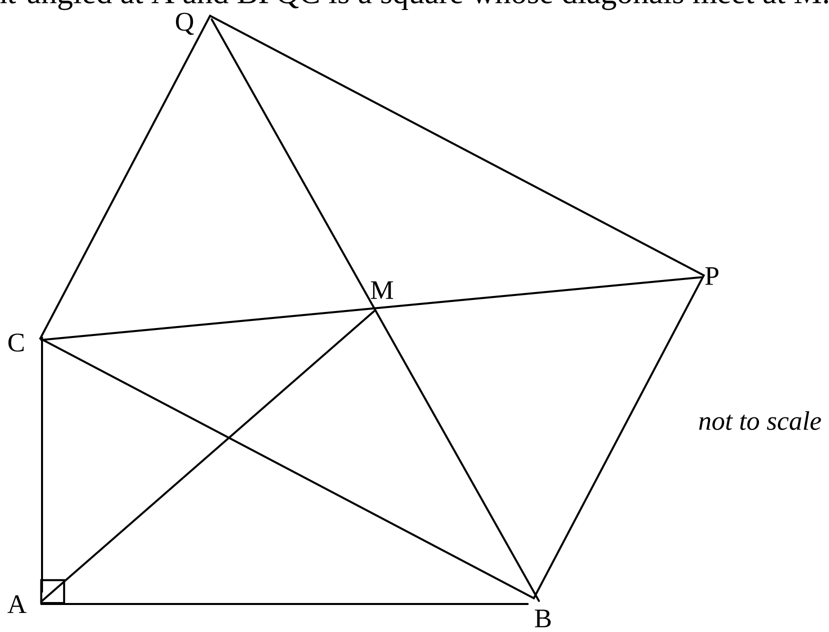
(ii) If $CD = x$, show that $AC = \sqrt{3}x$ **1**

(iii) Find the exact value of x. **3**

Marks

Question 4 (12 marks) Use a SEPARATE writing booklet.

(a) ΔABC is right-angled at A and BPQC is a square whose diagonals meet at M.



(i) Copy the diagram into your booklet.

(ii) Prove that ABMC is a cyclic quadrilateral. **2**

(iii) Prove that MA bisects $\angle$BAC **2**

(b) A particle is moving in simple harmonic motion on the *x*-axis. Its displacement, *x* metres, at any time *t* seconds, where $t \geq 0$, is given by $x = 10\cos nt$, n a constant.

(i) Show that the particle is initially at rest at $x = 10$. **2**

(ii) The period of the motion is T seconds and after $\frac{T}{3}$ seconds the particle is at position $x = b$.

Find the value of b. 2

(iii) The speed at $x = b$ is $20\sqrt{3}$ m/s. Find the period of the motion. **2**

(c) Simplify $\binom{n}{3} \div \binom{n-1}{2}$, $n \geq 3$ **2**

Marks

Question 5 (12 marks) Use a SEPARATE writing booklet.

(a) $P\left(2ap, ap^2\right)$, $Q\left(2aq, aq^2\right)$ are the end points of a focal chord of the parabola $x^2 = 4ay$.

(i) Show that the equation of the chord PQ is

$$y - \frac{1}{2}(p+q)x + apq = 0$$ **2**

(ii) Deduce that $q = -\frac{1}{p}$ **1**

(iii) Show that $PQ = 2a + a\left(p^2 + \frac{1}{p^2}\right)$ **2**

(iv) A circle is drawn with PQ as its diameter.

Prove that the directrix is a tangent to this circle. **3**

(b) A cube is expanding in such a manner that it maintains its cubic shape. Initially, each edge is 10cm and the surface area is expanding at a constant rate of 12.6 cm^2/s.

(i) Find an expression in terms of t for the surface area of the cube after t seconds. **1**

(ii) Hence, or otherwise, find the rate at which the volume of the cube is increasing after 10 seconds. **3**

Marks

Question 6 (12 marks) Use a SEPARATE writing booklet.

(a) Prove by mathematical induction for $n \geq 0$ that $E(n) = 9^{n+2} - 4^n$ is a multiple of 5. **3**

(b) (i) With the aid of a diagram, or otherwise,

solve $(x^2 - 1)(x^2 - 4) \leq 0$ **2**

(ii) A particle is moving along the x-axis with its acceleration at position x given by

$$\ddot{x} = 10x - 4x^3$$

When $x = \sqrt{2}$ its velocity $v = 2$

(α) Prove that the expression $\frac{1}{2}v^2 + x^4 - 5x^2$ is a constant for the motion and find this constant.

(v is its velocity at position x) **2**

(β) Describe the motion. **2**

(c) A random sample of 10 people is made. Assuming that either sex is equally likely, find the probability that

(i) there is an equal number of each sex **1**

(ii) there are more females **2**

Give your answers correct to three decimal places.

Marks

Question 7 (12 marks) Use a SEPARATE writing booklet.

(a) Solve $\dfrac{-2x}{x+1} > 0$ **2**

(b) Consider the curve $y = 1\text{n}\left(\dfrac{-2x}{x+1}\right)$

(i) Use (a) to explain why

$\ln\left(\dfrac{-2x}{x+1}\right) = \ln(-2x) - \ln(x+1)$ **1**

(ii) Show that the curve has no stationary points. **2**

(iii) Sketch the curve, showing the x-intercept. **2**

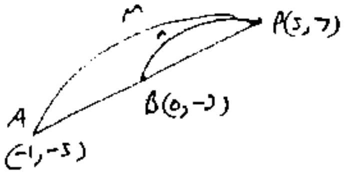
(iv) Find the inverse function of $y = \ln\left(\dfrac{-2x}{x+1}\right)$,
expressing your answer with y as subject. **3**

(v) Find the area of the region bounded by $y = \ln\left(\dfrac{-2x}{x+1}\right)$ and the y-axis
and the lines $y = 0$ and $y = 2$ **2**

End of Paper

Solutions TKS Trial HSC 2003 Extension 1

Qu 1

(a) 

$\Rightarrow$ $\boxed{m : n = 6 : 5}$

(b) $\dfrac{d}{dx} \tan^{-1}(1+x^2) = \dfrac{1}{1+(1+x^2)^2} \times 2x = \boxed{\dfrac{2x}{1+(1+x^2)^2}}$

(c) $\binom{8}{5}$ or, of course, $\binom{8}{3}$ $= \boxed{56}$

(d) gradients of lines are 2 and -3

$\therefore \tan\alpha = \left| \dfrac{2-(-3)}{1+2(-3)} \right| = \dfrac{5}{5} = 1$

$\therefore$ acute angle is $\boxed{45^\circ}$

(e) $P(-1) = 0 \Rightarrow (-1)^{2n+1} - (-1)^{2n} + b = 0$

ie $-1 - 1 + b = 0$ $\therefore \boxed{b = 2}$

(f) $\displaystyle\sum_{n=1}^{9} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(1 - \tfrac{1}{2}\right) + \left(\tfrac{1}{2} - \tfrac{1}{3}\right) + \dots + \left(\tfrac{1}{8} - \tfrac{1}{9}\right) + \left(\tfrac{1}{9} - \tfrac{1}{10}\right)$

$= 1 - \frac{1}{10}$

$= \boxed{\dfrac{9}{10}}$

Qu 2

(a) $f'(x) = 5 - 4\cos 4x$

$\geq 5 - 4(1)$ since $-1 \leq \cos 4x \leq 1$

$\Rightarrow f'(x) \geq 1 > 0 \quad \forall x$

$\therefore f(x)$ increases $\forall x$

(b) (i) $R\sin(x-\alpha) = R\cos\alpha \sin x - R\sin\alpha\cos x$

$\equiv \sin x - \sqrt{3}\cos x$

$\Rightarrow R\cos\alpha = 1$, $R\sin\alpha = \sqrt{3}$ $\quad \therefore \tan\alpha = \sqrt{3}$, $\boxed{\alpha = \frac{\pi}{3}}$

and $R = \sqrt{1^2 + \sqrt{3}^2} = \boxed{2}$

(ii) From (i), $2\sin\left(x - \frac{\pi}{3}\right) = \sqrt{2}$

$\therefore \sin\left(x - \frac{\pi}{3}\right) = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

$\therefore x - \frac{\pi}{3} = \frac{\pi}{4}$ or $\frac{3\pi}{4}$

ie $x = \boxed{\dfrac{7\pi}{12} \text{ or } \dfrac{13\pi}{12}}$

(c) (i) $u = 4 - x^2$

$\frac{du}{dx} = -2x$ or $du = -2x\,dx$ $\qquad x = 0,\ u = 4$; $x = \sqrt{3},\ u = 1$

$\therefore I = -\frac{1}{2}\displaystyle\int_4^1 \frac{du}{\sqrt{u}} = \frac{1}{2}\int_1^4 u^{-\frac{1}{2}}\,du$

$= \frac{1}{2} \cdot 2\left[u^{\frac{1}{2}}\right]_1^4$

$= 2 - 1 = 1$

(ii) $I = \displaystyle\int_0^{\sqrt{3}} \frac{4}{\sqrt{4-x^2}} - \frac{x}{\sqrt{4-x^2}}\,dx = 4\left[\sin^{-1}\frac{x}{2}\right]_0^{\sqrt{3}} - 1$, from (i)

$= 4\,\frac{\pi}{3} - 1$

Qn 3

(a) $f'(x) = 2x e^{x^2} - 1$

$\therefore x_1 = 1.2 - \dfrac{e^{1.44} - 1.2 - 3}{2.4e^{1.44} - 1} = 1.1977\ldots$

$\therefore$ two decimal approx^n = $\boxed{1.20}$

(b) (i) $\sin 2A = \dfrac{2t}{1+t^2}$

(ii) put $t = \tan A$,

then $\operatorname{cosec} 2A - 3\cot 2A = \dfrac{1+t^2}{2t} - 3\cdot\dfrac{1-t^2}{2t}$

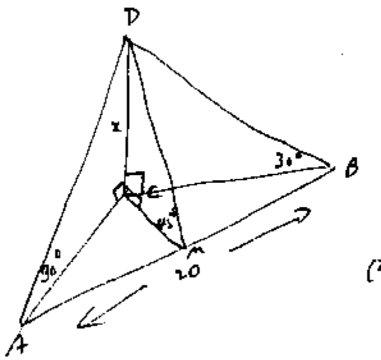
$= \dfrac{1+t^2-3+3t^2}{2t}$

$= \dfrac{4t^2-2}{2t}$

$= 2t - \dfrac{1}{t}$

$= 2\tan A - \cot A$

(c) (i)

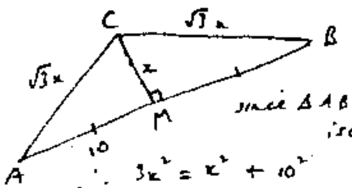


(ii) In ΔACD,

$\tan 30° = \dfrac{x}{AC} = \dfrac{1}{\sqrt{3}}$

$\Rightarrow AC = \sqrt{3}\,x$

(iii) From (ii), we have



since ΔABC is isosceles

$\therefore 3x^2 = x^2 + 10^2$

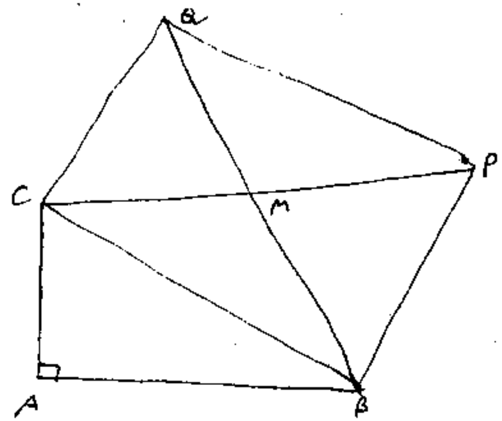
$2x^2 = 100$

$x^2 = 50$

$\therefore x = \sqrt{50}$ or $5\sqrt{2}$

Qn 4

(i)



(ii) $\angle CMB = 90°$, the diagonals of a square meet at right angles

$\therefore$ In $\Diamond ABMC$, $\angle A + \angle M = 180°$

$\Rightarrow \Diamond ABMC$ is cyclic, opposite angles are supplementary

(iii) $MC = MB$, equal diagonals in a square bisect each other.

$\therefore \angle CAM = \angle BAM$, $\angle$s at the circumference of a circle standing on equal arcs.

$\therefore MA$ bisects $\angle BAC$

(b) (i) $t = 0$, $x = 10\cos 0 = 10$

$\dot{x} = -10n\sin nt = -10n\sin 0$ at $t = 0$

$= 0$

$\therefore$ particle is initially at rest at $x = 10$

(ii) $T = \dfrac{2\pi}{n}$ $\therefore n = \dfrac{2\pi}{T}$

$\therefore b = 10\cos\left(\dfrac{2\pi}{T}\cdot\dfrac{T}{3}\right) = 10\cos\left(\dfrac{2\pi}{3}\right)$

ie $b = 10\left(-\dfrac{1}{2}\right) = -5$

(iii) $\dot{x} = -10n \sin nt = -10 \cdot \frac{2\pi}{T} \sin\left(\frac{2\pi t}{T}\right)$

$\therefore -20\sqrt{3} = -\frac{20\pi}{T} \sin\left(\frac{2\pi}{3}\right) = -\frac{20\pi}{T} \cdot \frac{\sqrt{3}}{2}$

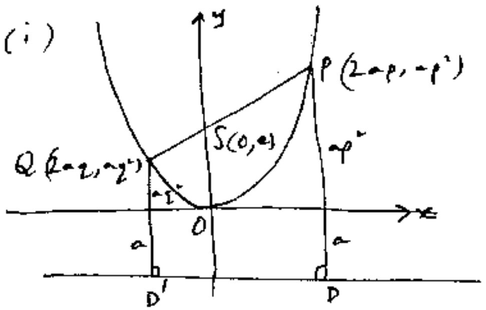
$\therefore T = \frac{\pi}{2}$ (seconds) ie period is $\frac{\pi}{2}$ s

(c) $\binom{n}{3} \div \binom{n-1}{2} = \frac{n!}{(n-3)!\,3!} \div \frac{(n-1)!}{(n-3)!\,2!}$

$= \frac{n!}{(n-3)!\,3!} \times \frac{(n-3)!\,2!}{(n-1)!} = \frac{n}{3}$

Qn 5

(a) (i)



gradient PQ
$= \frac{ap^2 - aq^2}{2ap - 2aq}$
$= \frac{a(p-q)(p+q)}{2a(p-q)}$
$= \frac{p+q}{2}$

$\therefore$ chord PQ is $y - ap^2 = \frac{1}{2}(p+q)(x - 2ap)$

or $y - ap^2 = \frac{1}{2}(p+q)x - ap(p+q)$

$\Rightarrow y - \frac{1}{2}(p+q)x + apq = 0$

(ii) Since $S(0,a)$ is on PQ, then

$a - 0 + apq = 0$

ie. $pq = -1$ or $q = -\frac{1}{p}$

(iii) $PQ = PS + QS = PD + QD'$, focus-directrix defn (see diagram)

$= a + ap^2 + a + aq^2, \quad q = -\frac{1}{p}$

$= 2a + a\left(p^2 + \frac{1}{p^2}\right)$

(iv) If PQ is a diameter, the radius is $a + \frac{a}{2}\left(p^2 + \frac{1}{p^2}\right)$, from (iii)

& the centre is $\left(\frac{2ap + 2aq}{2}, \frac{ap^2 + aq^2}{2}\right)$

$= \left(a\left(p - \frac{1}{p}\right), \frac{a}{2}\left(p^2 + \frac{1}{p^2}\right)\right), \quad q = -\frac{1}{p}$

$\therefore$ distance from centre to directrix $y = -a$ is

$\frac{a}{2}\left(p^2 + \frac{1}{p^2}\right) + a =$ radius

$\therefore$ directrix is a tangent to the circle

(b) (i) $A = 6\times10^2 + 12.6t = 600 + 12.6t$

(ii) If an edge is x, $A = 6x^2$, $V = x^3$

$\therefore$ From (i), $6x^2 = 600 + 12.6t$

$x^2 = 100 + 2.1t$

$\therefore V = (100 + 2.1t)^{3/2}$

& so, $\frac{dV}{dt} = \frac{3}{2}(100 + 2.1t)^{\frac{1}{2}}(2.1)$

$= 3.15 \times \sqrt{121}$ cm³/s when $t = 10$

$= 34.65$ cm³/s

Alternatively, using $A = 6x^2$, $V = x^3$, $\frac{dA}{dt} = 12.6$,

we have $\frac{dV}{dt} = \frac{dV}{dx}\cdot\frac{dx}{dt}$

$= \frac{dV}{dx}\cdot\frac{dx}{dA}\cdot\frac{dA}{dt}$

$= 3x^2 \cdot \frac{1}{12x}\cdot(12.6)$

$= 3.15x$

But, when $t = 10$, $A = 6\times10^2 + 12.6\times10 = 726$

$\therefore 6x^2 = 726 \Rightarrow x = 11$

$\therefore \frac{dV}{dt} = 3.15\times11$ cm³/s $= 34.65$ cm³/s

Qu 6

(a) $E(0) = 9^2 - 4^0 = 80$ is a multiple of 5

$\therefore$ assume $E(n) = 9^{n+2} - 4^n = 5q$, q an integer, $n \geq 0$

Then, $E(n+1) = 9^{n+3} - 4^{n+1}$

$= 9(9^{n+2}) - 4^{n+1}$

$= 9(5q + 4^n) - 4^{n+1}$, using the assumption

$= 5(9q) + 4^n(9-4)$

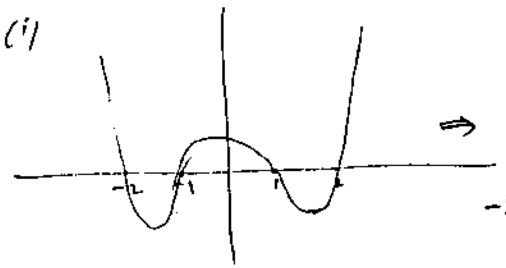
$= 5(9q + 4^n)$ is a multiple of 5 since $9q + 4^n$ is an integer

$\therefore$ if $E(n)$ is a multiple of 5, so is $E(n+1)$

But, $E(0)$ is a multiple of 5

$\therefore E(n)$ is a multiple of 5 by induction

(b) (i)



$\Rightarrow (x^2-1)(x^2-4) \leq 0$ has solutions $-2 \leq x \leq 1$ or $1 \leq x \leq 2$

(ii) (α) $\frac{d(\frac{1}{2}v^2)}{dx} = 10x - 4x^3$

$\therefore \frac{1}{2}v^2 = 5x^2 - x^4 + c$, c a constant

ie. $\frac{1}{2}v^2 + x^4 - 5x^2 = c$

When $x = \sqrt{2}$, $v = 2$

$\therefore 2 + 4 - 10 = c = -4$

(β) we have $\frac{1}{2}v^2 = 5x^2 - x^4 - 4$

or $\frac{1}{2}v^2 = -(x^4 - 5x^2 + 4)$

$= -(x^2-1)(x^2-4)$

Now, $\frac{1}{2}v^2 \geq 0 \Rightarrow (x^2-1)(x^2-4) \leq 0$

∴ Using (b)(i) and when $x = \sqrt{2}$, $v = 2$, we have the particle oscillates between $x = 1$ and $x = 2$

(c) (i) $P(5 \text{ males}, 5 \text{ females}) = \binom{10}{5}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^5$

$= 0.246$, 3 d.p.

(ii) P(more females) = P(more males), since $P(M) = P(F) = \frac{1}{2}$

∴ using (i), P(more females) $= \dfrac{1 - 0.246}{2} = 0.377$

Qn 7

(a) if $x+1 > 0$, then $-2x > 0$

i.e. $x > -1$ i.e. $x < 0$

∴ solution is $-1 < x < 0$

if $x < -1$, we'd have $x > 0 \Rightarrow$ no further solutions

∴ $-1 < x < 0$

(b)(i) We need $\dfrac{-2x}{x+1} > 0$ and $-2x > 0$ and $x + 1 > 0$.

All 3 inequalities are "true" from (a)

(ii) From (i), $y = \ln(-2x) - \ln(x+1)$

$\therefore \dfrac{dy}{dx} = \dfrac{-2}{-2x} - \dfrac{1}{x+1}$

$= \dfrac{1}{x} - \dfrac{1}{x+1} = \dfrac{x+1-x}{x(x+1)}$

$= \dfrac{1}{x(x+1)} \neq 0$ for any x

∴ there are no stationary points

(iii) Since $-1 < x < 0$, then $x(x+1) < 0$

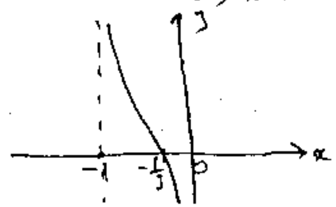
∴ From (ii) $\dfrac{dy}{dx} < 0$ for $-1 < x < 0$

i.e. curve is decreasing for $-1 < x < 0$

When $y = 0$, $\dfrac{-2x}{x+1} = 1$

∴ $-2x = x + 1$

$\Rightarrow$ x intercept is $-\frac{1}{3}$



(iv) since curve is decreasing, the inverse function

is $x = \ln\left(\frac{-2y}{y+1}\right)$

$\therefore \quad \frac{-2y}{y+1} = e^x$

$-2y = e^x y + e^x$

$\Rightarrow \quad y(e^x + 2) = -e^x$

ie inverse function is $y = -\frac{e^x}{e^x+2}$

(v) $A = \left| \int_0^2 x \, dy \right| = \int_0^2 \frac{e^y}{e^y+2} \, dy$, from (iv)

$= \left[\ln(e^y+2)\right]_0^2$

$= \ln(e^2+2) - \ln 3 \quad u^2$