THE KING'S SCHOOL

2007
Higher School Certificate
Trial Examination

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 84

- Attempt Questions 1-7
- All questions are of equal value

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Attempt Questions 1-7
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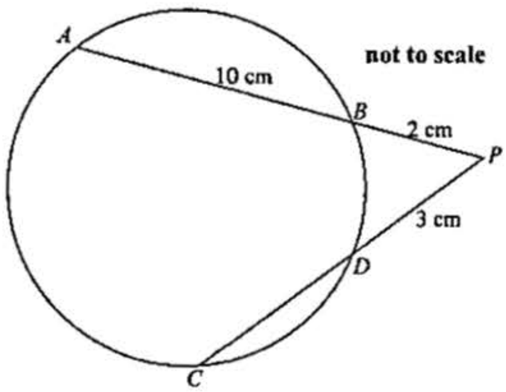
Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a)



The diagram shows the chord AB and CD intersecting externally at P.

If $AB = 10$ cm, $BP = 2$ cm and $DP = 3$ cm find the length of the chord CD. **2**

(b) $P\ (3, 5)$ divides the interval AB externally in the ratio $m : n$.

If $A = (-6, -13)$ and $B = (-1, -3)$, find the ratio $m : n$. **2**

(c) Differentiate $\dfrac{x}{1 + e^{2x}}$ **2**

(d) Find $\displaystyle\int 2\sin^2 x \, dx$ **2**

(e) Solve $\dfrac{x^2 + 2}{x + 2} < 1$ **3**

(f) Find $\displaystyle\lim_{x \to 0} \frac{x^2}{\sin x}$ **1**

End of Question 1

Question 2 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find the acute angle between the lines $y = \frac{2}{3}x + 1$ and $y = -\frac{1}{5}x - 1$ **2**

(b) Use the substitution $x = u^2, u > 0$, to evaluate $\int_1^4 \frac{dx}{\sqrt{x}(1 + \sqrt{x})}$ **4**

(c) The cubic equation $x^3 + ax^2 + bx + c = 0$ has roots $-\alpha, \alpha$ and $\frac{1}{\alpha}$

(i) Show that $ac = -1$ **2**

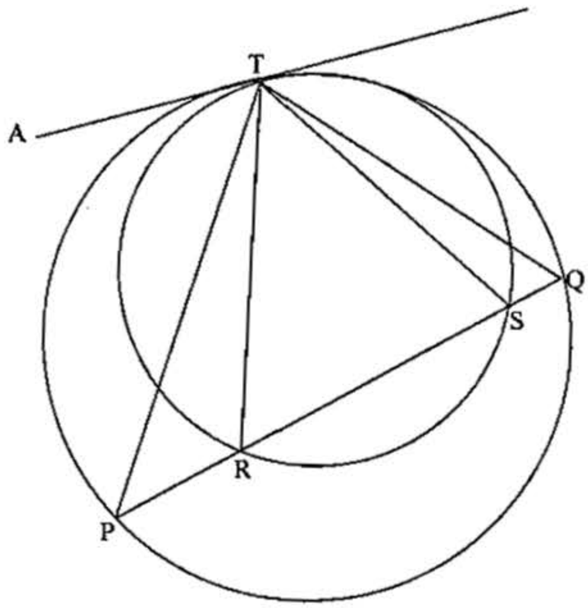
(ii) Show that $b + c^2 = 0$ **2**

(d) Find $\int_0^3 \frac{dx}{\sqrt{16 - x^2}}$. Give your answer correct to two decimal places. **2**

End of Question 2

Question 3 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a)



The diagram shows two circles touching internally at T. AT is the common tangent to the circles. The chord PQ of the outer circle meets the inner circle at R and S.

(i) Explain why $\angle ATP = \angle TQS$ **1**

(ii) Prove that $\angle PTR = \angle QTS$ **3**

(b) $f(x) = \tan^{-1} x - x + 0.3 = 0$ has a root near $x = 1.2$

Use one application of Newton's method to find a second approximation to this root.

Give your answer correct to 2 decimal places. **3**

(c) A particle moves on the x axis in simple harmonic motion. Its velocity v at any time $t \geq 0$ is given by $v = 6\sin 3t$.

Initially the particle is at rest at $x = -1$

(i) At what time is the particle next at rest? **1**

(ii) Hence or otherwise give the period of the motion. **1**

(iii) Find an expression for the displacement x in terms t. **2**

(iv) State when the particle is first at the position $x = 1$ **1**

End of Question 3

Question 4 **(12 marks)** Use a SEPARATE writing booklet. **Marks**

(a) (i) Show that $\cos\left(\frac{\pi}{4} - \frac{\pi}{12}\right) - \cos\left(\frac{\pi}{4} + \frac{\pi}{12}\right) = 2\sin\frac{\pi}{4}\sin\frac{\pi}{12}$ **1**

(ii) Hence prove that $\dfrac{\cos\frac{\pi}{6} - \cos\frac{\pi}{3}}{\sin\frac{\pi}{6} + \sin\frac{\pi}{3}} = \tan\frac{\pi}{12}$ **2**

(iii) Hence or otherwise show that $\tan\frac{\pi}{12} = 2 - \sqrt{3}$ **2**

(b) The surface area S of a sphere is increasing at a constant rate of k cm^2/s. Prove that the volume V of the sphere is increasing at a rate proportional to the radius r at any time t.

$\left[\, S = 4\pi r^2,\ V = \frac{4}{3}\pi r^3 \,\right]$ **3**

(c) Consider the function $f(x) = \frac{2}{\pi}\cos^{-1}(1 - x)$

(i) Find the domain of f. **2**

(ii) Sketch the function. **2**

End of Question 4

Question 5 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Prove by mathematical induction for positive integers n that

$$\left(1^3 + 3.1^5\right) + \left(2^3 + 3.2^5\right) + \ldots + \left(n^3 + 3n^5\right) = \tfrac{1}{2}n^3(n+1)^3$$

[You may assume $(n+2)^3 = n^3 + 6n^2 + 12n + 8$] **3**

(b) Let $(1+x)^n = c_0 + c_1x + c_2x^2 + \ldots + c_kx^k + \ldots + c_nx^n$

(i) Show that $\dfrac{c_k}{c_{k-1}} = \dfrac{n-k+1}{k}$, $1 \le k \le n$ **2**

(ii) Deduce that $\dfrac{c_1}{c_0} + \dfrac{2c_2}{c_1} + \dfrac{3c_3}{c_2} + \ldots + \dfrac{kc_k}{c_{k-1}} + \ldots + \dfrac{nc_n}{c_{n-1}} = \dfrac{n}{2}(n+1)$ **2**

(c) Let $f(x) = x^3 + (A+2)x^2 + Ax - 1$

(i) Show that $x+1$ is a factor of $f(x)$ for all values of A **1**

(ii) If $(x+1)^2$ is a factor of $f(x)$ find A **2**

(iii) If $x^2 - Ax - 1$ is a factor of $f(x)$ find A. **2**

End of Question 5

Question 6 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Let $f(x) = \ln\left(\tan\frac{x}{2}\right)$, $0 < x < \pi$

(i) Show that $f'(x) = \operatorname{cosec} x$ **3**

(ii) Sketch the curve showing the x intercept and any asymptotes. **2**

(iii) Find the inverse function $y = f^{-1}(x)$ **2**

(b) A particle moves on the x axis with its velocity v given by $\frac{dx}{dt} = 20 - x$

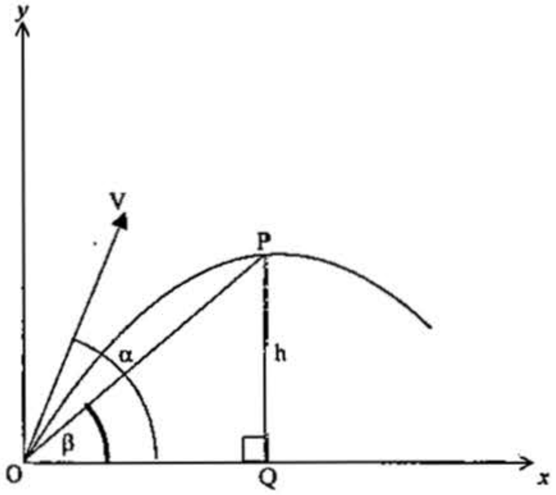
Initially the particle is at $x = 19$

(i) Prove that the acceleration is given by $\ddot{x} = -v$ **2**

(ii) Hence express v as a function of t. **3**

End of Question 6

(a)



A particle is projected from O with velocity V at an angle of α to the horizontal.

At time t the equations of motion are:

$$\dot{x} = V\cos\alpha \qquad \dot{y} = -10t + V\sin\alpha$$
$$x = (V\cos\alpha)\,t \qquad y = -5t^2 + (V\sin\alpha)\,t$$

[DO NOT PROVE THESE]

After time T the particle reaches its greatest height $h = PQ$ where $\angle POQ = \beta$

(i) Show that $T = \dfrac{V\sin\alpha}{10}$ **1**

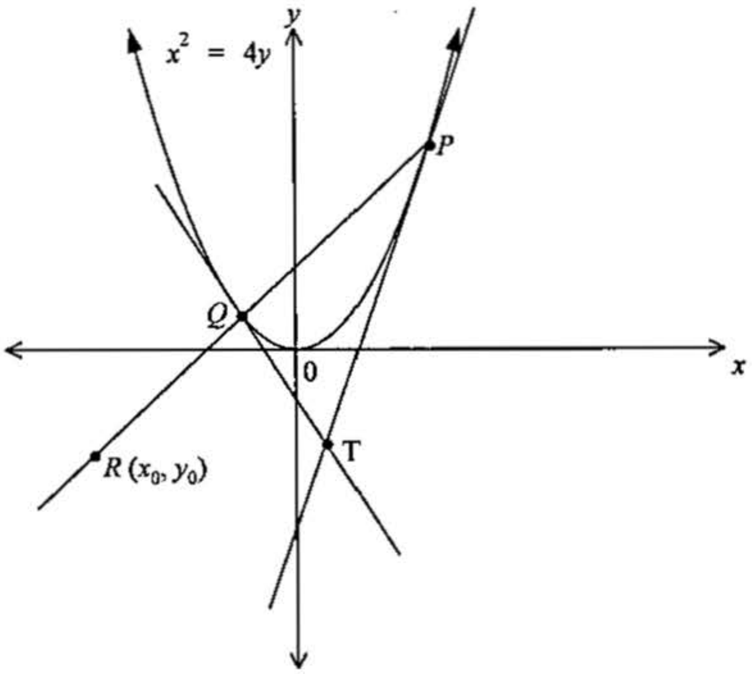
(ii) Deduce that $\tan\beta = \frac{1}{2}\tan\alpha$ **2**

(iii) Show that $20h = V^2\sin^2\alpha$ **1**

(iv) Deduce that $V^2 = 5h\left(4 + \cot^2\beta\right)$ **2**

Question 7 continues on next page

(b)



The diagram shows a variable chord PQ of the parabola $x^2 = 4y$ passing through the point $R\,(x_0\,,\,y_0)$. The tangents at P and Q meet at T.

The equation of the tangent at a point $(x_1\,,\,y_1)$ on $x^2 = 4y$ is $x_1 x = 2(y + y_1)$.

The equation of the chord of contact from $R\,(x_0\,,\,y_0)$ to $x^2 = 4y$ is $x_0 x = 2(y + y_0)$.

[DO NOT PROVE THESE]

(i) Let $P = \left(2p, p^2\right)$

Show that the equation of the tangent at P is $px = y + p^2$ **1**

(ii) Show that the tangents at $P\left(2p, p^2\right)$ and $Q\left(2q, q^2\right)$ meet at $T(p+q, pq)$ **2**

(iii) Show that the equation of the chord PQ is $(p+q)x = 2(y+pq)$ **1**

(iv) Hence prove that the locus of T lies on the line containing the chord of contact from $R\,(x_0\,,\,y_0)$ **2**

End of Examination

T.K.S. MATHEMATICS EXTENSION 1

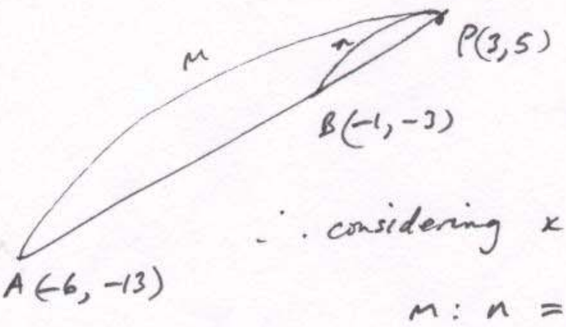
TRIAL PAPER 2007 SOLUTIONS

Question 1

(a) Let $CD = x$

Then $3(x+3) = 12 \times 2 \Rightarrow x = 5$

(b)



$\therefore$ considering x coordinates

$m : n = 9 : 4$

(c) $\dfrac{(1+e^{2x})1 - x(2e^{2x})}{(1+e^{2x})^2}$ will do

$= \dfrac{1 + e^{2x} - 2xe^{2x}}{(1+e^{2x})^2}$

(d) $= \displaystyle\int 1 - \cos 2x \, dx = x - \frac{\sin 2x}{2} \quad (+ c)$

(e) (Alternatives abound)

For $x + 2 > 0$ i.e. $x > -2$ we have $x^2 + 2 < x + 2$

$\Rightarrow x^2 - x < 0$

$x(x-1) < 0 \quad \therefore 0 < x < 1$

$\therefore 0 < x < 1$ is a solution as is $x < -2$

(f) $= \lim\limits_{x \to 0} x \cdot \dfrac{x}{\sin x} = 0 \times 1 = 0$

Question 2

(a) $m_1 = \frac{2}{3}$, $m_2 = -\frac{1}{5}$

$$\therefore \tan\alpha = \frac{\frac{2}{3} + \frac{1}{5}}{1 - \frac{2}{3}\cdot\frac{1}{5}} = \frac{10+3}{15-2} = 1 \quad \therefore \alpha = 45^\circ$$

(b) $x = u^2$: $x = 1, u = 1$

$\frac{dx}{du} = 2u$ $\quad x = 4, u = 2$

$$\therefore I = \int_1^2 \frac{2u\ du}{u(1+u)} = 2\int_1^2 \frac{1}{1+u}\ du$$

$$= 2\left[\ln(1+u)\right]_1^2$$

$$= 2(\ln 3 - \ln 2) = 2\ln 1\cdot 5$$

(c) (i) Sum of roots $= \frac{1}{\alpha} = -a$

Product of roots $= -\alpha = -c$

$$\therefore ac = \frac{1}{\alpha}(-\alpha) = -1$$

(ii) $-\alpha.\alpha - \alpha.\frac{1}{\alpha} + \alpha.\frac{1}{\alpha} = -\alpha^2 = b$

$\therefore$ From (i), $b + \alpha^2 = b + c^2 = 0$

(d) $I = \left[\sin^{-1}\frac{x}{4}\right]_0^3 = \sin^{-1}\left(\frac{3}{4}\right) = 0\cdot 85$, 2 d.p.

Question 3

(a)(i) alternate segment theorem in the outer circle

(ii) $\angle ATR = \angle TSR$, alt. seg. thm in inner circle

$\therefore \angle PTR = \angle QTS$ from (i) and exterior angle theorem in ΔQTS.

[Sadly (for the marker), lots of alternatives]

(b) $f'(x) = \frac{1}{1+x^2} - 1$

$\therefore x_1 = 1\cdot2 - \dfrac{\tan^{-1} 1\cdot2 - 1\cdot2 + 0\cdot3}{\frac{1}{1+1\cdot2^2} - 1} = 1\cdot16$, 2 d.p

(c) (i) $v = 0 \Rightarrow \sin 3t = 0$

$\therefore 3t = \pi$ for next at rest

i.e. $t = \frac{\pi}{3}$

(ii) From (i), period $= 2 \times \frac{\pi}{3} = \frac{2\pi}{3}$

(iii) $\frac{dx}{dt} = 6 \sin 3t \quad \therefore x = -\frac{6 \cos 3t}{3} + c$

$: -1 = -2 + c, \; c = 1$

$\therefore x = 1 - 2\cos 3t$

[$\Rightarrow$ particle oscillates about $x = 1$]

(iv) From (iii), at $x = 1$ when $t = \frac{1}{2} \cdot \frac{\pi}{3} = \frac{\pi}{6}$

[for first time]

Question 4

(a) (i) $LS = \cos\frac{\pi}{4}\cos\frac{\pi}{12} + \sin\frac{\pi}{4}\sin\frac{\pi}{12} - \left(\cos\frac{\pi}{4}\cos\frac{\pi}{12} - \sin\frac{\pi}{4}\sin\frac{\pi}{12}\right)$

$$= 2\sin\frac{\pi}{4}\sin\frac{\pi}{12}$$

(ii) $\sin\frac{\pi}{6} + \sin\frac{\pi}{3} = \sin\left(\frac{\pi}{4} - \frac{\pi}{12}\right) + \sin\left(\frac{\pi}{4} + \frac{\pi}{12}\right)$

$$= 2\sin\frac{\pi}{4}\cos\frac{\pi}{12}$$

$$\therefore LS = \frac{2\sin\frac{\pi}{4}\sin\frac{\pi}{12}}{2\sin\frac{\pi}{4}\cos\frac{\pi}{12}} = \tan\frac{\pi}{12}$$

(iii) From (ii), $\tan\frac{\pi}{12} = \dfrac{\frac{\sqrt{3}}{2} - \frac{1}{2}}{\frac{1}{2} + \frac{\sqrt{3}}{2}} = \dfrac{\sqrt{3}-1}{\sqrt{3}+1}$

$$= \frac{(\sqrt{3}-1)^2}{3-1} = \frac{4-2\sqrt{3}}{2} = 2-\sqrt{3}$$

(b) $\dfrac{dS}{dt} = k$ & $\dfrac{dV}{dt} = \dfrac{dV}{dr}\cdot\dfrac{dr}{dt} = \dfrac{dV}{dr}\cdot\dfrac{dr}{dS}\cdot\dfrac{dS}{dt}$

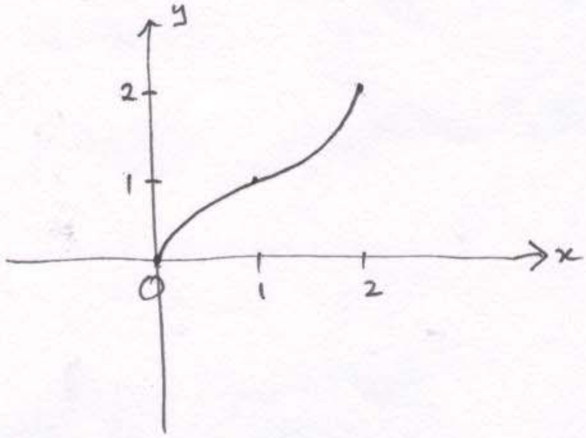
where $\dfrac{dS}{dr} = 8\pi r$ and $\dfrac{dV}{dr} = 4\pi r^2$

$$\therefore \frac{dV}{dt} = 4\pi r^2 \cdot \frac{1}{8\pi r}\cdot k = \frac{k}{2}\cdot r \propto r$$

(c) (i) We need $|1-x| \le 1$

$\Rightarrow -1 \le x-1 \le 1$ ie. domain is $0 \le x \le 2$

(ii)



Question 5

(a) For $n = 1$, $LS = 1^3 + 3 \times 1^5 = 4$

$RS = \frac{1}{2} \cdot 2^3 = 4$

$\therefore$ Assume $(1^3 + 3.1^5) + \dots + (n^3 + 3n^5) = \frac{1}{2} n^3 (n+1)^3$ for some integer $n \geq 1$

Then $(1^3 + 3.1^5) + \dots + (n^3 + 3n^5) + ((n+1)^3 + 3(n+1)^5)$

$= \frac{1}{2} n^3 (n+1)^3 + (n+1)^3 + 3(n+1)^5$, using the assumption

$= \frac{1}{2} (n+1)^3 \left(n^3 + 2 + 6(n+1)^2\right)$

$= \frac{1}{2} (n+1)^3 (n^3 + 6n^2 + 12n + 8)$

$= \frac{1}{2} (n+1)^3 (n+2)^3 = RS$ for $n+1$

$\therefore$ if correct for n it's correct for $n+1$

But, it is correct for $n = 1$ $\therefore$ by induction it's correct

(b) (i) $$\frac{c_k}{c_{k-1}} = \frac{\binom{n}{k}}{\binom{n}{k-1}} = \frac{n!\,(n-k+1)!\,(k-1)!}{(n-k)!\,k!\,n!} = \frac{n-k+1}{k}$$

(ii) From (i), $$\sum_{k=1}^{n} \frac{k c_k}{c_{k-1}} = \sum_{k=1}^{n} (n-k+1)$$

$= n + (n-1) + \dots + 2 + 1$, arithmetic series

$= \frac{n}{2}(n+1)$

(c) (i) $f(-1) = -1 + A + 2 - A - 1 = 0$

$\therefore x+1$ is a factor of $f(x)$ $\forall A$

(ii) From (i) and $f(x)$ we must have

$$f(x) = (x+1)^2(x-1)$$

$$\therefore f(1) = 1 + A + 2 + A - 1 = 0$$

$$\Rightarrow A = -1$$

(iii) From (i),

$$f(x) = (x+1)(x^2 - Ax - 1) \equiv x^3 + (A+2)x^2 + Ax - 1$$

$\therefore$ equating coefficients of x

$$-1 - A = A$$

$$\therefore A = -\tfrac{1}{2}$$

[Alternatives abound]

e.g. Sum of roots $= -1 + A = -(A+2)$

or put $x = 1$ ... $2(-A) = A + 2 + A$

etc

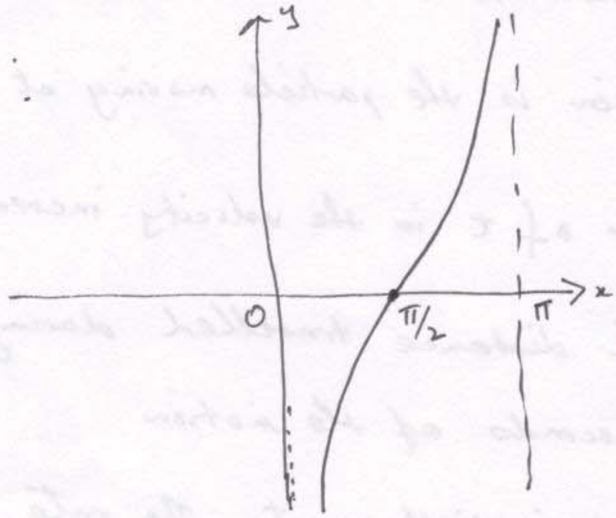
Question 6

(a) (i) $f'(x) = \dfrac{1}{\tan\frac{x}{2}} \cdot \frac{1}{2}\sec^2\frac{x}{2}$

$$= \frac{1+\tan^2\frac{x}{2}}{2\tan\frac{x}{2}} = \operatorname{cosec} x$$

since $\sin x = \dfrac{2t}{1+t^2}$ if $t = \tan\frac{x}{2}$

(ii) $y=0 \Rightarrow \tan\frac{x}{2} = 1 \Rightarrow \frac{x}{2} = \frac{\pi}{4}$, $x = \frac{\pi}{2}$



[Note, $\operatorname{cosec} x > 0$ if $0 < x < \pi$]

(iii) $f^{-1}(x)$: $x = \ln(\tan\frac{y}{2})$

$\therefore \tan\frac{y}{2} = e^x$

$\Rightarrow \frac{y}{2} = \tan^{-1}(e^x)$

ie $y = 2\tan^{-1}(e^x)$

(b) (i) $\frac{1}{2}v^2 = \frac{1}{2}(20-x)^2$

$$\therefore \ddot{x} = \frac{d(\frac{1}{2}v^2)}{dx} = \frac{1}{2}\cdot 2(20-x)(-1)$$

$$= -(20-x) = -v$$

(ii) $\therefore \frac{dv}{dt} = -v$

$$\Rightarrow \frac{dt}{dv} = -\frac{1}{v} \quad \text{or} \quad -\frac{dt}{dv} = \frac{1}{v}$$

$$\therefore -t = \ln v + c \quad : t=0,\ x=19,\ v=1$$

$$\therefore 0 = \ln 1 + c, \quad c=0$$

$$\therefore \ln v = -t$$

$$\Rightarrow v = e^{-t}$$

Question 7

(a) (i) at P, $\dot{y} = 0$

$$\therefore -10T + V\sin\alpha = 0 \quad \text{ie} \quad T = \frac{V\sin\alpha}{10}$$

(ii) $$\tan\beta = \frac{PQ}{OQ} = \frac{-5T + V\sin\alpha}{V\cos\alpha}$$

$$= \frac{-\frac{V\sin\alpha}{2} + V\sin\alpha}{V\cos\alpha}$$

$$= \frac{\sin\alpha}{2\cos\alpha} = \frac{1}{2}\tan\alpha$$

(iii) $$h = -5\left(\frac{V\sin\alpha}{10}\right)^2 + V\sin\alpha\left(\frac{V\sin\alpha}{10}\right)$$

$$= -\frac{V^2\sin^2\alpha}{20} + \frac{V^2\sin^2\alpha}{10} = \frac{V^2\sin^2\alpha}{20}$$

$$\therefore 20h = V^2\sin^2\alpha$$

(iv) From (iii) $$\frac{V^2}{5h} = \frac{4}{\sin^2\alpha} = 4\operatorname{cosec}^2\alpha$$

$$= 4(1 + \cot^2\alpha)$$

$$= 4 + (2\cot\alpha)^2$$

$$= 4 + \cot^2\beta, \text{ from (ii)}$$

$$\therefore V^2 = 5h(4 + \cot^2\beta)$$

(b) (i) From data, tangent at $P(2p, p^2)$ is

$$2p\,x = 2(y + p^2)$$

i.e. $px = y + p^2$

(ii) Tangent at Q is $qx = y + q^2$

$\therefore$ at T, $(p-q)x = p^2 - q^2 = (p-q)(p+q)$

$$\Rightarrow x = p + q$$

$$\therefore y = p(p+q) - p^2 = pq$$

i.e. $T = (p+q, pq)$

(iii) PQ is the chord of contact from T

$\therefore$ from data, PQ is $(p+q)x = 2(y + pq)$

(iv) $R(x_0, y_0)$ is on PQ

$\therefore$ from (iii), $(p+q)x_0 = 2(y_0 + pq)$

This last equation means that

$T(p+q, pq)$ is on the line $x_0 x = 2(y + y_0)$

ie T lies on the line containing the chord of contact from R