FKC

KNOX GRAMMAR SCHOOL
MATHEMATICS DEPARTMENT

2001
TRIAL HSC EXAMINATION

Mathematics Extension 1

Total marks (**84**)

- **General Instructions**

- Reading time – 5 minutes
- Working time – 2 hours
- Write using blue or black pen
- Board-approved calculators may be used
- A table of standard integrals is provided on page 10
- All necessary working should be shown in every question

- Attempt Questions 1–7
- All questions are of equal value
- Use a SEPARATE writing booklet for each question

NAME:____________________ TEACHER:____________________

Total marks (84)
Attempt questions 1 – 7
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) Evaluate $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$. **1**

(b) Find a primitive function of $\frac{1}{\sqrt{4-x^2}}$. **1**

(c) Show that $\frac{1+\cos 2\theta}{\sin 2\theta} \equiv \cot\theta$. **2**

(d) Solve $\frac{2}{x-4} \geq 1$. **3**

(e) Solve $\sin x - \cos x = 1$ for $0 \leq x \leq 2\pi$. **3**

(f) Find the acute angle between the lines $y = 2x - 1$ and $3x - 2y = 5$. **2**

Give your answer in radians correct to two decimal places.

Question 2 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) Find $\frac{d^2}{dx^2}\left(e^{x^2}\right)$. **2**

(b) (i) Express $\cos 2x$ completely in terms of $\sin x$. **1**

(ii) Hence or otherwise find $\int_0^{\frac{\pi}{2}} 2\sin^2 2x\, dx$. **2**

(c) Use the substitution $x = 1 - u^2$ to find $\int \frac{x}{\sqrt{1-x}}\, dx$. **3**

(d) If $\alpha, \beta,$ and γ are the roots of the cubic equation $x^3 + 2x^2 - 5x - 4 = 0$ then find the value of:

(i) $\alpha + \beta + \gamma$. **1**

(ii) $\alpha\beta\gamma$. **1**

(iii) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$. **1**

(iv) $\alpha^2 + \beta^2 + \gamma^2$. **1**

Question 3 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) Find $\int \frac{1}{4+9x^2}\,dx$. **2**

(b) Find the exact volume of the solid formed by rotating the area between the curve $y = \tan x$ and the x – axis from $x = 0$ to $x = \frac{\pi}{4}$, about the x – axis. **3**

(c) The polynomial $Q(x) = x^3 + 2x^2 + ax + b$ has a factor of $(x + 2)$. When $Q(x)$ is divided by $(x - 2)$ the remainder is 12.

Find the values of a and b. **2**

(d) Consider the function $f(x) = e^x + 4x$.

(i) Find $f'(x)$. **1**

(ii) Explain why $f(x)$ is increasing for all x? **1**

(iii) Show that $f(x) = 0$ has a root lying between $x = -1$ and $x = 0$. **1**

(iv) By letting $x = -0.5$ be an initial approximation to the real root of $f(x) = 0$, use Newton's method to find a second approximation, correct to two decimal places. **2**

Question 4 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) P is the point $(2at, at^2)$ on the parabola $x^2 = 4ay$ and the line l is tangent to the parabola at P.

(i) Represent this information on a clear and well-labelled diagram. **1**

(ii) Prove that the equation of the tangent line l is given by $y = tx - at^2$. **2**

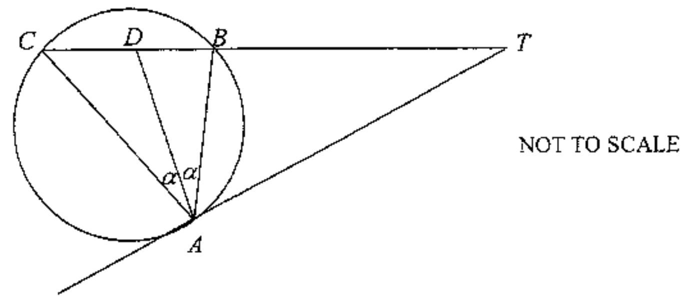
(iii) If l cuts the x – axis at A and the y – axis at B, then find the coordinates of A and B. **2**

(iv) In what ratio does the point P divide the interval AB externally? **1**

(v) Suppose Q is the midpoint of the interval PS where S is the focus of the parabola. **3**

Find the Cartesian equation of the locus of Q.

(b)



In the diagram above, TA is a tangent to the circle at A and DA bisects $\angle BAC$. **3**

Copy or trace this diagram into your writing booklet.

Prove, with reasons, that $TA = TD$.

Question 5 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) Consider the function $f(x)=(x-1)^2$.

(i) Sketch $y=f(x)$. **1**

(ii) Explain why $f(x)$ does not have an inverse function for all x in its domain? **1**

(iii) State a domain and range for which $f(x)$ has an inverse function $f^{-1}(x)$. **1**

(iv) For $x \geq 1$, find the equation of the inverse function $f^{-1}(x)$. **2**

(v) Hence on a new set of axes, sketch the graph of $y=f^{-1}(x)$. **1**

(b) A small rock is projected horizontally from the top of a vertical cliff 180 metres above sea level with a speed of projection of 35 metres per second. You may assume the acceleration g due to gravity is 10 m/s^2.

(i) Show that the equations of motion of the rock after t seconds in the horizontal and vertical directions can be given by $x=35t$ and $y=-5t^2$. **2**

(ii) Calculate the time for the rock to reach the ocean. **1**

(iii) Calculate the distance from the base of the cliff to the point where the rock strikes the surface of the ocean. **1**

(iv) Find, to the nearest degree, the angle at which the rock strikes the ocean. **2**

Question 6 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) Let T be the temperature inside a room at time t and let A be the temperature of its surrounding. Newton's Law of Cooling states that the rate of change of the temperature T is proportional to $(T-A)$.

(i) Verify that $T=A+Be^{kt}$ (where B and k are constants) satisfies Newton's Law of Cooling. **1**

(ii) The constant temperature of the surrounding is 4°C and an air conditioning system causes the temperature inside a room to drop from 25°C to 15°C in 45 minutes.

Find how long it takes for the inside room temperature to reach 8°C? **3**

(b) The displacement x (in metres) of a particle is given by $x=5\cos(4\pi t)$, where t is in seconds.

(i) Show that the acceleration of the particle can be expressed in the form: **2**

$$\ddot{x}=-n^2x.$$

(ii) State the period, T, of the motion. **1**

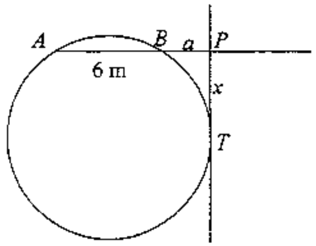
(iii) Determine the maximum velocity of the particle. **1**

(iv) Express v^2 completely in terms of x, where v is the velocity of the particle. **1**

(c) Use the Principle of Mathematical Induction to prove that $2^{10n+3}+3$ is divisible by 11 for all positive integers. **3**

Question 7 (12 marks) Use a SEPARATE writing booklet **Marks**

(a)

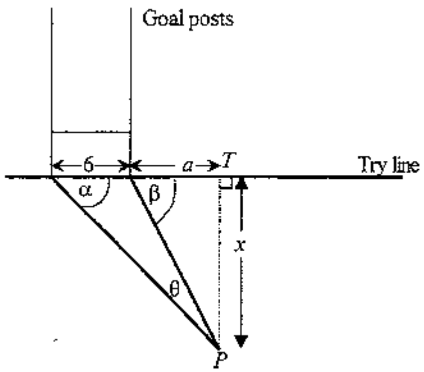


NOT TO SCALE

In the circle, the chord AB is 6 metres long. The chord is produced to the point P and BP is a metres. A tangent to the circle cuts the chord at P where PT is x metres

Show that $x = \sqrt{a(a+6)}$. **2**

(b) In a rugby game, teams score by placing the ball over the try line at the end of the field. A kicker may then take the ball back at right angles from the try line and attempt to kick the ball between the goal posts.



NOT TO SCALE

In the diagram above, a try has been scored a metres to the right of the goal posts. The kicker has brought the ball back to the point P to attempt his kick. The kicker wants to maximise θ, his angle of view of the goal posts.

Question 7(b) continues on page 9 – please turn over.

Question 7(b) continued

Let PT be x metres and assume that the goal posts are 6 metres wide.

(i) Show that $\tan\theta = \dfrac{6x}{a^2 + 6a + x^2}$. **3**

(ii) Letting $T = \tan\theta$, find the exact value of x for which T is a maximum. **2**

(iii) Hence show that the maximum angle, θ, is given by $\theta = \tan^{-1}\left(\dfrac{3}{\sqrt{a^2+6a}}\right)$. **2**

(iv) If a try is scored 10 metres to the right of the goal posts, find the maximum value of θ (to the nearest minute) and the corresponding value of x (to the nearest centimetre). **2**

(v) Explain why the goal kicker, to maximise his angle of view of the goal posts, should imagine himself at the point of contact of a tangent to the circle passing through the goal posts? **1**

End of Paper

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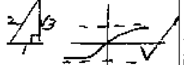
SOLUTIONS TO KNOX GRAMMAR SCHOOL MATHEMATICS EXTENSION I 2001 TRIAL HSC EXAMINATION

[✓ = ONE MARK]

QUESTION (1)



(a) $\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right) = -\frac{\pi}{3}$

(b) $\int \frac{1}{\sqrt{4-x^2}} = \sin^{-1}\frac{x}{2} + c$ ✓

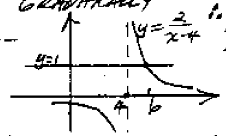
(c) $\frac{1+\cos 2\theta}{\sin 2\theta} = \cot\theta$

LHS $= \frac{1+(2\cos^2\theta - 1)}{2\sin\theta\cos\theta}$ ✓

$= \frac{2\cos^2\theta}{2\sin\theta\cos\theta} = \frac{\cos\theta}{\sin\theta} = \cot\theta = $ RHS ✓

(d) $\frac{2}{(x-4)} \geq 1$ OR Square both sides & Test

GRAPHICALLY ✓ ✓



$y = \frac{2}{x-4}$ ∴ $4 < x \leq 6$ ✓

(e) $\sin x - \cos x = 1$

Use "t" substitution OR ALTERNATIVE METHODS - GRAPHICALLY - $A\cos(x+\theta)$ TYPE

$\frac{2t}{(1+t^2)} - \frac{(1-t^2)}{(1+t^2)} = 1$

∴ $2t - (1-t^2) = 1+t^2$ ✓

$2t - 1 + t^2 = 1 + t^2$

$2t = 2$

$t = 1$

∴ $\tan\left(\frac{\theta}{2}\right) = 1$ ✓

∴ $\frac{\theta}{2} = \frac{\pi}{4}$,

$\theta = \frac{\pi}{2}$,

Test $\theta = \pi$, TRUE

∴ $\theta = \frac{\pi}{2}, \pi$ ✓

(f) $y = 2x - 1$ ∴ $m_1 = 2$

$3x - 2y = 5$ ∴ $m_2 = \frac{3}{2}$

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$= \left|\frac{2 - (3/2)}{1 + (2)(3/2)}\right| = \frac{\frac{1}{2}}{1+3} = \frac{1}{8}$ ✓

∴ $\theta = \tan^{-1}\left(\frac{1}{8}\right) \doteqdot 0.12435$ ✓

$\doteqdot 0.12$

QUESTION (2)

(a) $y = e^{x^2}$

$\frac{dy}{dx} = 2xe^{x^2}$ ✓

$\frac{d^2y}{dx^2} = 2x\left[e^{x^2} \cdot 2x\right] + [2]e^{x^2}$ ✓

$= 2e^{x^2}\{2x^2+1\}$ OR EQUIVALENT

(b) (i) $\cos 2x = 1 - 2\sin^2 x$ ✓

(ii) $\int_0^{\frac{\pi}{2}} 2\sin^2 2x = \int_0^{\frac{\pi}{2}} (1-\cos 4x)\,dx$ ✓

$= \left[x - \frac{1}{4}\sin 4x\right]_0^{\frac{\pi}{2}}$

$= \left[\left(\frac{\pi}{2} - \frac{1}{4}\sin 2\pi\right)\right] - [0] = \frac{\pi}{2}$ ✓

(c) $\int \frac{x}{\sqrt{1-x}}\,dx$ let $x = 1 - u^2$, $\frac{dx}{du} = -2u$ ✓

∴ $I = \int \frac{1-u^2}{\sqrt{1-(1-u^2)}}(-2u\,du)$ ✓

$= \int \frac{1-u^2}{\sqrt{u^2}}(-2u\,du)$ ✓

$= -2\int(1-u^2)\,du = -2\left[u - \frac{1}{3}u^3\right] + c$

$= -2\left[(1-x)^{\frac{1}{2}} - \frac{1}{3}(1-x)^{3/2}\right] + c$ ✓

(d) $x^3 + 2x^2 - 5x - 4 = 0$

(i) $\alpha + \beta + \gamma = \frac{-b}{a} = -2$ ✓

(ii) $\alpha\beta\gamma = \frac{-d}{a} = 4$ ✓

(iv) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma} = \frac{-5}{4}$ ✓

(iv) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\Sigma\alpha\beta)$

$= (-2)^2 - 2(-5)$

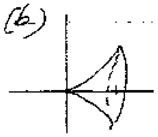
$= 4 + 10 = 14$ ✓

QUESTION (3)

(a) $\int \frac{1}{(4+9x^2)}\,dx = \frac{1}{9}\int \frac{dx}{\left(\frac{4}{9}+x^2\right)}$ ✓

$= \frac{1}{9}\left[\frac{1}{\left(\frac{2}{3}\right)}\tan^{-1}\frac{x}{\left(\frac{2}{3}\right)}\right] = \frac{1}{6}\tan^{-1}\frac{3x}{2} + C$ ✓

(b)



$V = \pi\int_a^b y^2\,dx$

$= \pi\int_0^{\frac{\pi}{4}} \tan^2 x\,dx$ ✓

∴ $V = \pi\int_0^{\frac{\pi}{4}} (\sec^2 x - 1)\,dx$ ✓

$= \pi\left[\tan x - x\right]_0^{\frac{\pi}{4}}$ ✓

$= \pi\left[\left(1 - \frac{\pi}{4}\right) - (0-0)\right] = \left(\pi - \frac{\pi^2}{4}\right)$ cubic units

(c) $Q(x) = x^3 + 2x^2 + ax + b$

$Q(-2) = (-2)^3 + 2(-2)^2 + a(-2) + b = 0$ — (i)

$-8 + 8 - 2a + b = 0$

∴ $-2a + b = 0$ ✓

$Q(2) = (2)^3 + 2(2)^2 + a(2) + b = 12$ — (ii)

$8 + 8 + 2a + b = 12$

∴ $2a + b = -4$ ✓

SOLVE (i) & (ii) $2b = -4$, $b = -2$

∴ $a = -1$

(d) $f(x) = e^x + 4x$

(i) $f'(x) = e^x + 4$ ✓

(ii) $f'(x) > 0$ for all values of x as $e^x > 0$ ✓

∴ $f(x)$ is increasing for all x

(iii) $f(-1) = e^{-1} + 4(-1) \doteqdot \frac{1}{e} - 4 < 0$

$f(0) = e^0 + 4(0) = 1 > 0$ ✓

∴ there is a root between -1 and 0

(iv) Let $x = -0.5$ be a_1

∴ $a_2 = a_1 - \frac{f(a_1)}{f'(a_1)}$

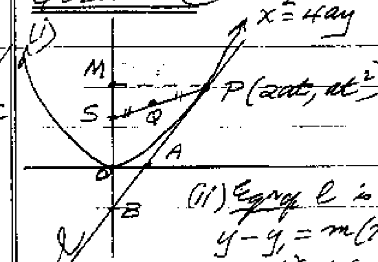
$= -0.5 - \left[\frac{e^{-.5} + 4(-.5)}{e^{-.5} + 4}\right]$ ✓

$= -0.5 - [\;]$

$\doteqdot -0.20$ (To 2dp) ✓

QUESTION (4)

(i)



(ii) Eqn of l is

$y - y_1 = m(x - x_1)$ ✓

$y - at^2 = t(x - 2at)$

∴ $y = tx - at^2$ ✓

(iii) A is $(at, 0)$ ✓

B is $(0, -at^2)$ ✓

(iv) $PA : PB = MO : MB = at^2 : 2at^2 = 1:2$ EXTERNALLY ✓

(v) Let Q be (x, y) where S is $(0, a)$

∴ $x = \frac{0 + 2at}{2}$ — (i) ∴ $x = at$ ✓

$y = \frac{a + at^2}{2}$ — (ii) ∴ $y = \frac{a(1+t^2)}{2}$

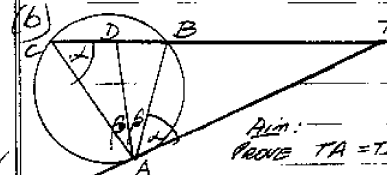
∴ $2y = a(1+t^2)$ ✓

HENCE $t = \frac{x}{a}$

∴ $2y = a\left[1 + \left(\frac{x}{a}\right)^2\right]$ ✓

OR EQUIVALENT

(b)



Aim: PROVE $TA = TD$

PROOF: Let $B\hat{A}T = \alpha$

∴ $B\hat{C}A = \alpha$ [Angle in Alternate Segment of a Circle Theorem] ✓

Let $C\hat{A}D = B\hat{A}D = \beta$

∴ $B\hat{D}A = \alpha + \beta$ [Exterior Angle of ΔCDA] ✓

∴ $B\hat{D}A = D\hat{A}T$ (Both $(\alpha+\beta)$)

∴ ΔTDA is ISOSCELES [2 Equal Angles] ✓

∴ $TA = TD$ [Opposite Equal Angles in an Isosceles Δ]

QUESTION 5

(a) $f(x) = (x-1)^2$

(i) Sketch

✓

(ii) It does not have an inverse because {ONE-TO-ONE OR ...} ✓

OR EQUIVALENT REASON

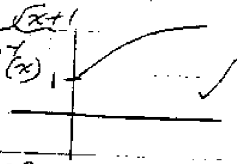
(iii) To have an inverse: $f(x)$ has $\begin{cases} D: x \geq 1 \\ R: y \geq 0 \end{cases}$ ✓

(iv) Inverse is $x = (y-1)^2$

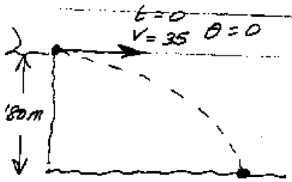
$\sqrt{x} = y - 1$ ✓

$f^{-1}(x)$: $y = \sqrt{x} + 1$

(v) Sketch $y = f^{-1}(x)$



✓



$g = 10$

(i) $\ddot{x} = 0$ — $\ddot{y} = -10$

$\dot{x} = C_1$ — $\dot{y} = -10t + C_3$

Initially, $t=0$, $\dot{x} = 35$ — Initially $t=0$, $\dot{y} = 0$

$\therefore \dot{x} = 35$ — $\therefore \dot{y} = -10t$

Hence $x = 35t + C_2$ — Hence $y = -\frac{10t^2}{2} + C_4$

Initially $x=0, t=0$ ✓ — Initially $t=0, y=0$

$\therefore x = 35t$ — $\therefore y = -5t^2$ ✓

(ii) $\therefore$ At the ocean, $y = -180$

$5t^2 = 180$, $\therefore t^2 = 36$ $\therefore t = 6$ ✓

$\therefore$ 6 seconds

(iii) DISTANCE: $x = 35t = 35(6) = 210$ metres ✓

$\therefore \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{-10t}{35} = \frac{-10 \times 6}{35} = -1\frac{5}{7}$ ✓✓

$\therefore \theta = 59.74 = 60°$ or $120°$

QUESTION 6

(a) $\frac{dT}{dt} \propto (T - A)$

$= k(T - A)$

(i) If $T = A + Be^{kt}$

$\frac{dT}{dt} = Bke^{kt} = k[Be^{kt}]$

$= k[T - A]$

$\therefore$ it satisfies the LAW ✓

(ii) If $A = 4$, $t = 0$, $T = 25$

$\therefore 25 = 4 + Be^0$

$\therefore B = 21$ $\quad \therefore T = 4 + 21e^{kt}$

Then $T = 15$, $t = 45$

$\therefore 15 = 4 + 21e^{45k}$

$11 = 21e^{45k}$

$\ln\left(\frac{11}{21}\right) = 45k$

$k = \frac{1}{45}\ln\left(\frac{11}{21}\right) \doteqdot -0.01144$ ✓

Then If $T = 8$

in $T = 4 + 21e^{kt}$

$\therefore 8 = 4 + 21e^{kt}$

$4 = 21e^{kt}$

$\ln\left(\frac{4}{21}\right) = kt$

$-1.6582 = -0.0144t$ ✓

$\therefore t \doteqdot 115$ minutes

(b) $x = 5\cos(4\pi t)$

(i) $\dot{x} = 5[-4\pi\sin 4\pi t] = -20\pi[\sin 4\pi t]$

$\ddot{x} = -20\pi[4\pi\cos 4\pi t] = -80\pi^2\cos 4\pi t$ ✓

$\therefore \ddot{x} = -16\pi^2[5\cos 4\pi t] = -16\pi^2 x$

which is in the FORM ✓

$\ddot{x} = -n^2 x$

(ii) Period $T = \frac{2\pi}{n} = \frac{2\pi}{4\pi} = \frac{1}{2}$ second ✓

QUESTION 6 CTD.

(b) (iii) Maximum Velocity

$\dot{x} = -20\pi[\sin 4\pi t]$

is when $\sin 4\pi t = \pm 1$ ✓

$\therefore \dot{x} = 20\pi$ is MAXIMUM ✓

(iv) $v^2 = n^2(a^2 - x^2)$

$= 16\pi^2(25 - x^2)$ ✓

OR EQUIVALENT

(c) MATHEMATICAL INDUCTION

Step 1: Test for $n = 1$

$\therefore (2^{10+3} + 3) = 8195 \div 11 = 745$ ✓

which is DIVISIBLE BY 11

Step 2: Assume True for $(n = k)$

$\therefore$ Assume

$2^{10k+3} + 3 = 11P$

where P is a Positive Integer.

Step 3 PROVE TRUE for $n = (k+1)$

Consider

$2^{10(k+1)+3} + 3 = 2^{10k+13} + 3$

$= (2^{10k+3})2^{10} + 3 = [11P - 3]2^{10} + 3$ ✓

$= 11P(2^{10}) - 3(2^{10}) + 3 = 11P(2^{10}) - 3069$

which is DIVISIBLE BY 11

Hence TRUE for $n = (k+1)$

Step 4 Hence since it is True for $n = 1$, it is True for $n = 2, 3, \ldots$

Hence $(2^{10n+3} + 3)$ is divisible by 11 for all positive integers. [OR EQUIVALENT STATEMENT] ✓

QUESTION 7

(a) Using THE SQUARE on the TANGENT EQUALS ... [OR USE SIMILAR TRIANGLES]

$(PT)^2 = (PA)(PB)$

$\therefore x^2 = (6+a)a$ ✓

$\therefore x = \sqrt{a(6+a)}$ ✓

(b) $\theta = (\beta - \alpha)$

(i) $\therefore \tan\theta = \tan(\beta - \alpha) = \frac{\tan\beta - \tan\alpha}{1 + \tan\beta\tan\alpha}$ ✓

$= \frac{\left(\frac{x}{a}\right) - \left(\frac{x}{a+6}\right)}{1 + \left(\frac{x}{a}\right)\left(\frac{x}{a+6}\right)} = \frac{x(a+6) - x(a)}{a(a+6) + x^2}$ ✓

$= \frac{6x}{a^2 + 6a + x^2}$ ✓

(ii) $\frac{dT}{dx} = \frac{(a^2+6a+x^2)[6] - 6x[2x]}{(a^2+6a+x^2)^2}$ ✓

$= \frac{6a^2 + 36a - 6x^2}{(\quad)^2} = 0$ ✓

$\therefore a^2 + 6a - x^2 = 0$

$a^2 + 6a = x^2$; $x = \sqrt{a(a+6)}$

* TEST (FOR MAXIMUM)

(iii) $T = \tan\theta = \frac{6\sqrt{a(a+6)}}{a^2 + 6a + a^2 + 6a}$

$= \frac{6\sqrt{a(a+6)}}{2(a^2+6a)} = \frac{3}{\sqrt{a^2+6a}}$ ✓

$\therefore \theta = \tan^{-1}\left(\frac{3}{\sqrt{a^2+6a}}\right)$ ✓

(iv) $x = 12.649110641 = \sqrt{160}$

$\doteqdot 12.65$ m or $4\sqrt{10}$ ✓

$\therefore \theta = 13°20'33'' \doteqdot 13°21'$ ✓

(v) Reason: Maximum θ occurs when $x = \sqrt{a(a+6)}$ ~~using the Result (a) we see that~~ the GOAL POSTS AND P all lie on a circle with PT as Tangent ✓

(OR A REASON CONNECTING (a) GEOMETRY WITH THE CONCEPT OF MAXIMUM AS IN (b))