THE KING'S SCHOOL

2011
Higher School Certificate
Trial Examination

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 84

- Attempt Questions 1-7
- All questions are of equal value

Examination Paper continues on the next page

Total marks – 84
Attempt Questions 1-7
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find the derivative of $\tan(\ln x)$. **2**

(b) A pair of parametric equations for a parabola is $x = 2t, \; y = -2t^2$

Find the Cartesian equation of the parabola. **2**

(c) Find the smallest integer n for which $0.2^n < 12^{-2011}$ **2**

(d) Let $P(x) = x^3 - Ax^2 + Ax + 5$

(i) Find the remainder when $P(x)$ is divided by $x - 1$ **1**

(ii) Find the value of A if $x + 1$ is a factor of $P(x)$ **2**

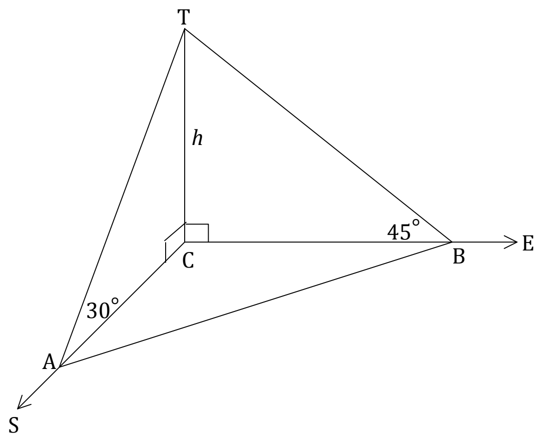
(e) The equation $f(x) = \dfrac{x^3}{3} + x^2 - 5x - 1 = 0$ has a root near $x = 3$.

Use one application of Newton's method to find an improved value for this root. **3**

End of Question 1

Question 2 (12 marks) Use a SEPARATE writing booklet **Marks**

(a)



A vertical tower CT of height h is due North of A and due West of B on horizontal ground ABC.

The elevations to the top T of the tower from A and B are 30° and 45°, respectively.

(i) Show that AC = $\sqrt{3}\ h$ **1**

(ii) Find the bearing of B from A **2**

(b) Sketch the graph of the function $y = -2\sin^{-1}\left(\frac{x}{2}\right)$ clearly indicating the domain and range. **3**

(c) Evaluate $\int_0^{\frac{\pi}{2}} 2\sin^2 x - 1 + \sin x\, dx$ **3**

(d) Solve the inequality $\frac{2x - 1}{x + 1} < -1$ **3**

End of Question 2

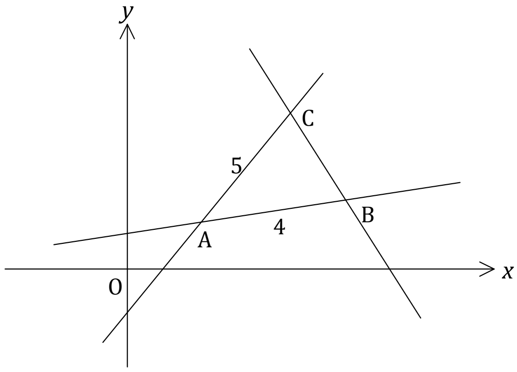
Question 3 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) AC has equation $y = 3x - 2$

AB has equation $y = \frac{1}{2}x + 1$

Lines AC and AB meet line BC so that AC = 5 and AB = 4

Find the area of ΔABC **3**

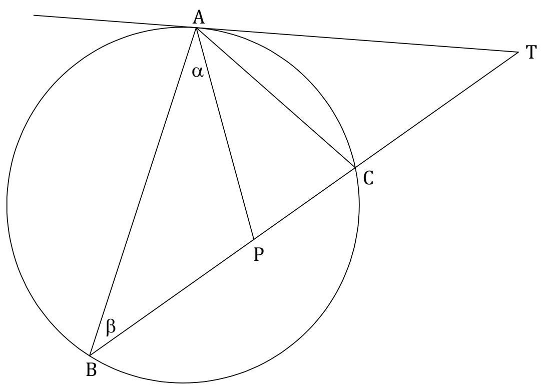


(b) AT is a tangent to the circle ABC

BCT is a straight line

P is the point on BCT so that PA bisects $\angle BAC$

Let $\angle BAP = \alpha$ and $\angle ABP = \beta$



(i) Explain why $\angle APC = \alpha + \beta$ **1**

(ii) Prove that PT = AT **3**

(iii) Explain why $PT^2 = BT \times TC$ **1**

(c) Use the substitution $x = u^2$ to show that $\int_1^3 \frac{6}{(1 + x)\sqrt{x}}\,dx = \pi$ **4**

End of Question 3

Question 4 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Let $f(x) = \dfrac{2\sqrt{x}}{x + 1}$

(i) State the domain of the function. **1**

(ii) Find any stationary points and determine their nature. **3**

(iii) Sketch the graph of $y = f(x)$

[YOU DO NOT NEED TO FIND POINTS OF INFLECTION] **2**

(b) $P(x) = x^3 + 4x^2 + Ax + 10 = 0$ has three real roots α, β, γ.

Two of these roots have a sum of 6.

Find the values of the roots. **3**

(c) The coefficient of x in the binomial expansion of $\left(x + \dfrac{a}{x^2}\right)^{10}$ is 15.

Find the value of a. **3**

End of Question 4

Question 5 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Prove by mathematical induction for positive integers n that
$1 + 3 + 7 + \ldots + (2^n - 1) = 2^{n+1} - (n + 2)$ **3**

(b) A particle is moving on the x axis with its velocity v given by $v^2 = 2(8 - 2x - x^2)$

(i) Prove that the motion is simple harmonic. **2**

(ii) State the period of the motion. **1**

(iii) Find the amplitude of the motion. **2**

(c) Let $f(x) = -\ln(\sqrt{x} - 1)$

(i) For what values of x is $f(x) > 0$? **2**

(ii) Given that $f(x)$ decreases for all values of x in its domain, find explicitly the inverse function $y = f^{-1}(x)$ **2**

End of Question 5

Question 6 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find the possible values of $\tan\theta$ if $9\cos 2\theta + 7\sin 2\theta = 11$ **2**

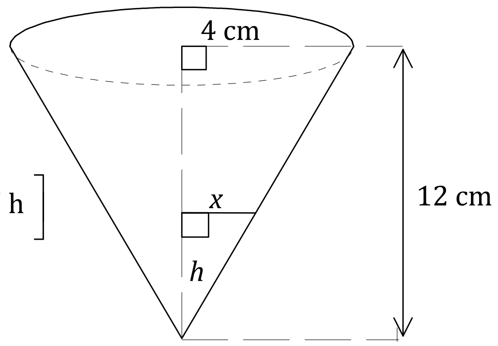
(b) The velocity, *v m/s*, for a particle moving on the *x* axis is given by $v = \frac{1}{6}\left(3 + 5e^{-2x}\right)$

Initially the particle is at $x = 0$

Find the time taken for the particle to travel 1 metre. **3**

(c)

$\left[V = \frac{1}{3}\pi r^2 h \right]$



A cone of radius 4 cm and height 12 cm is being filled with water at a constant rate of 2 cm³/s.

Let the depth of the water in the cone after *t s* be *h* cm and the radius of the surface water be *x* cm.

(i) Show that $x = \frac{h}{3}$ **1**

(ii) Find the rate at which the depth is increasing when the depth is 4 cm. **3**

(iii) Now suppose that the cone is being filled at a constant rate of *k* cm³/s and also is leaking at the vertex of the cone at a variable rate of $\frac{\sqrt{h}}{10}$ cm³/s.

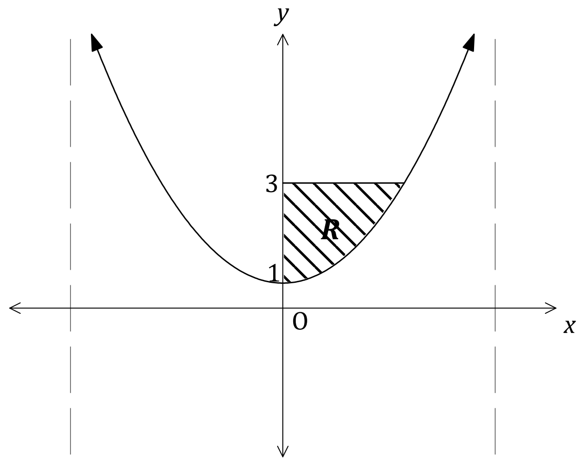
It is observed that when the depth is 4 cm the depth is increasing at the rate of 0.036 cm/s. **3**

Prove that the cone will eventually fill.

End of Question 6

Question 7 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a)



The sketch shows the shaded region ***R*** bounded by the curve $y = \dfrac{3}{\sqrt{\left(9 - 2x^2\right)}}$

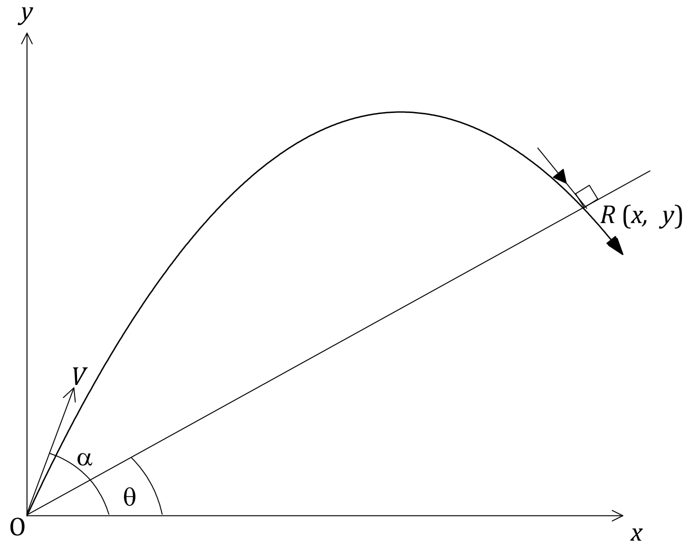
and the y axis between $y = 1$ and $y = 3$.

(i) Find the equations of the vertical asymptotes to the curve. **1**

(ii) Find the area of the region ***R***. **4**

Question 7 continues on the next page

(b)



A particle is projected from O at an angle of elevation α with velocity V up an inclined plane which makes an angle θ with the horizontal.

The particle hits the plane at right angles at $R(x,\ y)$ at time T.

In usual notations, you may assume that

$\dot{x} = V\cos\alpha$ $\qquad\qquad$ $\dot{y} = -gt + V\sin\alpha$

$x = V\cos\alpha\, t$ $\qquad\qquad$ $y = -g\dfrac{t^2}{2} + V\sin\alpha\, t$

(i) Show that at $R,\ y = x\tan\theta$ **1**

(ii) Deduce that $gT = 2V(\sin\alpha - \cos\alpha\tan\theta)$ **2**

(iii) Show that the horizontal component of velocity at $R(x,\ y)$ makes an angle θ with the inclined plane. **1**

(iv) Prove that $\tan\alpha = \cot\theta + 2\tan\theta$ **3**

End of Examination Paper

Standard Integrals

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

Note: $\ln x = \log_e x, \quad x > 0$

Student Number

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Question	Algebra and Number		Geometry		Functions		Trigonometry		Differential Calculus		Integral Calculus		Total
1	*c*	**2**			*b, d, e*	**8**			*a*	**2**			**12**
2	*d*	**3**			*b*	**3**	*a*	**3**			*c*	**3**	**12**
3			*b*	**5**	*a*	**3**					*c*	**4**	**12**
4					*a-i, iii, b, c*	**9**			*a-ii*	**3**			**12**
5	*a*	**3**			*c*	**4**			*b*	**5**			**12**
6							*a*	**2**	*b, c*	**10**			**12**
7					*a-i*	**1**			*b*	**7**	*a-ii*	**4**	**12**
Total		**8**		5		**28**		**5**		**27**		**11**	**84**

TKS Extension 1 Trial H.S.C. 2011 Solutions

1) (a) $\dfrac{\sec^2(\ln x)}{x}$

(b) $t = \dfrac{x}{2} \Rightarrow y = -2\left(\dfrac{x}{2}\right)^2$ i.e. $y = -\dfrac{x^2}{2}$ or $x^2 = -2y$

(c) $\therefore \ln 0{\cdot}2^n < \ln 12^{-2011}$

$\Rightarrow n \ln 0{\cdot}2 < -2011 \ln 12$

$\therefore n > \dfrac{-2011 \ln 12}{\ln 0{\cdot}2}$ since $\ln 0{\cdot}2 < 0$

$= 3104{\cdot}9\ldots$

$\therefore$ least n is 3105

(d) (i) $R = P(1) = 1 - A + A + 5 = 6$

(ii) $R = P(-1) = -1 - A - A + 5 = 0$

$\therefore 2A = 4$

$A = 2$

(e) $f'(x) = x^2 + 2x - 5$

$f(3) = 9 + 9 - 15 - 1 = 2$, $f'(3) = 9 + 6 - 5 = 10$

$\therefore x_1 = 3 - \dfrac{2}{10} = 2{\cdot}8$

Question 2

(a) (i) In ΔACT, $\tan 30^\circ = \frac{h}{AC} = \frac{1}{\sqrt{3}}$

$\therefore AC = h\sqrt{3}$

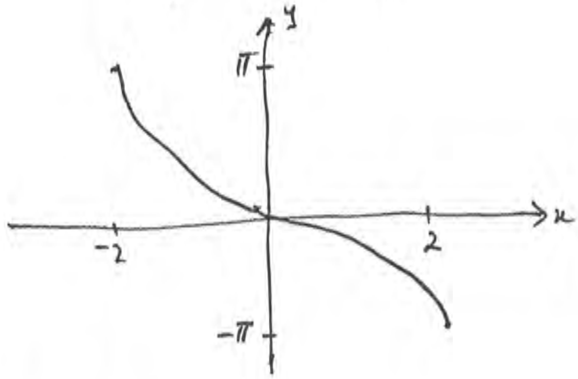
(ii) From ΔBCT, $CB = h$

$\therefore$ In ΔACB, $\tan A = \frac{h}{h\sqrt{3}} = \frac{1}{\sqrt{3}}$

$\therefore$ bearing of B from A is 30°

(b) $-1 \le \frac{x}{2} \le 1 \Rightarrow -2 \le x \le 2$

and range $-\pi \le y \le \pi$



(c) $I = \int_0^{\frac{\pi}{2}} -\cos 2x + \sin x \, dx$

$= \left[-\frac{\sin 2x}{2} - \cos x \right]_0^{\pi/2} = 0 - (0 - 1) = 1$

(d) (LOTS OF ALTERNATIVES)

if $x + 1 > 0$ i.e. $x > -1$ then $2x - 1 < -x - 1$

$\Rightarrow x < 0$ $\therefore -1 < x < 0$

& if $x < -1$ then $x > 0$ has no solution

$\therefore -1 < x < 0$

Question 3

(a) gradient $AC = 3$, grd $AB = \frac{1}{2}$

$$\therefore \tan \angle CAB = \frac{3 - \frac{1}{2}}{1 + 3.\frac{1}{2}} = \frac{6-1}{2+3} = 1 \Rightarrow \angle CAB = \frac{\pi}{4}$$

$$\therefore \text{Area } \Delta ABC = \frac{1}{2} . 5 . 4 . \sin\frac{\pi}{4} = \frac{10}{\sqrt{2}} \quad \text{will do}$$

$$= 5\sqrt{2}$$

(b) (i) $\angle APC = \angle BAP + \angle ABP$, ext $\angle$ thm in ΔABP

$= \alpha + \beta$

[or, simply, ext $\angle$ thm in ΔABP]

(ii) $\angle CAP = \angle BAP = \alpha$, PA bisects $\angle BAC$

$\angle TAC = \angle ABP = \beta$, alt seg thm

$\therefore$ From (i), in ΔAPT, $\angle P = \angle A = \alpha + \beta$

$\therefore \Delta APT$ is isosceles [base angles =]

$\Rightarrow PT = AT$

(iii) Now $TA^2 = BT \times TC$, intersecting chord + tangent thm

$= PT^2$ from (ii)

(c) $x = u^2$ $\quad$ $x = 1,\ u = 1$

$\frac{dx}{du} = 2u$ $\quad$ $x = 3,\ u = \sqrt{3}$

$$\therefore I = \int_1^{\sqrt{3}} \frac{6 . 2u}{u(1+u^2)}\, du$$

$$= 12\int_1^{\sqrt{3}} \frac{1}{1+u^2}\, du$$

$$= 12\left[\tan^{-1} u\right]_1^{\sqrt{3}}$$

$$= 12\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \pi$$

Question 4

(a) (i) $x \geq 0$

(ii) $f'(x) = \dfrac{(x+1).2.\frac{1}{2}.x^{-\frac{1}{2}} - 2\sqrt{x}.1}{(x+1)^2}$

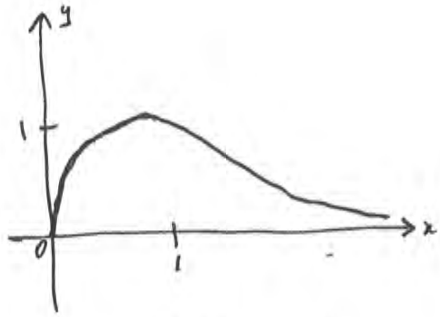
$= \dfrac{x+1-2x}{\sqrt{x}\,(x+1)^2} = \dfrac{1-x}{\sqrt{x}\,(x+1)^2} = 0$ if $x=1$, $f(1)=1$

Now $f'(x) > 0$ if $x<1$ and $f'(x)<0$ if $x>1$ $\Rightarrow$ ╱╲

ie maximum turning point at $(1,1)$

(iii) $f(0)=0$ & clearly $f(x)>0$ for $x>0$

Further, $\lim\limits_{x\to\infty} \dfrac{2\sqrt{x}}{x+1} = \lim\limits_{x\to\infty} \dfrac{2\sqrt{x}}{x} = \lim\limits_{x\to\infty} \dfrac{2}{\sqrt{x}} = 0$



(b) $\alpha+\beta+\gamma = 6+\gamma = -4 \quad \therefore \gamma = -10$

product of roots $= \alpha\beta\gamma = -10\alpha\beta = -10$

$\therefore \alpha+\beta = 6$ and $\alpha\beta = 1$

$\therefore \alpha(6-\alpha) = 1$ ie $\alpha^2 - 6\alpha + 1 = 0$

$\therefore \alpha = \dfrac{6 \pm \sqrt{36-4}}{2} = \dfrac{6 \pm 4\sqrt{2}}{2} = 3 \pm 2\sqrt{2}$

ie roots are -10, $3 \pm 2\sqrt{2}$

(c) $u_{k+1} = \binom{10}{k} x^{10-k} \left(\dfrac{a}{x^2}\right)^k = \binom{10}{k} a^k x^{10-k-2k}$

$= \binom{10}{k} a^k x^{10-3k} \Rightarrow$ coefficient of x occurs when $10-3k=1$ or $k=3$

$\therefore \binom{10}{3} a^3 = 15 \Rightarrow a^3 = \dfrac{15}{120} = \dfrac{1}{8}$

$\therefore a = \dfrac{1}{2}$

Question 5

(a) For $n=1$, $LS = 2-1 = 1$, $RS = 4-3 = 1$

$\therefore$ Assume $1+3+7+\dots+(2^n-1) = 2^{n+1}-(n+2)$ for some integer $n \geq 1$

Then $1+3+7+\dots+(2^n-1)+(2^{n+1}-1)$

$= 2^{n+1}-(n+2)+2^{n+1}-1$ using the assumption

$= 2.2^{n+1}-(n+3)$

$= 2^{n+2}-(n+3)$

$\therefore$ by induction it's true.

(b) (i) $\frac{1}{2}v^2 = 8-2x-x^2$

$\therefore \ddot{x} = \frac{d(\frac{1}{2}v^2)}{dx} = -2-2x = -2(x+1)$

is of the form $-n^2(x-b)$ $\therefore$ SHM

(ii) $n^2 = 2$, $n = \sqrt{2}$ $\therefore$ period $= \frac{2\pi}{\sqrt{2}} = \sqrt{2}\,\pi$

(iii) $v = 0 \Rightarrow x^2+2x-8 = 0$

$(x-2)(x+4) = 0$

ie $x = 2, -4$

$\therefore$ amp $= \frac{1}{2}.6 = 3$

(c) (i) We need $\ln(\sqrt{x}-1) < 0 \Rightarrow 0 < \sqrt{x}-1 < 1$

or $1 < \sqrt{x} < 2$

$\therefore$ for $1 < x < 4$

(ii) $y = f^{-1}(x)$: $x = -\ln(\sqrt{y}-1)$

ie $\ln(\sqrt{y}-1) = -x$

$\therefore \sqrt{y}-1 = e^{-x}$

$\sqrt{y} = 1+e^{-x}$

ie $f^{-1}(x)$: $y = (1+e^{-x})^2$

Question 6

(a) For ease, put $\tan\theta = t$

Then $9\dfrac{(1-t^2)}{1+t^2} + 7\cdot\dfrac{2t}{1+t^2} = 11$

$\therefore\ 9 - 9t^2 + 14t = 11 + 11t^2$

or $20t^2 - 14t + 2 = 0$

ie $10t^2 - 7t + 1 = 0$

$\therefore\ (5t - 1)(2t - 1) = 0$

ie $\tan\theta = \frac{1}{5}$ or $\frac{1}{2}$

(b) $\dfrac{dx}{dt} = \dfrac{3 + 5e^{-2x}}{6}$

$\therefore\ \dfrac{dt}{dx} = \dfrac{6}{3 + 5e^{-2x}} = \dfrac{6e^{2x}}{3e^{2x} + 5}$

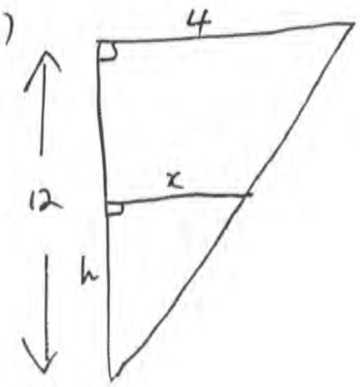
$\therefore\ t = \displaystyle\int_0^1 \frac{6e^{2x}}{3e^{2x} + 5}\,dx$ * Note $v > 0\ \forall\ x, t$

$= \left[\ln(3e^{2x} + 5)\right]_0^1$

$= \ln(3e^2 + 5) - \ln 8$ will do

$= \ln\left(\dfrac{3e^2 + 5}{8}\right)$

(c) (i)



From similar Δs,

$\dfrac{x}{4} = \dfrac{h}{12} \Rightarrow x = \dfrac{h}{3}$

(ii) $V = \frac{1}{3}\pi r^2 h \Rightarrow V = \frac{1}{3}\pi\left(\frac{h}{3}\right)^2 h = \frac{\pi h^3}{27}$

$$\therefore \frac{dV}{dh} = \frac{\pi h^2}{9}$$

Now $\frac{dh}{dt} = \frac{dh}{dV} \cdot \frac{dV}{dt}$

$$= \frac{9}{\pi h^2} \cdot 2 = \frac{18}{\pi \times 4^2} \quad \text{when } h = 4$$

$$= \frac{9}{8\pi} \text{ cm/s}$$

(iii) Now $\frac{dV}{dt} = k - \frac{\sqrt{h}}{10}$

and $\therefore \frac{dh}{dt} = \frac{9}{\pi h^2}\left(k - \frac{\sqrt{h}}{10}\right)$

$$\therefore 0.036 = \frac{9}{\pi \times 4^2}\left(k - \frac{2}{10}\right)$$

$$\Rightarrow k = 16\pi \times 0.004 + 0.2 = 0.401\ldots\ldots$$

$\therefore \frac{dh}{dt} = \frac{9}{\pi h^2}\left(0.4 - \frac{\sqrt{h}}{10}\right)$ very nearly

But max $h = 12$ & $\frac{\sqrt{12}}{10} = 0.346\ldots\ldots < 0.4$

$\therefore$ for all values of $0 < h \le 12$, $\frac{dh}{dt} > 0$

$\therefore$ cone will eventually fill

Question 7

(a) (i) Vertical asymptotes occur when $9 - 2x^2 = 0$

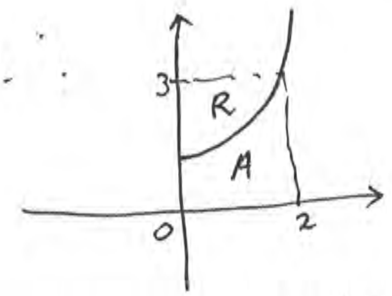
i.e. $2x^2 = 9$

$\therefore$ asymptotes are $x = \pm \dfrac{3}{\sqrt{2}}$

(ii) when $y = 3$, $9 - 2x^2 = 1$

or $2x^2 = 8$, $x^2 = 4 \Rightarrow x = 2$ for R

$\therefore$



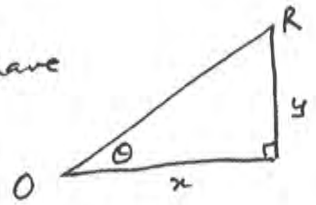
$\therefore$ area $R = 3 \times 2 - A$

$$\Rightarrow \text{area } R = 6 - \int_0^2 \frac{3}{\sqrt{9-2x^2}}\,dx$$

$$= 6 - \frac{3}{\sqrt{2}}\left[\sin^{-1}\frac{\sqrt{2}x}{3}\right]_0^2$$

$$= 6 - \frac{3}{\sqrt{2}}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$$

(b) (i) at R we have



$\therefore \tan\theta = \dfrac{y}{x}$

i.e. $y = x\tan\theta$

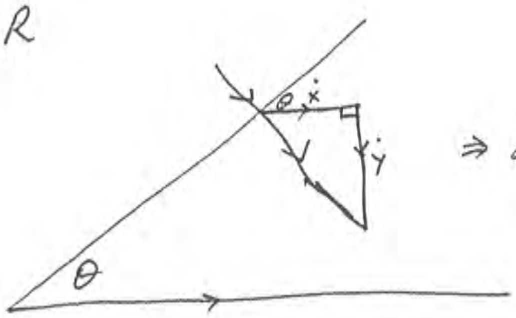
(ii) at R, $x = V\cos\alpha\, T$ and $y = -g\dfrac{T^2}{2} + V\sin\alpha\, T = x\tan\theta$

$$\therefore -g\frac{T^2}{2} + V\sin\alpha\, T = V\cos\alpha\, T\tan\theta$$

$$\Rightarrow gT = 2V\sin\alpha - 2V\cos\alpha\tan\theta$$

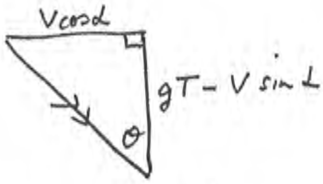
i.e. $gT = 2V(\sin\alpha - \cos\alpha\tan\theta)$

(iii) at R



⇒ angle between plane & $\dot{x}$ is θ,
corresponding ∠s in // lines
[lots of others]

(iv) at R, $\dot{x} = V\cos\alpha$, $\dot{y} = -gT + V\sin\alpha < 0$

⇒ 

$$\therefore \tan\theta = \frac{V\cos\alpha}{gT - V\sin\alpha}$$

$$\therefore gT - V\sin\alpha = \frac{V\cos\alpha}{\tan\theta} = V\cos\alpha\cot\theta$$

$\therefore$ from (ii), $V\cos\alpha\cot\theta = V\sin\alpha - 2V\cos\alpha\tan\theta$

$\therefore$ $\cot\theta = \tan\alpha - 2\tan\theta$, dividing by $\cos\alpha$

ie. $\tan\alpha = \cot\theta + 2\tan\theta$