THE KING'S SCHOOL

2005
Higher School Certificate
Trial Examination

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided
- All necessary working should be shown in every question

Total marks – 84

- Attempt Questions 1-7
- All questions are of equal value

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Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (12 marks) Use a SEPARATE writing booklet.

(a) Find $\int \frac{dx}{\sqrt{9-4x^2}}$ **2**

(b) $P(3,7)$ divides the interval AB in the ratio $m:n$,
where $A=(-1,2)$ and $B=(11,17)$.

Find the ratio $m:n$. **2**

(c) Let $f(x)=\ln(\tan x)$, $0 < x < \frac{\pi}{2}$

Show that $f'(x)=2\operatorname{cosec}2x$ **3**

(d) Two lines have gradients $2+\sqrt{3}$ and 1.

Find the size of the acute angle between these lines. **2**

(e) Use the substitution $u=1+x^2$ to evaluate $\int_1^{\sqrt{3}} 6x\sqrt{1+x^2}\,dx$ **3**

End of Question 1

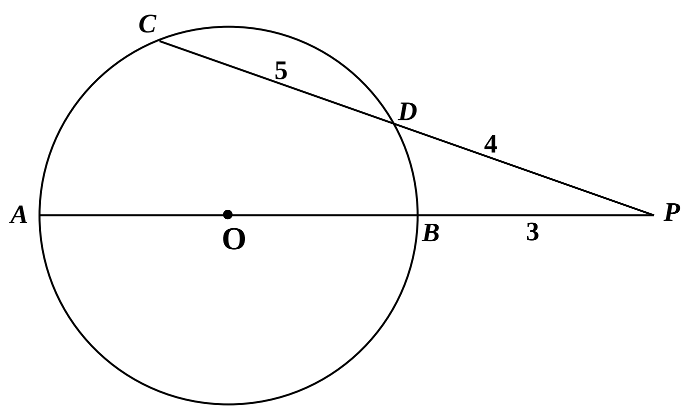
Question 2 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Use the expansion of $\cos(A+B)$ to show that $\cos 2x = 1 - 2\sin^2 x$ **2**

(ii) Evaluate $\lim_{x \to 0} \dfrac{\cos 2x - 1}{2x^2}$ **2**

(b) Find $\int 4\sin^2 x \, dx$ **2**

(c)



AB is the diameter of a circle, centre O.

AB produced meets the secant CD at P.

$CD = 5$, $DP = 4$ and $BP = 3$

Find the diameter of the circle. **2**

(d) (i) Solve $\dfrac{x+1}{x^2+1} > 1$ **2**

(ii) Without solving, explain why $x < -1$ is a solution of $\dfrac{x^2+1}{x+1} < 1$ **1**

(iii) Hence, or otherwise, solve $\dfrac{x^2+1}{x+1} < 1$ **1**

End of Question 2

Question 3 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) The roots of $x^3 - 6x + 1 = 0$ are α, β, γ.

Find values for

(i) $\alpha + \beta + \gamma$ **1**

(ii) $\alpha^2 + \beta^2 + \gamma^2$ **2**

(b) Use the substitution $t = \tan\frac{x}{2}$ to solve $\sin x - 7\cos x - 5 = 0$, $0° < x < 360°$, giving your answers to the nearest degree. **3**

(c) (i) A particle is moving along the x axis so that at any time $t \geq 0$ its velocity is v and its acceleration is $\frac{dv}{dt}$.

Prove that $\frac{dv}{dt} = \frac{d\left(\frac{1}{2}v^2\right)}{dx}$ **2**

(ii) A particle is moving along the x axis in simple harmonic motion with its velocity v given by the equation $v^2 = 4x - 2x^2$.

Find

(α) its amplitude **2**

and (β) its period. **2**

End of Question 3

Question 4 (12 marks) Use a SEPARATE writing booklet. **Marks**

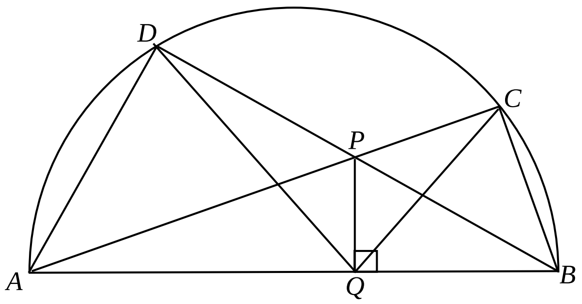
(a) $f(x) = 5\sqrt{1+x^6} - 7x$ has a zero near $x = 1$.

Use Newton's method once to find a two decimal place approximation to this zero. **3**

(b) Use mathematical induction to prove that for all integers $n \geq 1$,

$$(1^2+1)\,1! \; + \; (2^2+1)\,2! \; + \; \ldots \; + (n^2+1)\,n! \; = n(n+1)\,!$$ **4**

(c)



AB is the diameter of a semi-circle $ABCD$.

DB meets AC at P and $PQ \perp AB$ at Q.

(i) Explain why $\angle DAC = \angle CBD$ **1**

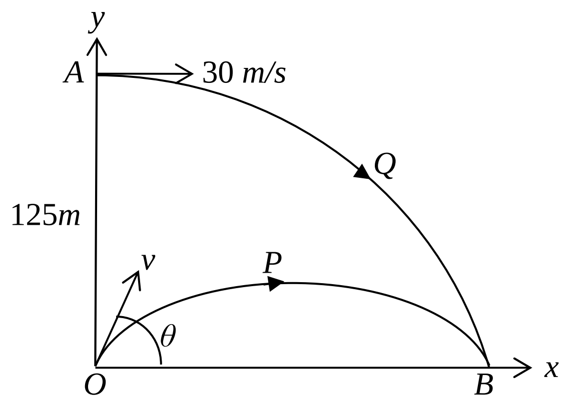
(ii) Prove that $QBCP$ is a cyclic quadrilateral. **2**

(iii) Deduce that PQ bisects $\angle DQC$. **2**

End of Question 4

Question 5 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a)



Two particles P and Q are projected at the same time in the same vertical plane. The only force acting on the particles is due to gravity, where $g = 10\ m/s^2$.

Taking axes of reference as in the diagram, particle P is projected from O on horizontal ground with velocity $v\ \ m/s$ at an elevation of θ and particle Q is projected horizontally at $30\ m/s$ from a point A $125\ m$ above O. The two particles meet at the same time t at a point B on ground level.

You may assume the equations of motion for particle P are:

$\ddot{x} = 0$
$\dot{x} = v\cos\theta$
$x = (v\cos\theta)\,t$

$\ddot{y} = -10$
$\dot{y} = -10t + v\sin\theta$
$y = -5t^2 + (v\sin\theta)\,t$

[DO NOT PROVE THESE]

(i) Write down the equations of motion for particle Q **2**

(ii) Find the time taken for particle Q to reach B and determine the horizontal distance it has travelled. **2**

(iii) Find the velocity of projection and angle of elevation for particle P, i.e. find v and θ **3**

(iv) Find the cartesian equation of the parabolic path of particle P. **2**

Question 5 continues on the next page

(b) You are given $P(x) = (2x^2 - x - 3)^3 \equiv 8x^6 - 12x^5 - 30x^4 + 35x^3 + Ax^2 - 27x - 27$

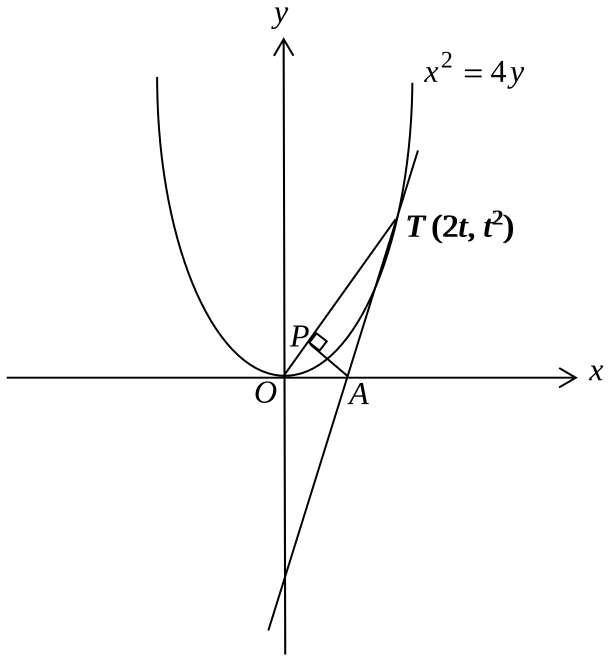
(i) Find the value of A. **1**

(ii) Find two of the roots of $P''(x) = 0$ **2**

End of Question 5

Question 6 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a)



The tangent at $T(2t, t^2)$, $t \neq 0$, on the parabola $x^2 = 4y$ meets the x axis at A.

$P(x, y)$ is the foot of the perpendicular from A to OT, where O is the origin.

The equation of the tangent at T is $y = tx - t^2$

(i) Prove that the equation of AP is $y = -\frac{2}{t}(x - t)$ **2**

(ii) Show that the equation of OT is $t = \frac{2y}{x}$ **1**

(iii) Hence, or otherwise, prove that the locus of $P(x, y)$ lies on a circle with centre $(0,1)$ and give its radius. **3**

Question 6 continues on the next page

(b) The surface area S of a spherical bubble is changing at a constant rate of $k\ cm^2 / s$.

Prove that the volume is changing at a rate proportional to the radius at any time t.

$$\left[S = 4\pi r^2, \quad V = \frac{4}{3}\pi r^3 \right]$$ **3**

(c) If $\log_6 9 = A$, express $\log_3 2$ in terms of A. **3**

End of Question 6

Question 7 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Consider the function $f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

(i) Show that $f(-x) = -f(x)$ **1**

(ii) Show that $f(x) = 1 - \dfrac{2}{e^{2x} + 1}$ **1**

(iii) Explain why $f(x) < 1$ for all values of x **1**

(iv) Hence, or otherwise, sketch $y = f(x)$ in the number plane. **2**

(v) Explain why the inverse function $f^{-1}(x)$ exists without any restriction on the domain of $f(x)$ **1**

(vi) Sketch $y = f^{-1}(x)$ in the number plane. **1**

(vii) For $y = f^{-1}(x)$, express y in terms of x **2**

(b) (i) Show that $\dfrac{\binom{3n}{k}}{\binom{3n}{k-1}} = \dfrac{3n - k + 1}{k}$ **1**

(ii) Find the greatest term in the expansion of $\left(1 + \dfrac{x}{2}\right)^{3n}$, n a positive integer, when $x = 1$ **2**

End of Examination

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \, \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

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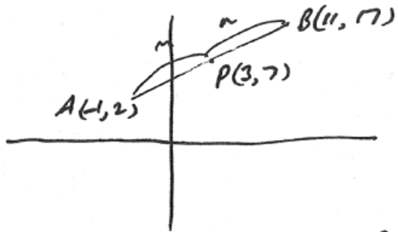
Qu 1

(a) $\frac{1}{2}\sin^{-1}\frac{2x}{3}$ $(+c)$

(b)



$\therefore m:n = 4:8 = 1:2$

(c) $f'(x) = \frac{\sec^2 x}{\tan x} = \frac{1}{\sin x \cos x}$

$= \frac{2}{\sin 2x} = 2\operatorname{cosec} 2x$

(d) $\tan\alpha = \frac{2+\sqrt{3} \quad -1}{1 + 2+\sqrt{3}} = \frac{1+\sqrt{3}}{3+\sqrt{3}}$

$= \frac{1+\sqrt{3}}{\sqrt{3}(1+\sqrt{3})} = \frac{1}{\sqrt{3}}$

$\Rightarrow \alpha = \frac{\pi}{6}$ or 30°

(e) $u = 1+x^2$

$\frac{du}{dx} = 2x$

$: x=1, u=2$

$x=\sqrt{3}, u=4$

$\therefore I = \int_2^4 3\sqrt{u}\, du$

$= 3.2\left[\frac{u^{3/2}}{3}\right]_2^4$

$= 2\left(8 - 2\sqrt{2}\right)$

$= 4(4-\sqrt{2})$

Qu 2

(a) (i) $\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\Rightarrow \cos 2x = \cos^2 x - \sin^2 x$

$= 1-\sin^2 x - \sin^2 x \quad = 1-2\sin^2 x$

(ii) $\lim_{x\to 0} \dfrac{-2\sin^2 x}{2x^2} = -\lim_{x\to 0}\left(\dfrac{\sin x}{x}\right)^2 = -1$

(b) $I = 2\int 1-\cos 2x \, dx = 2\left(x - \dfrac{\sin 2x}{2}\right) \quad (+c)$

$= 2x - \sin 2x \quad (+c)$

(c) $\Rightarrow 3(d+3) = 9\times 4$

$\therefore d+3 = 12$

ie diameter $= 9$

(d) (i) $\therefore x+1 > x^2+1$

ie $x^2 - x < 0$

$x(x-1) < 0$

$\Rightarrow 0 < x < 1$

(ii) $x < -1 \Rightarrow x+1 < 0$ and since $x^2+1 > 0$,

$\therefore \dfrac{x^2+1}{x+1} < 0 \Rightarrow \dfrac{x^2+1}{x+1} < 1$

(iii) From (i) & (ii), $x < -1$ or $0 < x < 1$

Qu 3

(a) (i) $\alpha+\beta+\gamma=0$

(ii) $\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)$

$=0-2(-6)=12$

(b) $t=\tan\frac{x}{2} \Rightarrow \frac{2t}{1+t^2}-\frac{7(1-t^2)}{1+t^2}-5=0$

i.e. $2t-7(1-t^2)-5(1+t^2)=0$

$\therefore 2t^2+2t-12=0$

i.e. $t^2+t-6=0$

$(t-2)(t+3)=0$

$\Rightarrow \tan\frac{x}{2}=2$ or -3 , $0^\circ<\frac{x}{2}<180^\circ$

$\therefore x=127^\circ$ or 217°

(c) (i) $\frac{dv}{dt}=\frac{dv}{dx}\cdot\frac{dx}{dt}$

$=v\frac{dv}{dx}$

$=\frac{d(\frac{1}{2}v^2)}{dv}\cdot\frac{dv}{dx}=\frac{d(\frac{1}{2}v^2)}{dx}$

(ii) (α) $v=0 \Rightarrow 2x(2-x)=0$

i.e. $x=0,2$

$\therefore$ amplitude $=1$

(β) $\frac{1}{2}v^2=2x-x^2$

$\therefore \ddot{x}=2-2x=-2(x-1)$

$\Rightarrow n^2=2$ in usual notation

i.e. $n=\sqrt{2}$

$\therefore$ period $T=\frac{2\pi}{\sqrt{2}}=\sqrt{2}\,\pi$

Qu 4

(a) $f'(x) = 5 \cdot \frac{1}{2}(1+x^6)^{-\frac{1}{2}} \cdot 6x^5 - 7 = \frac{15x^5}{\sqrt{1+x^6}} - 7$

$\therefore x_1 = 1 - \frac{5\sqrt{2} - 7}{\frac{15}{\sqrt{2}} - 7} \approx 0.98$

(b) For $n=1$, $LS = (1^2+1)1! = 2$

$RS = 1(2!) = 2$

$\therefore$ Assume $(1^2+1)1! + \cdots + (n^2+1)n! = n(n+1)!$ for some integer $n \geq 1$

Then $(1^2+1)1! + \cdots + (n^2+1)n! + ((n+1)^2+1)(n+1)!$

$= n(n+1)! + (n^2+2n+2)(n+1)!$, using the assumption

$= (n+1)!\,[n + n^2 + 2n + 2]$

$= (n+1)!\,(n^2+3n+2)$

$= (n+1)!\,(n+1)(n+2)$

$= (n+1)(n+2)! = RS$ for $n+1$

$\therefore$ ....

(c) (i) angles in the same segment

(ii) $\angle ACB = 90^\circ$, $\angle$ in a semi-circle

$= \angle PQB$, given

$\therefore QBCP$ is a cyclic ◇, opp. angles supplementary

(iii) Similarly to (ii), AQPD is a cyclic quad

$\therefore \angle PQD = \angle DAP$, $\angle s$ in same segment

& $\angle PQC = \angle CBP$, "

But $\angle DAP = \angle CBP$, given

$\therefore \angle PQD = \angle PQC$ ie. PQ bisects $\angle DQC$

Qn 5

(a) (i) $\ddot{x} = 0$ $\qquad$ $\ddot{y} = -10$

$\dot{x} = 30$ $\qquad$ $\dot{y} = -10t$

$x = 30t$ $\qquad$ $y = -5t^2 + 125$

(ii) at B, $y=0$ $\therefore$ $5t^2 = 125$

or $t^2 = 25$ $\therefore t = 5$

& so $x = 30 \times 5 = 150$

i.e. 5 s to reach B 150 m horizontally

(iii) From (ii), $150 = 5v\cos\theta$ i.e. $v\cos\theta = 30$

and $0 = -125 + 5v\sin\theta$ i.e. $v\sin\theta = 25$

$\therefore v^2 = 30^2 + 25^2 \Rightarrow v = \sqrt{30^2 + 25^2}$

$= 5\sqrt{61}$ m/s [or $\sqrt{1525}$ or 39.....]

and $\tan\theta = \frac{25}{30} \Rightarrow \theta = \tan^{-1}\left(\frac{5}{6}\right)$ [$\approx 39.8°$ or 0.7]

(iv) from (iii), $x = 30t$, $y = -5t^2 + 25t$

$\therefore$ Cartesian path is $y = -5\left(\frac{x}{30}\right)^2 + 25\left(\frac{x}{30}\right)$ [will do]

i.e. $y = -\frac{x^2}{180} + \frac{5x}{6}$

(b)(i) Put $x = -1$, then $(2+1-3)^3 = 8 + 12 - 30 - 35 + A = 0$

$\therefore A = 45$

(ii) $P(x) = (2x^2 - x - 3)^3 = ((x+1)(2x-3))^3$

$\Rightarrow x = -1$ and $\frac{3}{2}$ are triple roots of $P(x) = 0$

$\therefore$ two roots of $P''(x) = 0$ are $x = -1, \frac{3}{2}$

Qu 6

(a) (i) Tangent at T is $y = tx - t^2$

$\therefore$ at A, $0 = tx - t^2 \Rightarrow x = t$

ie $A = (t, 0)$

gradient of $OT = \frac{t^2}{2t} = \frac{t}{2}$

$\therefore$ AP is $y = -\frac{2}{t}(x - t)$

(ii) OT is $y = \frac{t}{2}x$ or $t = \frac{2y}{x}$

(iii) From (ii), locus of $P(x, y)$ is

$$y = -\frac{2x}{2y}\left(x - \frac{2y}{x}\right)$$

or $y^2 = -x^2 + 2y$

$\Rightarrow x^2 + y^2 - 2y = 0$

ie $x^2 + (y-1)^2 = 1$,

a circle centre $(0, 1)$, radius 1 [except the point $(0,0)$]

OTHERWISE using $y = -\frac{2}{t}(x - t)$ and $y = \frac{t}{2}x$

we have $\frac{t}{2}x = -\frac{2}{t}(x - t)$

or $t^2 x = -4(x - t) \Rightarrow x = \frac{4t}{t^2+4}$

$\therefore y = \frac{t}{2} \cdot \frac{4t}{t^2+4} = \frac{2t^2}{t^2+4}$

$\therefore x^2 + (y-1)^2 = \frac{16t^2}{(t^2+4)^2} + \left(\frac{2t^2}{t^2+4} - 1\right)^2$

$$= \frac{16t^2 + (t^2-4)^2}{(t^2+4)^2} = \frac{t^4 + 8t^2 + 16}{(t^2+4)^2} = \frac{(t^2+4)^2}{(t^2+4)^2} = 1$$

$\Rightarrow$ circle, centre $(0, 1)$, radius 1

(b) $\frac{dS}{dt} = k$

$$\therefore \frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$$

$$= \frac{dV}{dr} \cdot \frac{dr}{dS} \cdot \frac{dS}{dt} \qquad : \frac{dS}{dr} = 8\pi r$$

$$= 4\pi r^2 \cdot \frac{1}{8\pi r} \cdot k$$

$$= \frac{k}{2} r$$

$\propto r$ since k is constant

(c) $$\log_6 9 = \frac{\log_3 9}{\log_3 6} = \frac{2}{\log_3 3 + \log_3 2}$$

$$\Rightarrow A = \frac{2}{1 + \log_3 2}$$

$$\therefore 1 + \log_3 2 = \frac{2}{A}$$

or $\log_3 2 = \frac{2}{A} - 1$ or $\frac{2 - A}{A}$

Qu 7

(a) (i) $f(-x) = \dfrac{e^{-x} - e^{x}}{e^{-x} + e^{x}} = -\left(\dfrac{e^{x} - e^{-x}}{e^{x} + e^{-x}}\right)$

$= -f(x)$

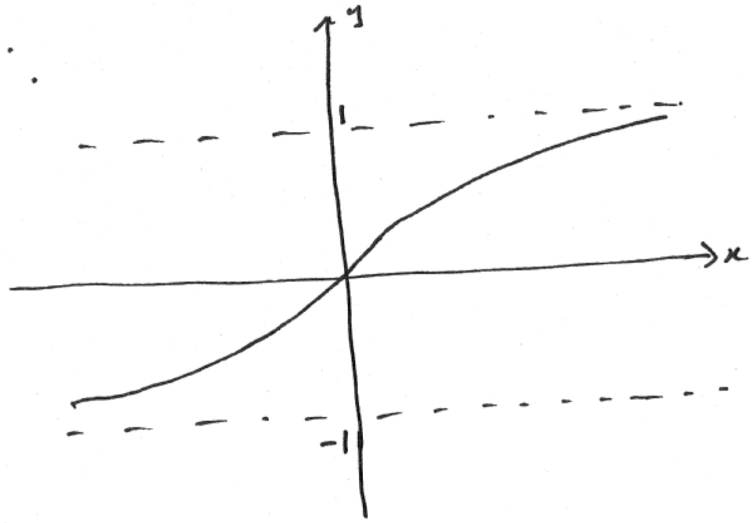
[$\Rightarrow$ odd function]

(ii) $f(x) = \dfrac{e^{2x} - 1}{e^{2x} + 1} = \dfrac{(e^{2x}+1) - 2}{e^{2x} + 1}$

$= 1 - \dfrac{2}{e^{2x}+1}$

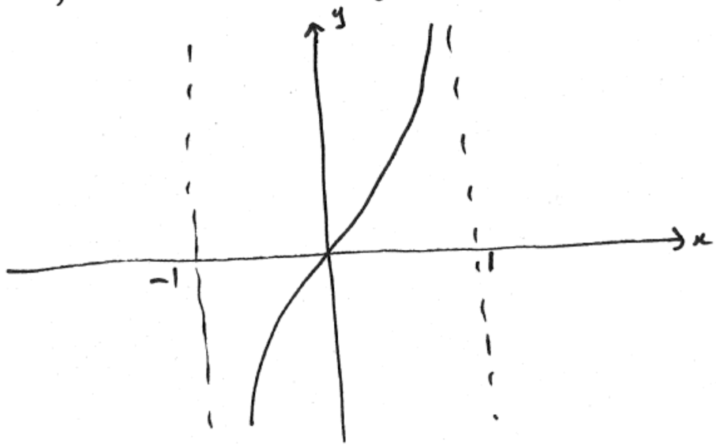
(iii) $\dfrac{2}{e^{2x}+1} > 0 \;\forall x \quad \therefore f(x) < 1$ from (ii)

(iv) $\therefore$



(v) $\because f(x)$ is increasing $\forall x$

(vi)



(vii) f^{-1} : $x = 1 - \dfrac{2}{e^{2y}+1}$

$$\Rightarrow \frac{2}{e^{2y}+1} = 1-x$$

or $e^{2y}+1 = \dfrac{2}{1-x}$

$\therefore e^{2y} = \dfrac{2}{1-x} - 1 = \dfrac{1+x}{1-x}$

$\therefore 2y = \ln\left(\dfrac{1+x}{1-x}\right)$

ie $y = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$

(b) (i) $\dfrac{\binom{3n}{k}}{\binom{3n}{k-1}} = \dfrac{(3n)!\,(3n-k+1)!\,(k-1)!}{(3n-k)!\,k!\,(3n)!}$

$= \dfrac{3n-k+1}{k}$

(ii) Now, $\dfrac{u_{k+1}}{u_k} = \dfrac{3n-k+1}{k} \cdot \dfrac{x}{2}$ from (i)

$\therefore$ for $x=1$, $\dfrac{u_{k+1}}{u_k} \geq 1 \Rightarrow 3n-k+1 \geq 2k$

$\Rightarrow 3k \leq 3n+1$

or $k \leq n + \frac{1}{3}$

$\Rightarrow u_{k+1} > u_k$ for $k \leq n$

ie greatest term $= \binom{3n}{n} \cdot \dfrac{1}{2^n}$ when $x=1$

[u_{n+1} will do]