Manly High School

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

1999

MATHEMATICS

3 UNIT (ADDITIONAL)
AND
3/4 UNIT (COMMON)

Time Allowed - Two hours
(Plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- ALL questions are of equal value.
- Write your student Name / Number on every page of the question paper and your answer sheets.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Standard integrals are supplied.
- Board approved calculators may be used.
- The answers to the seven questions are to be handed in separately clearly marked Question 1, Question 2, etc..
- *The question paper must be handed to the supervisor at the end of the examination.*

STUDENT NUMBER / NAME..

Question 1 (Start a new page) **Marks**

a. Two dice are rolled. If you know that at least one of the dice is a 5, what is the probability of getting a total of 8? **2**

b. Consider the parabola with equation $y^2 = 4(x - 3)$. **2**

(i) Find the coordinates of the vertex of the parabola.

(ii) Find the coordinates of the focus of the parabola.

c. The point C(−1, −4) divides the interval AB externally in the ratio 3:1. If the coordinates of A are (3, 2), find the coordinates of B. **2**

d. Evaluate $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin x \cos^3 x \, dx$ using the substitution $u = \cos x$ **3**

e. Find the exact value of $\int_0^{\frac{\pi}{4}} \cos^2 \frac{1}{2}x \, dx$ **3**

Question 2 (Start a new page)

a. Solve $\dfrac{1}{x + 1} \geq 1 - x$ **3**

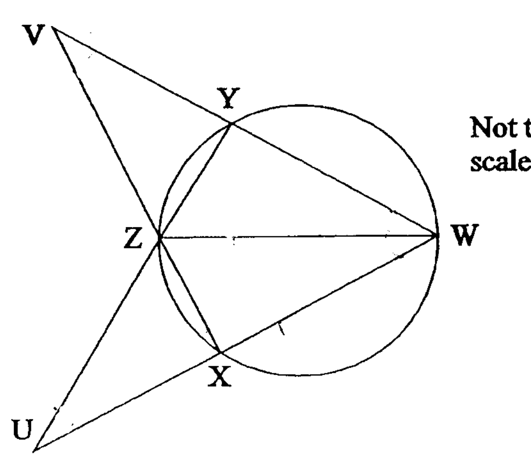
b. Find $\displaystyle\int_0^{\frac{2}{5}} \frac{dx}{\sqrt{16 - 25x^2}}$ **3**

c. The points P($2ap$, ap^2) and Q($2aq$, aq^2) lie on the parabola $x = 2at$, $y = at^2$. **3**

i. Find M, the midpoint of PQ.

ii. Show that, if the gradient of PQ is constant, the locus of M is a line parallel to the y–axis.

d. In the diagram, UZY, XZV, VYW and UXW are all straight lines. Given ZW bisects ∠XWY and ∠WUZ = ∠WVZ, prove that XW = YW. **3**



Question 3 (Start a new page) **Marks**

a. Show that $\frac{2x+1}{x+2} = 2 - \frac{3}{x+2}$ 3

Hence or otherwise, find the exact value of $\int_0^1 \frac{2x+1}{x+2}\,dx$

b. Solve $\cos x - \sqrt{3}\sin x + 1 = 0$ for $0 \le x \le 2\pi$ 3

c. i. Show that the solution of $x\ln x - 1 = 0$ lies between $x = 1$ and $x = 2$. 3

ii. Using $x = 2$ as a first approximation, apply Newton's method once to obtain a better approximation. Give your answer to one decimal place.

d. Beginning in 1960, Ranger Smith planted 1 000 trees at the start of each year. Initially the average mass of each tree is 5 kilograms. This increased at the rate of 20% pa. The trees should not be harvested until their average mass reaches 3 000 kilograms. 3

(i) Find the minimum number of years that the first trees must be left before harvesting, correct to the nearest year.

(ii) After the initial waiting time, calculated in (i), the trees are harvested at the rate of 1 000 per year, in the same order as the trees were planted. Find the total tonnage harvested in the 40th year.

Question 4 (Start a new page)

a. Two circles, C_1 and C_2, are members of the set of circles defined by the equation $x^2 + y^2 - 6x + 2ky + 3k = 0$, where k is real. 4

The centre of C_1 lies on the line $x - 3y = 0$ and C_2 touches the x–axis.

Find the equations of C_1 and C_2.

b. The acceleration, a, of a particle is given in terms of its position, x, by the equation $a = 2x^3 + 2x$. 4

i. If $v = 2$ when $x = 1$, show that $v^2 = (1 + x^2)^2$

ii. Show that, if $x = \frac{1}{\sqrt{3}}$ when $t = 0$, then $t = \frac{\pi}{6}$ when $x = \sqrt{3}$

c. Prove by Mathematical Induction that $5^{2n} - 1$ is divisible by 6 when n is a positive integer 4

Question 5 (Start a new page) **Marks**

a. At 9 am, an ultralight aircraft flies directly over Daryl's head at 500 metres. It maintains a constant speed of 20 ms^{-1} and a constant altitude. 5

If x is the horizontal distance travelled by the plane and θ is the angle of elevation from Daryl to the plane,

i. show that $\dfrac{dx}{d\theta} = -500\operatorname{cosec}^2\theta$.

ii. Hence show that $\dfrac{d\theta}{dt} = -\dfrac{1}{25}\sin^2\theta$.

iii. Find the rate of change of the angle of elevation at 9:01 am.

b. Two groups of terrorists are 150 metres from their target. 7

The first group, Group A, is on the same horizontal level as the target and can fire their missiles in any direction at a speed of 50 ms^{-1}.

i. Show that Group A can hit the target and calculate the angle(s) at which their missiles are to be fired. [Use $g = 10$ ms^{-2}]

The second group, Group B, is positioned in a building 30 metres above the horizontal level of the target and can fire their missile only horizontally through a small window and at 55 ms^{-1}.

ii. Determine whether Group B can hit their target. [Use $g = 10$ ms^{-2}]

Question 6 (Start a new page) **Marks**

a. The displacement, x cm, of an object from the origin is given by **5**

$$x = 2\sin t - 3\cos t, \quad t \geq 0$$

where time, t, is measured in seconds.

i. Show that the object is moving in Simple Harmonic Motion.

ii. Find the amplitude of the motion.

iii. At what time does the object first reach its maximum speed?

b. A cup of soup at temperature T°C loses heat when placed in the lounge room. It cools according to the law: **7**

$$\frac{dT}{dt} = k(T - T_0)$$

where t is the elapsed time in minutes and T_0 is the temperature of the room in degrees centigrade.

i. Show that the equation $T = T_0 + Ae^{kt}$ satisfies the above law of cooling.

ii. A cup of soup at 95°C is placed in the freezer at −10°C for 5 minutes and cools to 65°C. Find the exact value of k

iii. The same cup, at 65°C, is then taken into the lounge room where the surrounding temperature is 26°C. Assuming k remains the same, find, to the nearest degree, the temperature of the soup after another 5 minutes.

Question 7 (Start a new page)

Marks

a. Find the constant term in the expansion of $\left(3x - \frac{1}{x^2}\right)^6$ **3**

b. i. Solve the equation $x^4 + x^2 - 1 = 0$, giving your answer(s) to two decimal places. **9**

ii. On the same axes, draw the graphs of $y = \tan^{-1} x$ and $y = \cos^{-1} x$, showing all important features. Mark the point, P, where the curves intersect.

iii. Show that, if $\tan^{-1} x = \cos^{-1} x$, then $x^4 + x^2 - 1 = 0$. Hence find the coordinates of P.

iv. Find to two decimal places the area enclosed by the curves and the y-axis.

Q1 a. Possibilities are

1,5
2,5
3,5
4,5
5,1 5,2 5,3 5,4 5,5 5,6
6,5

$\therefore$ Probability of total of 8 $= \frac{2}{11}$

b. ~~Let p = prob of supporting A = 3/10~~
~~q = prob of supporting other = 7/10~~
~~n = no. of A supporters~~
~~Then $P(X=r) = {}^7C_r\left(\frac{3}{10}\right)^r\left(\frac{7}{10}\right)^{7-r}$~~

~~$\therefore P(X=4) = {}^7C_4\left(\frac{3}{10}\right)^4\left(\frac{7}{10}\right)^3$~~

~~$= 0.0972405$~~
~~$\doteqdot 0.1$~~

c. $x = \frac{kx_2 + lx_1}{k+l}$ $\qquad y = \frac{ky_2 + ly_1}{k+l}$

$-1 = \frac{-3\times x_2 + 1\times 3}{-3+1}$ $\qquad -4 = \frac{-3\times y_2 + 1\times 2}{-3+1}$

$2 = -3x_2 + 3$ $\qquad 8 = -3y_2 + 2$

$x_2 = \frac{1}{3}$ $\qquad y_2 = -2$

$\therefore B\left(\frac{1}{3}, -2\right)$

d. $u = \cos x$

$du = -\sin x \, dx$

if $x = \frac{\pi}{2}$, $u = 0$

if $x = \frac{\pi}{3}$, $u = \frac{1}{2}$

$\therefore I = \int_{.5}^{0} -u^3 \, du$

$= \left[\frac{u^4}{4}\right]_0^{.5}$

$= \frac{.5^4}{4} - 0 = \frac{1}{64}$

e. $\int_0^{\pi/4} \cos^2 \frac{1}{2}x \, dx$

$= \frac{1}{2}\int_0^{\frac{\pi}{4}} 1 + \cos x \, dx$

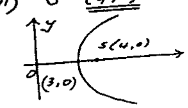
$= \frac{1}{2}\left[x + \sin x\right]_0^{\pi/4}$

$= \frac{1}{2}\left(\frac{\pi}{4} + \frac{1}{\sqrt{2}}\right)$

(b) $(y-0)^2 = 4\times 1(x \ldots$

(i) Vertex $(3,0)$

(ii) S $(4,0)$



These suggested answers/marking schemes are issued as a guide only
-offerred as an assistance in constructing your own marking format
(individual teachers/schools find many other acceptable responses)

Q2 a. $\frac{1}{x+1} \geq 1 - x$

Critical points at $x = -1$ and

$\frac{1}{x+1} = 1 - x$

$1 = 1 - x^2$

$\Rightarrow x = 0$

Test $x = -2$ False

Test $x = -\frac{1}{2} \Rightarrow 2 \geq 1\frac{1}{2}$ $\therefore$ True

$x = 1 \Rightarrow \frac{1}{2} \geq 0$ $\therefore$ True

Solution: $x > -1$

b. $\int_0^{2/5} \frac{dx}{\sqrt{16 - 25x^2}}$

$= \int_0^{2/5} \frac{dx}{5\sqrt{\frac{16}{25} - x^2}}$

$= \frac{1}{5}\left[\sin^{-1}\frac{x}{4/5}\right]_0^{2/5}$

$= \frac{1}{5}\left[\sin^{-1}\frac{5x}{4}\right]_0^{2/5}$

$= \frac{1}{5}\left(\sin^{-1}\frac{1}{2} - \sin^{-1} 0\right)$

$= \frac{1}{5}\cdot\frac{\pi}{6} = \frac{\pi}{30}$

c. (i) $M\left(a(p+q), \frac{a(p^2+q^2)}{2}\right)$

(ii) $M_{PQ} = \frac{p+q}{2} = k$, a constant

Then, for the point M,

$x = a(p+q)$

$= a.2k$

$x = 2ak$

Since a and k are constant, the locus of M is a line paral to the y-axis

d. $\angle U = \angle V$ (given)

$\angle UZX = \angle VZY$ (vertically opp

Now $\angle ZXW = \angle UZX + \angle U$ (exterior an triangle)

and $\angle ZYW = \angle VZY + \angle V$ (ditto)

$\therefore \angle ZXW = \angle ZYW$ (equal to sum equal angles

In ΔXZW & ΔYZW,

ZW is common

$\angle ZXW = \angle ZYW$ (above)

$\angle XWZ = \angle YWZ$ (given ZW bisec $\angle YWX$)

$\therefore \Delta XZW \equiv \Delta YZW$ (AAS)

and XW = YW

Q3. (a) $2 - \frac{3}{x+2} = \frac{2(x+2)-3}{x+2}$

$= \frac{2x+1}{x+2}$

$\therefore \int_0^1 \frac{2x+1}{x+2}\,dx$

$= \int_0^1 2 - \frac{3}{x+2}\,dx$

$= \left[2x - 3\ln(x+2)\right]_0^1$

$= (2 - 3\ln 3) - (0 - 3\ln 2)$

$= 2 + 3\ln(2/3)$

(b) Let $\cos x - \sqrt{3}\sin x = A\cos(x+\theta)$

$= A\cos x\cos\theta - A\sin x\sin\theta$

then $A\cos\theta = 1$

$A\sin\theta = \sqrt{3}$

$\Rightarrow \tan\theta = \sqrt{3}$ and $\theta = \pi/3$

and $A = 2$

$\therefore 2\cos\left(x + \frac{\pi}{3}\right) + 1 = 0$

$\cos\left(x+\frac{\pi}{3}\right) = -1/2$

$x + \frac{\pi}{3} = \ldots\ \frac{4\pi}{3}, \frac{5\pi}{3}, \ldots$

$x = \pi, \frac{4\pi}{3}$ in given domain

(c)(i) Let $f(x) = x\ln x - 1$

$f(1) = 1.\ln 1 - 1 < 0$

$f(2) = 2\ln 2 - 1 > 0$

$\therefore$ a solution exists between $x=1$ & $x=2$ (assuming $f(x)$ is continuous)

(ii) $f'(x) = x.\frac{1}{x} + \ln x = \ln x + 1$

By Newton's method,

$x_1 = x - \frac{f(x)}{f'(x)}$

$= x - \frac{x\ln x - 1}{\ln x + 1}$

If $x=2$, $x_1 = 2 - \frac{2\ln 2 - 1}{\ln 2 + 1}$

$= +1.77184832$

$= 1.8$

[illegible]

(i) $5(1.2)^n = 3000$

$1.2^n = 600$

$n \log 1.2 = \log 600$

$n = \frac{\log 600}{\log 1.2}$

$= 35.0859$

$n = 35$ yrs

(ii) $T = 1000 \times 3000$ kg

$=$ 3000 tonnes

(iii) Each yr (after 35 yrs) 1000 planted, 1000 removed.

$\therefore$ Stable $T = 5000(1.2)^{35} + 5000(1.2)^{34} \ldots + 5000(1.2)$

$= 5000(1.2)\frac{[(1.2)^{35} - 1]}{1.2 - 1}$

$= 17690047$ kg

$=$ 17690 tonnes

Q4(a) $x^2 + y^2 - 6x + 2ky + 3k = 0$

Completing the squares:

$(x-3)^2 + (y+k)^2 = k^2 - 3k + 9$

If the centre $(3, -k)$ is on the line $x - 3y = 0$, then

$3 - 3\times -k = 0 \Rightarrow k = -1$

$\therefore C_1: (x-3)^2 + (y-1)^2 = 13$

If C_2 touches the x-axis, the radius is k

$\therefore \sqrt{k^2 - 3k + 9} = k$

$k^2 - 3k + 9 = k^2$

$\Rightarrow k = 3$

$\therefore C_2: (x-3)^2 + (y+3)^2 = 9$

(b)(i) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 2x^3 + 2x$

$\therefore \frac{1}{2}v^2 = \frac{1}{2}x^4 + x^2 + C$

If $v=2$, $x=1$

$\therefore \frac{1}{2}.2^2 = \frac{1}{2}.1 + 1 + C \Rightarrow C = \frac{1}{2}$

& $\frac{1}{2}v^2 = \frac{1}{2}x^4 + x^2 + \frac{1}{2}$

$v^2 = x^4 + 2x^2 + 1$

$v^2 = (x^2+1)^2$

(ii) so $v = \pm(x^2+1)$

but $v = 2\ (>0)$ when $x = 1$

$\therefore v = +(x^2+1)$

$\frac{dt}{dx} = \frac{1}{x^2+1}$

so $t = \tan^{-1}x + C$

Now $x = \frac{1}{\sqrt{3}}$ when $t = 0$

$\therefore C = -\tan^{-1}\frac{1}{\sqrt{3}} = -\frac{\pi}{6}$

so $t = \tan^{-1}x - \frac{\pi}{6}$

when $x = \sqrt{3}$, $t = \tan^{-1}\sqrt{3} - \frac{\pi}{6}$

$= \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$

(c) Let $S(n)$: $5^{2n} - 1 = 6I$, whe

I is an integer.

$S(1)$: LHS $= 5^2 - 1 = 24 = 6\times 4$

$\therefore S(1)$ is true

Assume $S(k)$: $5^{2k} - 1 = 6I$ (I, in

Consider $S(k+1)$:

LHS $= 5^{2k+2} - 1$

$= 5^{2k}.5^2 - 1$

$= 25(5^{2k} - 1) - 1 + 25$

$= 25.6I + 24$ by $S(k$

$= 6[25I + 4]$

Now I is integer, $\therefore 25I + 4$ is int

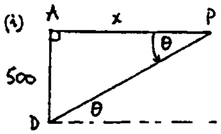
Hence, if $S(k)$ is true, $S(k+1)$

But $S(1)$ is true, so $S(2)$ is true

and then $S(3)$ is true and so

for all integer values of n.

Q5(a)(i)



$\tan\theta = \frac{500}{x}$

$\therefore x = 500\cot\theta$

$$\frac{dx}{d\theta} = -500\,\mathrm{cosec}^2\theta$$

(ii) $\frac{d\theta}{dt} = \frac{d\theta}{dx} \times \frac{dx}{dt}$

$= \frac{1}{-500\,\mathrm{cosec}^2\theta} \times 20$

$= -\frac{1}{25}\sin^2\theta$

(iii) At 9:01, $t=60$, $x=1200$

Then $PD=1300$ (Pythagoras' Theorem)

so $\sin\theta = \frac{500}{1300} = \frac{5}{13}$

$\therefore \frac{d\theta}{dt} = -\frac{1}{25} \times \left(\frac{5}{13}\right)^2$

$= -\frac{1}{169}$ degrees/sec.

(b)(i) $\ddot{x} = 0$ $\qquad \ddot{y} = -10$

$\dot{x} = C_1$ $\qquad \dot{y} = -10t + C_2$

Initially $\dot{x} = 50\cos\alpha$ $\quad \therefore \dot{x} = 50\cos\alpha$

and $\dot{y} = 50\sin\alpha$ $\quad \therefore \dot{y} = -10t + 50\sin\alpha$

$x = 50t\cos\alpha + C_3$ $\qquad y = -5t^2 + 50t\sin\alpha + C_4$

+ since $x=0$ when $t=0$, and $y=0$ when $t=0$

$C_3 = 0$ $\qquad C_4 = 0$

$\therefore x = 50t\cos\alpha$ $\qquad \therefore y = -5t^2 + 50t\sin\alpha$

when $x=150$, $150 = 50t\cos\alpha$

so $3 = t\cos\alpha$ ... (1)

when $y=0$, $0 = -5t^2 + 50t\sin\alpha$

$= -5t(t - 10\sin\alpha)$

$\Rightarrow t = 10\sin\alpha$ (2)

Solving (1) + (2):

$3 = 10\sin\alpha\cos\alpha$

$= 5\sin 2\alpha$

$\therefore \sin 2\alpha = 3/5$

$2\alpha = 36°52', 143°08'$

$\therefore \alpha = 18°26'$ or $71°29'$

(ii) $\ddot{x} = 0$ $\qquad \ddot{y} = -10$

$\dot{x} = C_1$ $\qquad \dot{y} = -10t + C_2$

Initially, $\dot{x} = 55\cos\alpha$, $\dot{y} = 55\sin\alpha$

$\therefore \dot{x} = 55\cos\alpha$ $\qquad \dot{y} = -10t + 55\sin\alpha$

$\dot{x} = 55$ $\qquad \dot{y} = -10t$ since $\alpha = 0$

Then $x = 55t + C_3$ $\qquad y = -5t^2 + C_4$

when $t=0$, $x=0$ and $y=30$

$\Rightarrow C_3 = 0$ $\qquad C_4 = 30$

$\therefore x = 55t$ $\qquad y = -5t^2 + 30$

Now when $y=0$, $-5t^2 + 30 = 0$

$\therefore t^2 = 6$

+ $t = \sqrt{6}$

At $t = \sqrt{6}$, $x = 55\sqrt{6}$

≈ 135 m

$\therefore$ Group B cannot reach the target

Q6(a)(i) $x = 2\sin t - 3\cos t$

$\dot{x} = 2\cos t + 3\sin t$

$\ddot{x} = -2\sin t + 3\cos t$

$= -(2\sin t + 3\cos t)$

$= -x$

$\therefore$ motion is simple harmonic.

(ii) Amplitude $= \sqrt{2^2 + 3^2}$

$\sqrt{13}$ cm

(iii) $\dot{x} = 2\cos t + 3\sin t$

$\ddot{x} = -2\sin t + 3\cos t$

Max velocity when $\ddot{x} = 0$

$-2\sin t + 3\cos t = 0$

$3\cos t = 2\sin t$

$3/2 = \tan t$

$t = 0{\cdot}983, 4.1243 \ldots$ etc

$\therefore$ reaches maximum velocity when $t = 0{\cdot}983$

(b)(i) $T = T_0 + Ae^{kt}$

$\frac{dT}{dt} = k.Ae^{kt}$

$= k(T - T_0)$

(ii) When $t=0$, $T=95$, $T_0=-10$

$\Rightarrow A = 105$

when $t=5$, $T=65$

$\therefore 65 = -10 + 105e^{5k}$

$e^{5k} = 75/105 = 5/7$

$\ln e^{5k} = \ln 5/7$

$\therefore k = \frac{1}{5}\ln\frac{5}{7}$

(iii) When $t=0$, $T_0 = 26$ + $T = 65$

$\therefore 65 = 26 + Be^{k.0}$

$\therefore B = 39$

Therefore, at $t=5$,

$T = 26 + 39e^{5k}$ with $k = \frac{1}{5}$

so $T = 53{\cdot}86°$

$= 54°$ (to the nearest deg

MATHS 3U ANSWERS - 1999

Q7(a) $\left(3x-\frac{1}{x^2}\right)^6=\sum_{r=0}^{6} {}^6C_r\,(3x)^{6-r}\left(-\frac{1}{x^2}\right)^r$

Typical term, T_r, is

$T_r = {}^6C_r\, 3^{6-r}\cdot x^{6-r}\cdot(-1)^r\cdot(x^{-2})^r$

$= {}^6C_r\, 3^{6-r}(-1)^r\, x^{6-3r}$

Constant term when $6-3r=0$

$r=2$

then $T_2={}^6C_2\,3^4\,(-1)^2$

$=1215$

(b)(i) $x^4+x^2-1=0$

$x^2=\frac{-1\pm\sqrt{1-4\times1\times-1}}{2}$

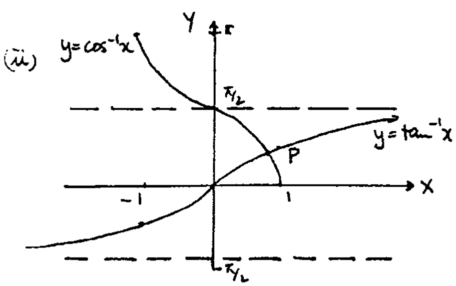
$=\frac{-1\pm\sqrt5}{2}$

$\therefore x^2=\cancel{\frac{-1-\sqrt5}{2}}$ or $\frac{-1+\sqrt5}{2}$

$x^2=0.618033988$

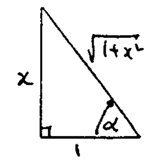
$x=\pm\,0.786151377$

$=\pm\,0.79$

(ii) 

(iii) let $\tan^{-1}x=\alpha$

$\therefore x=\tan\alpha$



At P, $\cos^{-1}x=\tan^{-1}x=\alpha$

$\therefore$ at P $\cos^{-1}x=\alpha$ + $x=\cos\alpha$

But $\cos\alpha=\frac{1}{\sqrt{1+x^2}}$ (from diagram)

$\therefore\ x=\frac{1}{\sqrt{1+x^2}}$

Squaring, $x^2=\frac{1}{1+x^2}$

+ $x^4+x^2=1$

$x^4+x^2-1=0$

$\therefore x=0.79$ (from (i))

and $y=\tan^{-1}0.79=0.6686$

so $P(0.79,\ 0.67)$

(iv) $A=\int_0^{0.67}\tan y\,dy+\int_{0.67}^{\pi/2}\cos y\,dy$

$=\left[-\ln|\cos y|\right]_0^{0.67}+\left[\sin y\right]_{0.67}^{\pi/2}$

$=-\ln|\cos 0.67|+\sin\frac{\pi}{2}-\sin 0.67$

$=0.62258$

$=0.62$ (to 2-decimal places)