THE KING'S SCHOOL

2012
Higher School Certificate
Trial Examination

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 70

Section I

10 marks

- Attempt Questions 1-10
- Answer on the Multiple Choice Answer Sheet provided.
- Allow about 15 minutes for this Section.

Section II

60 marks

- Attempt Questions 11-14
- Answer in the examination booklets provided, unless otherwise instructed.
- Start a new booklet for each question.
- Allow about 1 hour 45 minutes for this Section.

Section I
Total marks (10)
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the Multiple Choice Answer Sheet provided.
Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Example: $2 + 4 = ?$

(A) 2

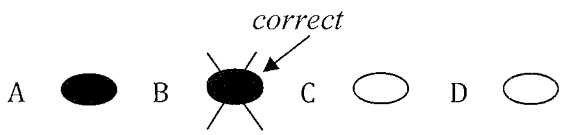
(B) 6

(C) 8

(D) 9

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

If you change your mind and have crossed out what you consider to be the correct answer, then indicate this by writing the word *correct* and drawing an arrow as follows:



1 An asymptote for the curve $y = x^2 - \dfrac{3}{x^2 - 3} - 3$ is

(A) $y = x^2$

(B) $y = x^2 - 3$

(C) $y = -\sqrt{3}$

(D) $y = -3$

2 For the polynomial equation $3 - 2x + 5x^2 - 4x^3 = 0$, the sum of its roots, when divided by the product of its roots, would be:

(A) $-\frac{4}{3}$

(B) $\frac{5}{3}$

(C) $-\frac{1}{2}$

(D) $\frac{5}{4}$

3 Find $\lim\limits_{x \to 0} \dfrac{3sin\ 7x}{5x}$

(A) $\frac{21}{5}$

(B) 3

(C) $\frac{15}{7}$

(D) 0

4 $\dfrac{d}{dx}[\cos(\ln x)] =$

(A) $-\sin(\ln x)$

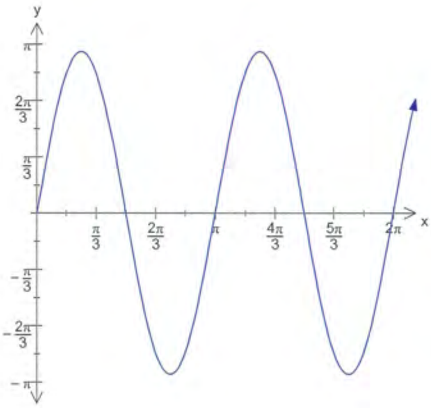
(B) $\dfrac{\cos(\ln x)}{x}$

(C) $\sin(\ln x)$

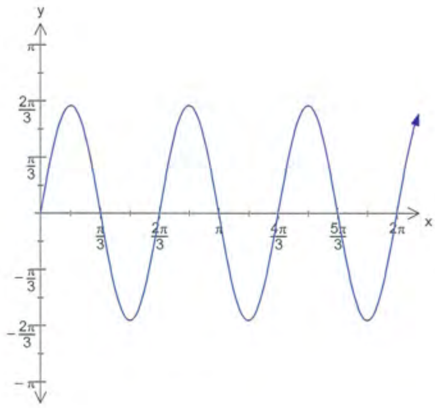
(D) $\dfrac{-\sin(\ln x)}{x}$

5 Which graph represents the curve $y = 3\sin 2x$?

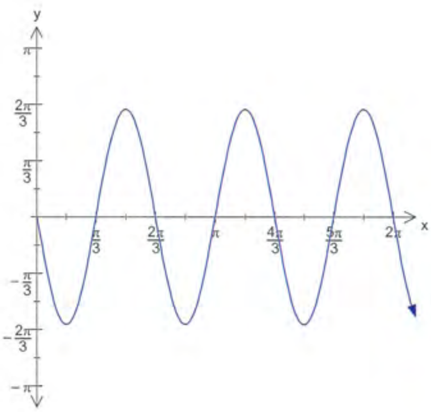
(A)



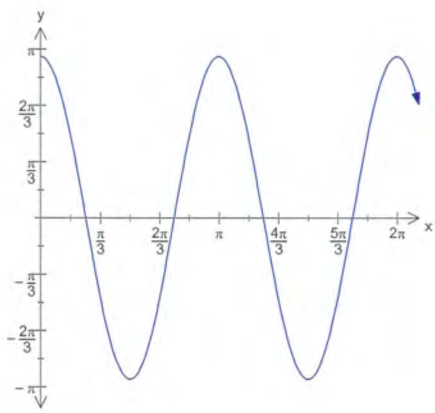
(B)



(C)



(D)



6 Find an approximation of the root of $e^x - 3x^2 = 0$ by using Newton's Method once and starting with an approximation of $x = 3.8$. Answer correct to two decimal places.

(A) 3.74

(B) 3.86

(C) 3.38

(D) 4.22

7 For the function $f(x) = \sin 2x - \cos x$, there is a zero between $x = 1$ and $x = 2$.
A reasonable approximation to this root, using the method of *Halving the Interval*, would be:

(A) 0.07

(B) 1.25

(C) 1.75

(D) 1.875

8 What is the smallest value possible for the expression $\sqrt{2}\cos x - 3\sin x$?

(A) $\sqrt{2} - 3$

(B) $-\sqrt{5}$

(C) $-\sqrt{11}$

(D) -11

9 The domain for the derivative of $y = \cos^{-1} 2x$ is:

(A) $-\frac{1}{2} \leq x \leq \frac{1}{2}$

(B) $-\frac{1}{2} < x < \frac{1}{2}$

(C) $x < \frac{1}{2}$

(D) $x < \pm\frac{1}{2}$

10 We can express $\sin x$ and $\cos x$ in terms of $\tan \frac{x}{2}$, for all values of x except ... ...

(A) $x = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}, \pm\frac{5\pi}{4}, \ldots$

(B) $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$

(C) $x = \pm\pi, \pm3\pi, \pm5\pi\ldots$

(D) $x = \pm2\pi, \pm6\pi, \pm8\pi\ldots$

End of Section I

Section II

Total marks (60)
Attempt Questions 11-14
Start a new booklet for each Question
Allow about 1 hour 45 minutes for this section

START A NEW BOOKLET
Question 11 (15 marks) **Marks**

(a) Find the obtuse angle between the lines, to the nearest degree:

$2x + 3y = 8$ and $x - 2y = -5$ **2**

(b) Given that $x = 5sin\theta$ and $y = 5cos\theta + 1$, show the equation relating x and y by eliminating θ is $y^2 + x^2 - 2y - 24 = 0$ **2**

(c) Solve the inequation $\dfrac{2x}{x^2 - 9} \geq 0$ **3**

(d) Find $\int 3x\sqrt{4 - x}\, dx$ using the substitution $u = 4 - x$ **2**

(e) Find the value of the term independent of x in the expansion $\left(2x^3 - \dfrac{1}{x}\right)^{12}$ **2**

(f) Evaluate $\int_0^{\frac{\pi}{4}} \sin x \cos^2 x \ dx$, as a surd. **2**

(g) ABCDE are points, in order, on a circle and $\angle DBC = \angle DAE$.

Draw a diagram to represent this information and prove that the triangle formed by the points CDE is isosceles. **2**

End of Question 11

START A NEW BOOKLET
Question 12 (15 marks) **Marks**

(a) Prove by induction, that

$4^n > 1 + 3n$ for $n > 1$, where n is an integer. **3**

(b) Sketch the graph of $y = cos^{-1}(x + 3)$ **2**

(c) (i) Prove that $\int_0^{\frac{\pi}{4}} sin^2 x\, dx = \frac{\pi}{8} - \frac{1}{4}$ **2**

(ii) Prove that $\frac{d}{dx}\left(x\, sin^2 x\right) \equiv x\, sin\, 2x + sin^2 x$ **1**

(iii) Hence, or otherwise, prove $\int_0^{\frac{\pi}{4}} x\, sin\, 2x\, dx = \frac{1}{4}$ **2**

(d) Calculate the exact volume generated by the solid formed when $y = \ln x - 1$ is rotated about the y-axis between $y = 0$ and $y = 1$. **2**

(e) $P(x) = x^4 - 2x^3 + 5x^2 - 16x + 12$

(i) Show that $(x - 1)(x - 2)$ is a factor of $P(x)$. **1**

(ii) Hence find the remaining factor of $P(x)$. **2**

End of Question 12

START A NEW BOOKLET
Question 13 (15 marks) **Marks**

(a) Evaluate $\int_0^1 \frac{1}{\sqrt{4 - 2x^2}}\,dx$ in exact form. **2**

(b) Storm is making a toffee dessert. The rate at which the toffee cools is proportional to the difference between the temperature of the toffee (T) and room temperature (R).

$$\text{ie, } \frac{dT}{dt} = -k(T - R)$$

(i) Show that $T = R + Ae^{-kt}$, where A is a constant, is a solution of this differential equation. **1**

(ii) Storm notices that a 2L pot of toffee initially cools from 540 °C to 100 °C in 50 minutes in a room whose temperature is 20 °C. Storm cannot put the toffee into the dessert until it reaches 40 °C.

How much longer does Storm need to wait to be able to add the toffee and finish her dessert, to the nearest minute? **3**

(iii) Show, by calculation, whether it would take more or less time to create this dessert if the room temperature were 25 °C, and k were to remain the same. **1**

(c) A particle moves in a straight line and its position at time (t) is given by:

$$x = 4 + \frac{\sin 4t}{\sqrt{3}} - \cos 4t$$

(i) Express $\frac{\sin 4t}{\sqrt{3}} - \cos 4t$ in the form $R \sin(4t - \alpha)$

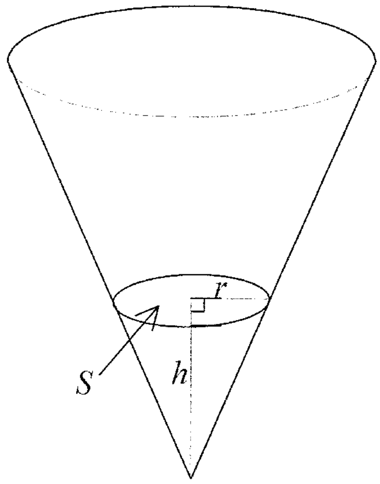
where $0 < \alpha < \frac{\pi}{2}$ and $R > 0$. **2**

(ii) The particle is undergoing Simple Harmonic Motion. Show the equation for acceleration is $\ddot{x} = -16(x - 4)$ **1**

(iii) When does the particle first reach its maximum speed? **2**

Question 13 continues on the next page

(d)



A right conical vessel, whose height is three times its radius, is inverted so that the liquid it contains escapes from its vertex at a constant rate of 12 cm^3 per second.

At time t seconds, the depth of water is h cm and the radius of the surface area of the water S cm^2, is r cm.

At what rate is S decreasing when the depth of water is 4 cm? **3**

End of Question 13

START A NEW BOOKLET

Question 14 (15 marks) **Marks**

(a) Zanthie bought a Splat Blaster that fires paint balls at a velocity of 20 ms^{-1}. A target has been placed on a tree, with its centre 2.5 m from the ground. The base of the tree is 25 m horizontally away from Zanthie.

Zanthie holds the Splat Blaster at a height of 1.5 m and wants to hit the centre of the target with a paint ball. Assume acceleration due to gravity is given by $g = 9.8\ ms^{-2}$.

(i) The equation of horizontal motion is given by $x = 20t\cos\theta$. Derive the equations of vertical motion. **1**

(ii) To avoid overhead power lines, Zanthie must fire at an angle less than $45°$. At what angle should she fire the paint ball to hit the target on the tree? Give your answer to the nearest degree. **3**

(b) A particle, which starts at the origin, with velocity $v = 2ms^{-1}$, has its acceleration described as $\frac{1}{1 + 9x^2}\ ms^{-2}$.

Find an expression for v^2 as a function of x. **2**

(c) The chord of contact of the tangents to the parabola $x^2 = 4ay$ from an external point $A(x_1, y_1)$ passes through the point $B(0, 2a)$.

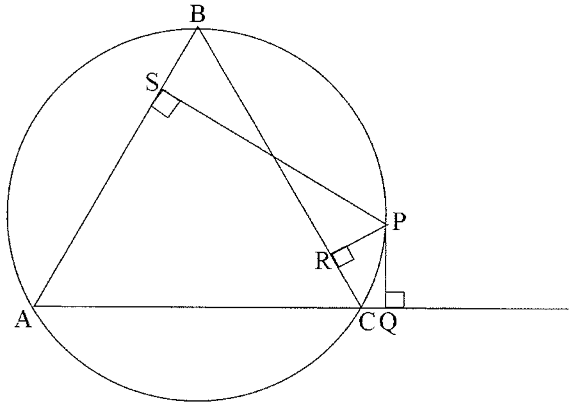
Find the equation of the locus of the midpoint of AB. **2**

(d) Given that $\sin^{-1}x$ and $\cos^{-1}x$ are acute, show that

$\sin\left(\sin^{-1}x - \cos^{-1}x\right) \equiv 2x^2 - 1$ **2**

Question 14 continues on the next page

(e)



The triangle ABC is inscribed in a circle, with P a point on the arc BC. Chord AC is produced to Q such that PQ is perpendicular to AC. PR is perpendicular to BC and PS is perpendicular to AB, as shown on the diagram.

(i) Prove that RCQP and BSRP are cyclic quadrilaterals. **2**

(ii) Prove that S, R and Q are collinear points. **3**

End of Examination

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Student Number

Multiple Choice Answer Sheet

Section I
Total marks (10)
Attempt Questions 1-10
Allow about 15 minutes for this section

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

---

1 (A) (B) (C) (D)

2 (A) (B) (C) (D)

3 (A) (B) (C) (D)

4 (A) (B) (C) (D)

5 (A) (B) (C) (D)

6 (A) (B) (C) (D)

7 (A) (B) (C) (D)

8 (A) (B) (C) (D)

9 (A) (B) (C) (D)

10 (A) (B) (C) (D)

Student Number

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Mathematics Extension 1

Question	Algebra and Number	Differential Calculus	Functions	Geometry	Integral Calculus	Trigonometry	Total
1-10		4, 9 / 2	1, 2, 6, 7 / 4			3, 5, 8, 10 /4	/10
11	c, e /5		b / 2	g /2	d, f / 4	a /2	/15
12	a /3		b, e / 5		c, d / 7		/15
13		b, c(ii), (iii), d /11			a / 2	c(i) /2	/15
14		a, b / 6	c, d / 4	e /5			/15
Total	/8	/19	/15	/7	/13	/8	/70

2012 Trial HSC Examination - Mathematics Extension 1
Multiple Choice Answer Sheet

Name ANSWERS

Completely fill the response oval representing the most correct answer.

1.	A ○	B ●	C ○	D ○
2.	A ○	B ●	C ○	D ○
3.	A ●	B ○	C ○	D ○
4.	A ○	B ○	C ○	D ●
5.	A ●	B ○	C ○	D ○
6.	A ●	B ○	C ○	D ○
7.	A ○	B ○	C ●	D ○
8.	A ○	B ○	C ●	D ○
9.	A ○	B ●	C ○	D ○
10.	A ○	B ○	C ●	D ○

2012 Ext 1 THSC SOLUTIONS

Section I

MULTIPLE CHOICE: (1 each)

1. as $x \to \infty$, $\frac{3}{x^2-3} \to 0$ $\therefore y \to x^2 - 3$
$\therefore y = x^2 - 3$ is a curved asymptote. $\therefore$ B

2. $\frac{\text{Sum of Roots}}{\text{Product of Roots}} = \frac{-b/a}{-d/a} = \frac{b}{d} = \frac{5}{3}$ $\therefore$ B

3. $\lim_{x \to 0} \frac{3\sin 7x}{5x} = \frac{3}{5}\left(\lim_{x \to 0} \frac{\sin 7x}{7x}\right) \cdot 7 = \frac{21}{5}(1) = \frac{21}{5}$ $\therefore$ A

4. $\frac{d}{dx}\cos(\ln x) = -\sin(\ln x) \cdot \frac{1}{x}$ (by Chain Rule) $\therefore$ D

5. amplitude $= 3$; Period $= \frac{2\pi}{2} = \pi$; Starts at O. $\therefore$ A

6. $f(x) = e^x - 3x^2$ $\therefore f'(x) = e^x - 6x$
$\therefore f(3 \cdot 8) = e^{3\cdot 8} - 3(3 \cdot 8)^2 \doteqdot 1 \cdot 381184$ $f'(3\cdot 8) = e^{3\cdot 8} - 6(3\cdot 8) \doteqdot 21 \cdot 90118$
$x_2 = 3\cdot 8 - \frac{f(3\cdot 8)}{f'(3\cdot 8)} \doteqdot 3.8 - \frac{1\cdot 381184}{21\cdot 90118} \doteqdot 3\cdot 7369$ $\therefore$ A

7. $f(1) = \sin 2 - \cos 1 = 0 \cdot 37$
$f(2) = \sin 4 - \cos 2 = -0 \cdot 34$
} 1st approx is $x = 1\cdot 5$. $f(1\cdot 5) = \sin 3 - \cos 1\cdot 5 \doteqdot 0 \cdot 07 > 0$
$\therefore$ New Interval $(1\cdot 5, 2)$
$\therefore$ 2nd Approx is $x = 1\cdot 75$
NB: $f(1\cdot 75) = \sin 3\cdot 5 - \cos 1\cdot 75 = -0\cdot 17 < 0$ $\therefore$ C
$\therefore$ Next approx would be in interval $(1\cdot 5, 1\cdot 75)$ & not $> 1\cdot 75$

8. If $\sqrt{2}\cos x - 3\sin x \equiv R\cos(x+\alpha)$
then $R = \sqrt{(\sqrt{2})^2 + 3^2} = \sqrt{11}$ $\therefore$ Min. value is $-\sqrt{11}$ $\therefore$ C

9. $\frac{d}{dx}\cos^{-1} 2x = -\frac{1}{\sqrt{1-4x^2}}$ $\therefore$ Domain is $1 - 4x^2 > 0$
$(1-2x)(1+2x) > 0$
$\therefore -\frac{1}{2} < x < \frac{1}{2}$ $\therefore$ B

10. Odd multiples of π, for x, will locate the undefined $\tan\frac{x}{2}$, where asymptotes would be at $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2} \ldots$ $\therefore$ C

10/10

Section II:

QUESTION 11:

(a) $m_1 = -\frac{2}{3}$ and $m_2 = \frac{1}{2}$

∴ METHOD 1:

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$= \left|\frac{-\frac{2}{3} - \frac{1}{2}}{1 + (-\frac{2}{3})(\frac{1}{2})}\right|$ 1

$= \frac{7}{4}$

∴ ACUTE $\theta = \tan^{-1}\left(\frac{7}{4}\right)$

$= 60^\circ$

∴ OBTUSE $\theta = 180 - 60^\circ$

$= 120^\circ$ 1

OR METHOD 2:

Acute $\theta = \left|\tan^{-1}(-\frac{2}{3}) - \tan^{-1}(\frac{1}{2})\right|$ 1

$= 60^\circ$

∴ Obtuse $\theta = 120^\circ$ 1

(/2)

(b) $x = 5\sin\theta$ $y = 5\cos\theta + 1$

∴ $\sin\theta = \frac{x}{5}$ ∴ $\cos\theta = \frac{y-1}{5}$ 1

Now, $\sin^2\theta + \cos^2\theta \equiv 1$

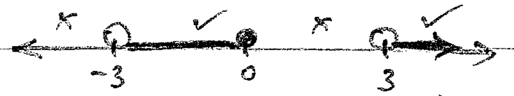
∴ $\frac{x^2}{25} + \frac{(y-1)^2}{25} = 1$

∴ $x^2 + y^2 - 2y + 1 = 25$

∴ $y^2 + x^2 - 2y - 24 = 0$ (QED) 1 (/2)

(c) $\frac{2x}{x^2 - 9} \geq 0$ [NB: $x \neq \pm 3$]

CRITICAL POINTS at $x = \pm 3$ and $x = 0$. 1



Test points in each region (eg −4, −1, 1, 4)

∴ Solution is

$-3 < x \leq 0$ or $x > 3$

1 1

(/3)

QUESTION 11 (continued)

(d) Let $I = \int 3x\sqrt{4-x}\,dx$ and let $u = 4-x$

$\therefore \frac{du}{dx} = -1$

$\therefore dx = -du$

Also, $x = 4-u$

$\therefore I = -3\int (4-u)\sqrt{u}\,du$ 1

$= 3\int (u-4)(u^{1/2})\,du$

$= 3\int (u^{3/2} - 4u^{1/2})\,du$

$= 3\left[\frac{2u^{5/2}}{5} - \frac{8u^{3/2}}{3}\right] + c$

$= \frac{6\sqrt{(4-x)^5}}{5} - 8\sqrt{(4-x)^3} + c$ 1 (/2)

$= \frac{6}{5}(4-x)^2\sqrt{4-x} - 8(4-x)\sqrt{4-x} + c$

(e) $T_{r+1} = \binom{12}{r}(2x^3)^{12-r}\left(-\frac{1}{x}\right)^r$

$= (-1)^r\binom{12}{r}\,2^{12-r}\,x^{36-4r}$ 1

Let $36 - 4r = 0$

$\therefore r = 9$

$\therefore T_{10} = (-1)^9\binom{12}{9}\,2^3$

$= -1760$ 1 (/2)

(which is independent of x)

(f) $\int_0^{\pi/4} \sin x \cos^2 x\,dx$

$= -\frac{1}{3}\left[\cos^3 x\right]_0^{\pi/4}$ 1

$= -\frac{1}{3}\left[\cos^3\left(\frac{\pi}{4}\right) - \cos^3 0\right]$

$= -\frac{1}{3}\left[\left(\frac{1}{\sqrt{2}}\right)^3 - 1\right]$

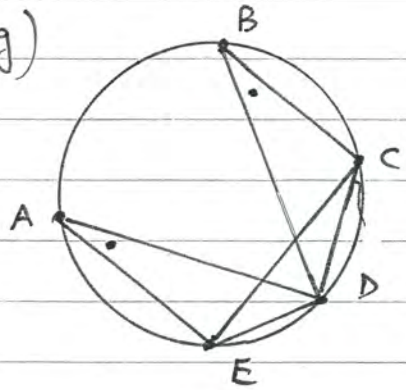
$= -\frac{1}{3}\left[\frac{1}{2\sqrt{2}} - 1\right]$

$= \frac{1}{3} - \frac{1}{6\sqrt{2}}$ 1 (/2)

$= \frac{4-\sqrt{2}}{12}$

QUESTION 11 (Cont.)

(g)



1

CD = DE (Equal chords subtending equal ∠'s at the circumference, ∠A = ∠B (Data)).

∴ △ CDE is isosceles 1
(2 equal sides, CD = DE). /2

Q11.	A:	c, e	/5
	F:	b	/2
	G:	g	/2
	I:	d, f	/4
	T:	a	/2

/15

QUESTION 12

(a) RTP: $4^n > 1 + 3n$ for $n > 1$

Show true for $n = 2$

$4^2 > 1 + 3(2)$

$16 > 7$

$\therefore$ It is true for $n = 2$. 1

Assume $S(n)$ is true

ie $4^n - 3n - 1 > 0$

If $S(n)$ is true, prove $S(n+1)$ is also true.

Now $S(n+1)$

$= 4^{n+1} - 3(n+1) - 1$ 1

$= 4(4^n) - 3n - 4$

$= 4(4^n - 3n - 1) + 9n$

> 0

Since $4^n - 3n - 1 > 0$, by the assumption

an $9n > 0 \because n > 1$.

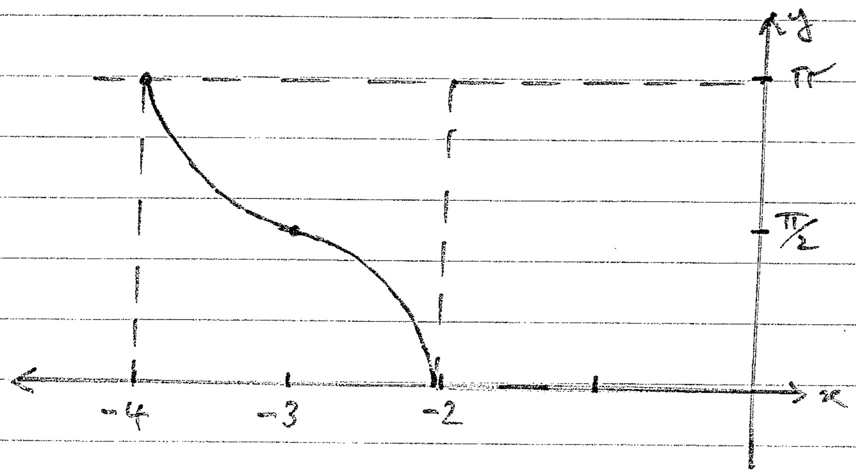
$\therefore$ If true for n, it is also true for $(n+1)$.

$\therefore$ Since true for $n = 2$, by induction it is true 1

for all integers $n > 1$.

(/3)

(b)



1 for shape and location

1 for correct Domain & Range.

(/2)

QUESTION 12 (Cont.)

(c)(i) $\int_0^{\pi/4} \sin^2 x \, dx$

$= \frac{1}{2}\int_0^{\pi/4} (1 - \cos 2x)\, dx$ 1

$= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{\pi/4}$

$= \frac{1}{2}\left[\frac{\pi}{4} - \frac{1}{2}\sin\frac{\pi}{2}\right]$

$= \frac{\pi}{8} - \frac{1}{4}$ (QED) 1 (/2)

(ii) $\frac{d}{dx}(x\sin^2 x)$ (Using the product rule)

$= \sin^2 x(1) + x(2\sin x\cos x)$

$= \sin^2 x + x\sin 2x$ (QED) (/1)

(iii) Now, from (ii),

$x\sin 2x = \frac{d}{dx}(x\sin^2 x) - \sin^2 x$

$\therefore \int_0^{\pi/4} x\sin 2x \, dx = \left[x\sin^2 x\right]_0^{\pi/4} - \left(\frac{\pi}{8} - \frac{1}{4}\right)$ from (i) 1

$= \frac{\pi}{4}\sin^2\frac{\pi}{4} - 0 - \frac{\pi}{8} + \frac{1}{4}$

$= \frac{\pi}{4}\left(\frac{1}{2}\right) - \frac{\pi}{8} + \frac{1}{4}$

$= \frac{1}{4}$ (QED) 1 (/2)

(d) $V = \pi\int_c^d x^2\, dy$ where $y = \ln x - 1$

$\therefore e^{y+1} = x$

$\therefore x^2 = e^{2y+2}$

$\therefore V = \pi\int_0^1 e^{2y+2}\, dy$ 1

$= \frac{\pi}{2}\left[e^{2y+2}\right]_0^1$

$= \frac{\pi}{2}\left(e^4 - e^2\right)$ units^3 1 (/2)

QUESTION 12 (Cont.)

(e)(i) $P(1) = 1 - 2 + 5 - 16 + 12 = 0$ $\therefore (x-1)$ is a factor of $P(x)$

$P(2) = 2^4 - 2(2^3) + 5(2^2) - 16(2) + 12$

$= 16 - 16 + 20 - 32 + 12$

$= 0$ $\therefore (x-2)$ is a factor of $P(x)$

$\therefore (x-1)(x-2)$ is a factor of $P(x)$. (/1)

(ii) Method 1 (By Inspection)

$(x-1)(x-2) = x^2 - 3x + 2$

$\therefore (x^2 - 3x + 2)(ax^2 + bx + c) \equiv x^4 - 2x^3 + 5x^2 - 16x + 12$

coeff of $x^4 \Rightarrow a = 1$

Constant term $\Rightarrow 2c = 12$ $\therefore c = 6$ 1

Coeff of $x \Rightarrow -3c + 2b = -16$

$\therefore -18 + 2b = -16$

$\therefore b = 1$

$\therefore$ Other factor is $(x^2 + x + 6)$ 1 (/2)

(which is irreducible to linear factors, where $x \in \mathbb{R}$)

Method 2 (By Polynomial Division)

$(x-1)(x-2) = x^2 - 3x + 2$

$$
\begin{array}{r}
x^2 + x + 6 \\
x^2 - 3x + 2 \,\overline{\big)\, x^4 - 2x^3 + 5x^2 - 16x + 12} \\
\underline{x^4 - 3x^3 + 2x^2} \\
x^3 + 3x^2 - 16x \\
\underline{x^3 - 3x^2 + 2x} \\
6x^2 - 18x + 12 \\
\underline{6x^2 - 18x + 12} \\
0
\end{array}
$$

1

$\therefore$ The other factor is $(x^2 + x + 6)$ 1

—//—

Q12. A: a /3
F: b, e /5
I: c, d /7

/15

QUESTION 13

(a) $\int_0^1 \frac{1}{\sqrt{4-2x^2}}\,dx$

$= \int_0^1 \frac{1}{\sqrt{2(2-x^2)}}\,dx$

$= \frac{1}{\sqrt{2}}\int_0^1 \frac{dx}{\sqrt{2-x^2}}$

$= \frac{1}{\sqrt{2}}\left[\sin^{-1}\left(\frac{x}{\sqrt{2}}\right)\right]_0^1$ 1

$= \frac{1}{\sqrt{2}}\left[\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) - \sin^{-1}0\right]$

$= \frac{1}{\sqrt{2}}\left(\frac{\pi}{4} - 0\right)$

$= \frac{\pi}{4\sqrt{2}}$

$= \frac{\pi\sqrt{2}}{8}$ 1 (2/2)

(b) (i) $T = R + Ae^{-kt}$

$\therefore \frac{dT}{dt} = -k(Ae^{-kt})$ where $Ae^{-kt} = T - R$

$= -k(T-R)$ (QED) (1/1)

(ii) $t=0,\ T=540,\ R=20 \Rightarrow 540 = 20 + Ae^0$

$\therefore A = 520$ 1

$\therefore T = 20 + 520e^{-kt}$

$\left.\begin{array}{l} t=50 \\ T=100 \end{array}\right\} \Rightarrow 100 = 20 + 520e^{-50k}$

$\frac{80}{520} = e^{-50k}$

$-\frac{1}{50}\ln\left(\frac{80}{520}\right) = k$ 1

$\therefore k \doteqdot 0.03743604$

$\therefore T = 20 + 520e^{-0.037436t}$

$T = 40 \Rightarrow 40 = 20 + 520e^{-0.037436t}$

$\frac{20}{520} = e^{-0.037436t}$

$t = -\frac{1}{0.037436}\ln\left(\frac{1}{26}\right) = 87$

$\therefore$ Extra time is $87 - 50 =$ 37 minutes 1 (3/3)

(iii) $40 = 25 + 515e^{-0.037436t}$

$\therefore t = \frac{1}{-0.037436}\ln\left(\frac{15}{515}\right) = 94$ minutes

$\therefore$ 7 minutes longer to cool (1/1)

QUESTION 13 (Cont.)

(c) (i) $\frac{1}{\sqrt{3}}\sin 4t - \cos 4t \equiv R\sin(4t-\alpha)$

$= R\sin 4t\cos\alpha - R\cos 4t\sin\alpha$

$\therefore R\cos\alpha = 1/\sqrt{3}$ and $R\sin\alpha = 1$

$R = \sqrt{\frac{1}{3}+1}$

$= \frac{2}{\sqrt{3}}$

$\therefore \tan\alpha = \frac{1}{1/\sqrt{3}} = \sqrt{3}$

$\therefore \alpha = \frac{\pi}{3}$

$\therefore \frac{1}{\sqrt{3}}\sin 4t - \cos 4t \equiv \frac{2}{\sqrt{3}}\sin\left(4t-\frac{\pi}{3}\right)$

1 for R
1 for α

/2

(ii) $x = 4 + \frac{2}{\sqrt{3}}\sin\left(4t-\frac{\pi}{3}\right)$ from (i)

$\therefore \dot{x} = \frac{8}{\sqrt{3}}\cos\left(4t-\frac{\pi}{3}\right)$

$\ddot{x} = -\frac{32}{\sqrt{3}}\sin\left(4t-\frac{\pi}{3}\right)$

$= -16\left[\frac{2}{\sqrt{3}}\sin\left(4t-\frac{\pi}{3}\right)\right]$

$= -16(x-4)$ (QED)

/1

(iii) Max speed at centre of motion, ie $x=4$.

$\therefore 4 = 4 + \frac{2}{\sqrt{3}}\sin\left(4t-\frac{\pi}{3}\right)$

$\therefore \sin\left(4t-\frac{\pi}{3}\right) = 0$ 1

$\therefore 4t-\frac{\pi}{3} = 0, \pi, 2\pi, \ldots$

$\therefore$ First reaches max speed when

$4t-\frac{\pi}{3}=0$

$4t=\frac{\pi}{3}$

$t=\frac{\pi}{12}$ seconds. 1

/2

QUESTION 13 (Cont.)

(d) $\frac{dV}{dt} = -12$ and $h = 3r$

$\frac{dS}{dt} = \frac{dS}{dh} \cdot \frac{dh}{dV} \cdot \frac{dV}{dt}$ where $S = \pi r^2$

$= \pi\left(\frac{h}{3}\right)^2$

$= \frac{\pi}{9}h^2$

$\therefore \frac{dS}{dh} = \frac{2\pi h}{9}$ 1

and $V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi\left(\frac{h^2}{9}\right)h$

$= \frac{\pi h^3}{27}$

$\therefore \frac{dV}{dh} = \frac{\pi h^2}{9}$

$\therefore \frac{dh}{dV} = \frac{9}{\pi h^2}$ 1

Triple Chain, using h:

$\therefore \frac{dS}{dt} = \frac{2\pi h}{9} \cdot \frac{9}{\pi h^2} \cdot (-12)$ when $h = 4$

$= -\frac{24}{4}$

$= -6$

$\therefore S$ is decreasing at 6 cm²/s 1 (/3)

Alternate Triple Chain: (in r)

OR $\frac{dS}{dt} = \frac{dS}{dr} \cdot \frac{dr}{dV} \cdot \frac{dV}{dt}$

$= 2\pi r \cdot \frac{1}{3\pi r^2} \cdot (-12)$

$= -\frac{8}{r}$ when $r = 4/3$

$= -8 \cdot \frac{3}{4}$

$= -6$ cm/s.

Max 1, if differentiated with both r & h.

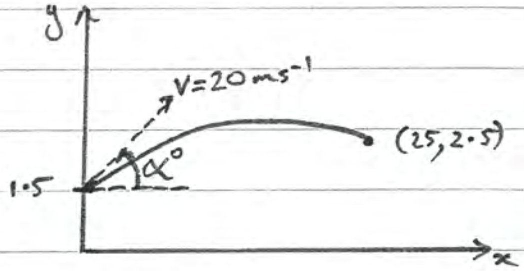
D: b, cii, iii /11

I: a /2

T: c(i) /2 /15

QUESTION 14

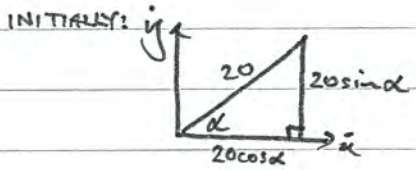
(a) (i) Vertical motion:



$$\ddot{y} = -g = -9 \cdot 8$$

$$\therefore \dot{y} = -\int 9 \cdot 8 \, dt$$

$$= -9 \cdot 8t + C_1$$



$t = 0$, $\dot{y} = 20\sin\alpha$ $\Rightarrow$ $C_1 = 20\sin\alpha$

$$\therefore y = \int (20\sin\theta - 9 \cdot 8t) \, dt$$

$$= 20t\sin\alpha - 4 \cdot 9t^2 + C_2$$

$t = 0$, $y = 1 \cdot 5$ $\Rightarrow$ $1 \cdot 5 = C_2$

[NB: $C_2 = 0$, if Origin placed at Starting Point ... and therefore $(25, 1)$ is the position of the target]

$$\therefore y = 20t\sin\alpha - 4 \cdot 9t^2 + 1 \cdot 5$$

(1)

(ii) $x = 20t\cos\alpha$ and $y = 20t\sin\alpha - 4 \cdot 9t^2 + 1 \cdot 5$

$$\therefore t = \frac{x}{20\cos\alpha} \Rightarrow \therefore y = 20\sin\alpha\left(\frac{x}{20\cos\alpha}\right) - 4 \cdot 9\left(\frac{x^2}{400\cos^2\alpha}\right) + 1 \cdot 5 \quad 1$$

$$= x\tan\alpha - \frac{49x^2}{4000}(1 + \tan^2\alpha) + 1 \cdot 5$$

$$(25, 2 \cdot 5) \Rightarrow 2 \cdot 5 = 25\tan\alpha - \frac{49(25)^2}{4000} - \frac{49(25)^2}{4000}\tan^2\alpha + 1 \cdot 5$$

$$\therefore 7 \cdot 65625\tan^2\alpha - 25\tan\alpha + 8 \cdot 65625 = 0$$

$$\therefore \tan\alpha = \frac{25 \pm \sqrt{25^2 - 4(7 \cdot 65625)(8 \cdot 65625)}}{2(7 \cdot 65625)} \quad 1$$

$$= 2 \cdot 87 \text{ or } 0 \cdot 39$$

$$\therefore \alpha = 70°47' \text{ or } 21°29'$$

$\therefore$ Zanthie must shoot at $21°$ 1

(under $45°$, to avoid the power lines)

(3)

QUESTION 14 (Cont.)

(b) $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{1}{1+9x^2}$

$\therefore \frac{1}{2}v^2 = \int \frac{1}{1+9x^2}\,dx$ 1

$= \frac{1}{9}\int \frac{1}{\frac{1}{9}+x^2}\,dx$

$= \frac{1}{9}\cdot 3\tan^{-1}(3x) + c$

$\therefore v^2 = \frac{2}{3}\tan^{-1}(3x) + k$

$\left.\begin{matrix}x=0\\ v=2\end{matrix}\right\} \Rightarrow \therefore 4 = \frac{2}{3}\tan^{-1}0 + k$

$\therefore k = 4$

$\therefore v^2 = \frac{2}{3}\tan^{-1}(3x) + 4$ 1 (/2)

(c) Chord of Contact is $xx_1 = 2a(y+y_1)$

$(0, 2a) \Rightarrow 0 = 2a(2a+y_1)$

$\therefore y_1 = -2a$ 1

$\therefore A(x_1, -2a)$ and $B(0, 2a)$

$\therefore$ Midpt of $AB = \left(\frac{x_1+0}{2}, \frac{-2a+2a}{2}\right)$

$= \left(\frac{x_1}{2}, 0\right)$

Since the y value is a constant (zero) the locus is $y=0$ 1 (/2)

Q14. D: ab /6

I: cd /4

G: e /5

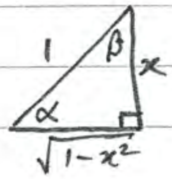
/15

QUESTION 14 (Cont.)

(d) Now, $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$

$\therefore \alpha = \sin^{-1}x$ and $\beta = \cos^{-1}x$ are complementary angles.

$\therefore \sin\alpha = x$ and $\cos\beta = x$



1

$\therefore \sin(\sin^{-1}x - \cos^{-1}x)$

$= \sin(\alpha - \beta)$

$= \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$= x.x - (\sqrt{1-x^2})(\sqrt{1-x^2})$

$= x^2 - (1-x^2)$

$= 2x^2 - 1$ (QED) 1 (/2)

(e)(i) $\angle PRC = 90°$ ($PR \perp BC$, data) and $\angle PQC = 90°$ ($PQ \perp AC$, data)

$\therefore \angle PRC + \angle PQC$
$= 90° + 90°$
$= 180°$

RCQP is a cyclic quad (one pair opposite ∠'s supplementary). 1

$\angle BSP = \angle BRP = 90°$ (perpendicular lines, data)

and since these are equal ∠'s, standing on arc PB [OR DIAMETER PB]

BSRP is a cyclic quad (equal ∠'s in same segment) 1

[OR: ∠'s in a semicircle] (/2)

(ii) Let $\angle QCP = x°$

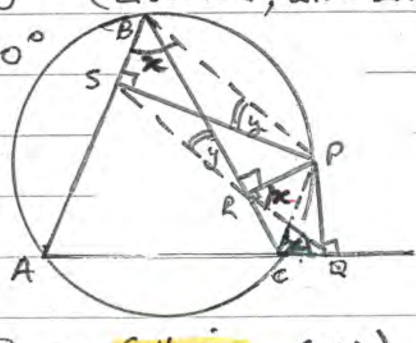
$\therefore \angle PRQ = \angle QCP = x°$ (∠'s in same segment on chord PQ in circle RCQP)

Also, $\angle ABP = \angle QCP = x°$ (exterior ∠ of Cyclic quad ABPC is equal to the interior-opposite ∠) 1

Also, $\angle SRB = \angle SPB = y°$ (∠'s in same segment, on chord SB in circle BSRP)

$\therefore$ In $\triangle SBP$, $\angle SBP + \angle SPB = 90°$ (∠ sum △, with $\angle BSP = 90°$)

ie $x° + y° = 90°$ 1



$\therefore$ At R, $\angle SRQ$

$= \angle QRP + \angle SRB + \angle BRP$

$= (x° + y°) + 90°$

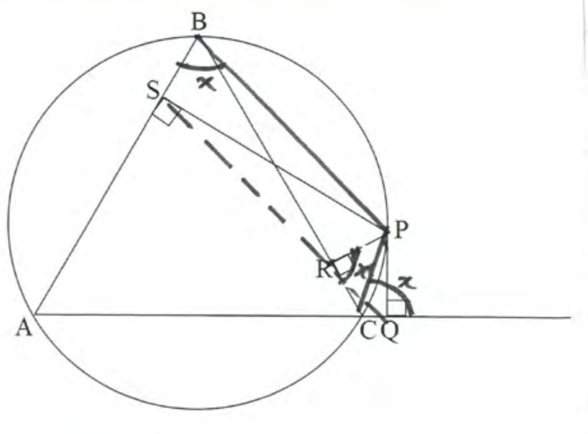
$= 90° + 90°$

$= 180°$ (/3)

$\therefore \angle SRQ$ is a straight ∠ $\therefore$ S, R & Q are Collinear (QED) 1

14(e)(ii) (Cont).

A "Better" Solution:



[As Before:]

Let $\angle QCP = x^\circ$

$\therefore \angle PRQ = \angle QCP = x^\circ$

($\angle$'s in same segment on chord PQ, in circle RCQP)

Also, $\angle ABP = \angle QCP = x^\circ$

(Exterior $\angle$ of Cyclic quad ABPC)

[The new "bit" (no need for y°).]:

$\angle SBP + \angle SRP = 180^\circ$ (opp. $\angle$'s of cyclic quad BSRP)

$\therefore \angle SRP = 180^\circ - x^\circ$

$\angle SRQ = \angle SRP + \angle PRQ$

$= (180^\circ - x^\circ) + x^\circ$

$= 180^\circ$

$=$ straight $\angle$

$\therefore$ S, R and Q are collinear

END OF EXAMINATION

Q14. D: ab /6

I: cd /4

G: e /5

/15