QUESTION 1

a) Use the table of Standard Integrals to find:

$$\int \frac{1}{t^2+25}\,dt$$ (1)

b) The roots of a monic polynomial P(x) of degree 3 are 2, -2 and 1. Find an equation of the polynomial (1)

c) Differentiate $x^2 \sin^{-1} 4x$ (3)

d) Use the substitution $u = 4 - x^2$ to find $\int 2x\sqrt{4-x^2}\,dx$ (3)

e) Solve the equation $2\sin^2\theta = \sin 2\theta$ for $0 \le \theta \le 2\pi$ (4)

QUESTION 2 (Please start a new page)

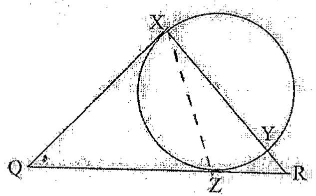
a) Is $x - 2$ a factor of $x^3 - 5x + 12$? Give a reason for your answer. (1)

b) Solve $\frac{1}{x} + \frac{x}{4} \le 1$ (3)

c) Solve the equation $\sqrt{3}\cos\theta - \sin\theta = \sqrt{3}$, $0 \le \theta \le 2\pi$ (4)

d) XY is a diameter of a circle XYZ. The tangents at X and Z meet at Q. The lines QZ and XY are produced to meet at R.

(Draw this diagram onto your answer sheet)



(i) Prove $\angle ZXQ = 90° - \angle YZR$ (2)
(ii) Hence, or otherwise, prove that $\angle XQZ = 2.\angle YZR$ (2)

Student name/number: ____________

2002
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

General Instructions

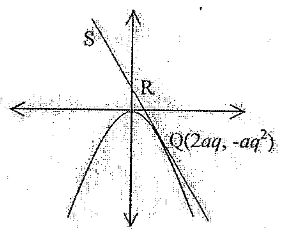
- Reading time – 5 minutes
- Working time – 2 hours
- Write using blue or black pen
- Board-approved calculators may be used
- A table of standard integrals is provided on last page
- All necessary working should be shown in every question

Total marks (84)
- Attempt Questions 1 – 7
- All questions are of equal value

Please note that this is a Trial paper only and cannot in any way guarantee the format or the content of the Higher School Certificate Examination.

QUESTION 4 CONTINUED

b) The point $Q(2aq, -aq^2)$ is a variable point on the parabola $x^2 = -4ay$. The tangent at Q meets the y axis at R. The point S lies on the tangent and divides QR externally in the ratio 3:1.



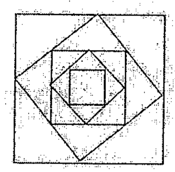
i) Show that the equation of the tangent at Q is $qx + y = aq^2$ (2)

ii) Find the coordinates of the points R. (1)

iii) Show the coordinates of S are $(-aq, 2aq^2)$ (2)

iv) Show that the locus of S is a parabola. (1)

c) The midpoints of a square side 4a are joined to give a second square. The midpoints of the sides of this square are joined to give a third square and this process is repeated indefinitely. Calculate the limit of the sum of the areas of all the squares. (3)



QUESTION 3 (Please start a new page)

a) i.) Prove that a root of the equation $x^3 - 6x + 1 = 0$ lies between $x = 2$ and $x = 3$. (1)

ii) By taking a first approximation $x_1 = 2{\cdot}5$ use one application of Newton's method to find a better approximation of the root. (3)

b) Three boys and five girls are at a birthday party.

i) The children are asked to form a queue to collect some food. In how many ways can the queue be formed? (1)

ii) After eating the children are asked to sit in a circle for the party games. In how many ways can the children be seated around the circle? (1)

iii) For the '*Pass the Parcel*' game the children remain in a circle, but two of the boys are asked not to sit together. In how many ways may this occur? (2)

c) For the expansion $(4 + 5x)^{17}$:

i) Show the ratio of the coefficients of consecutive terms ~~T_r and T_{r+1}~~ T_{r+1} T_r is $\dfrac{5(18-r)}{4r}$. (2)

ii) Determine which pair of terms has equal coefficients. (2)

QUESTION 4 (Please start a new page)

a) A side of lamb at room temperature T is placed in a cooling room at a time t where A is the constant temperature of the cooling room. Newton's Law of Cooling states that the rate of change of temperature T is proportional to $(T - A)$

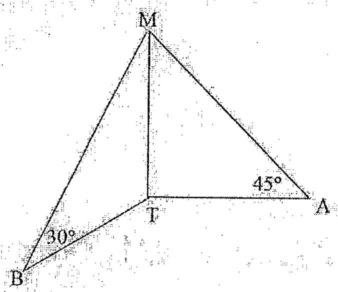
i) Show that $T = A + Ce^{kt}$ (where C and k are constants) satisfies Newton's Law of Cooling. (1)

ii) The temperature of the room is 20°C whilst the cooling room temperature is −5°C. How long will it take for the side of lamb to reach freezing point (0°) if it drops to 10° in 3 hours? (2)

QUESTION 7 (Please start a new page)

a) The angle of elevation of a mobile phone tower MT of height h metres at a point A due east is 45°. From another point B, the bearing of the mobile phone tower is 061°T and the angle of elevation is 30°. The points A and B are 20 metres apart.

Calculate to 2 significant figures the height of the tower. (5)



b) An arrow fired from ground level at velocity 40 m/sec, strikes the ground 80 metres away.

Derive the equations of motion and hence find the angle at which the arrow was ~~be~~ fired. (Assume $g = 10\text{m/sec}^2$) (7)

QUESTION 5 (Please start a new page)

a) Prove by induction that for a positive integer

$$\frac{1}{1\times5}+\frac{1}{5\times9}+\ldots\ldots+\frac{1}{(4n-3)(4n+1)} = \frac{n}{4n+1}$$ (4)

b) A particle is moving in a straight line with Simple Harmonic Motion. The velocity of the particle is respectively $\sqrt{20}\ \text{ms}^{-1}$ and 4ms^{-1} at distances of 1 metre and 2 metres from the centre of motion. Find the period and amplitude of the motion. (4)

c) Evaluate $\displaystyle\int_0^{\frac{3}{4}} \frac{dx}{\sqrt{3-4x^2}}$ (4)

QUESTION 6 (Please start a new page)

a) A spherical hot air balloon is heated so that its radius is expanding at the rate of 0·4 metres per second. At what rate will the volume be increasing when the radius is 3·4 metres? (3)

b) Let $f(x)=\dfrac{x^2-4}{x}$

i) For what values is $f(x)$ undefined? (1)
ii) Show that $y=f(x)$ is an odd function. (1)
iii) Discuss the behaviour of the curve as $x \to \pm\,\infty$ (2)
iv) Show that $f'(x) > 0$ at all values of x for which the function is defined. (2)
v) Hence, or otherwise, sketch $y=f(x)$. (3)

Question 1. 2008 Manly HS. 3U Trial. Daniel Ong

(a) $\int_{}^{3} \frac{dt}{t^2+25} = \frac{1}{5}\tan^{-1}\left(\frac{x}{5}\right)$. ✓

(b). $P(x) = (x-2)(x+2)(x-1)$
$= (x^2-4)(x-1)$ ✓
$= x^3 - x^2 - 4x + 4$.

(c). $x^2 \sin^{-1} 4x$
$u = x^2 \quad v = \sin^{-1} 4x$
$u' = 2x \quad v' = \frac{4}{\sqrt{1-16x^2}}$ ✓

$\frac{d}{dx} = vu' + uv'$
$= 2x\sin^{-1}4x + \frac{4x^2}{\sqrt{1-16x^2}}$ ✓
$= 2x\left[\sin^{-1}4x + \frac{2x}{\sqrt{1-16x^2}}\right]$

(d). $\int 2x\sqrt{4-x^2}\,dx$
Let $u = 4-x^2$
$\frac{du}{dx} = -2x$ ✓

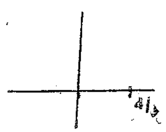
$I = -\int \sqrt{u}\,du$
$= -\frac{2u^{3/2}}{3}$
$= \frac{-2(4-x^2)\sqrt{4-x^2}}{3} + C$. ✓

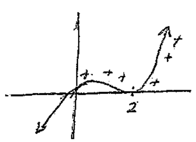
(e). $2\sin^2\theta = \sin 2\theta$
$2\sin^2\theta = 2\sin\theta\cos\theta$
$2\sin^2\theta - 2\sin\theta\cos\theta = 0$. ✓
$2\sin\theta[\sin\theta - \cos\theta] = 0$.
$\sin\theta = 0$. or $\tan\theta = 1$.
$\theta = 0, \pi, 2\pi$ ✓ $\quad \theta = \frac{\pi}{4}, \frac{5\pi}{4}$. ✓

Question 2.

(a). $P(x) = x^3 - 5x + 12$
$P(2) = 8 - 10 + 12$ ✓
$= 10$
$\neq 0$.
$\therefore (x-2)$ is not a factor. ✓

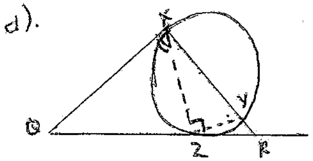
(b). $\frac{1}{x} + \frac{x}{4} \leq 1 \qquad x \neq 0$.
$\frac{4+x^2}{4x} \leq 1$
$4x(4+x^2) \leq (4x)^2$
$16x + 4x^3 \leq 16x^2$
~~$12x^2 - 16x > 0$.~~
~~$x(3x-4) > 0$.~~
$\Rightarrow 4x^3 - 16x^2 + 16x \leq 0$
$x^3 - 4x^2 + 4x \leq 0$ ~~$x \leq 0 \cap x > 4/3$.~~
$x(x^2 - 4x + 4) \leq 0$
$x(x-2)^2 \leq 0 \qquad x \geq 0$.





(c). $\sqrt{3}\cos\theta - \sin\theta = \sqrt{3}$.
$R\cos(\theta - \alpha) = \sqrt{3}$. $\Rightarrow 2\cos\left(\theta + \frac{\pi}{6}\right) = \sqrt{3}$ for $0 \leq \theta \leq 2\pi$
$R = 2 \quad \tan\alpha = \frac{1}{\sqrt{3}}$
$\alpha = \pi/6$. ✓
$\cos\left(\theta + \frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \qquad \frac{\pi}{6} \leq \theta + \frac{\pi}{6} \leq \frac{13\pi}{6}$
S | A ✓
T | C ✓
$\theta + \frac{\pi}{6} = \frac{\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}$
$\therefore \theta = 0; \frac{5\pi}{3}, 2\pi$

(d).



(i). $Z\hat{X}Q = X\hat{Y}Z$ (L's in the alternate segment). ✓
But $X\hat{Y}Z = \frac{\pi}{2} - Z\hat{X}Y$ (since $X\hat{Z}Y = \frac{\pi}{2}$, semicircle & Angle sum).
But $Z\hat{X}Y = Y\hat{Z}R$ ✓ (L's in the Alternate Segment)
$\therefore Y\hat{Z}R = \pi/2 - Y\hat{Z}R$

(ii). $Q\hat{X}Z = Q\hat{Z}X$ (Angles are equal as base of Iso △; since $XQ = ZQ$ (tangents))
$\therefore X\hat{Q}Z = \pi - 2Z\hat{X}Q$ (Angle sum)
$= \pi - 2[\pi/2 - Y\hat{Z}X]$ ✓
$= \pi - \pi + 2 \cdot Y\hat{Z}R$
$= 2 \cdot 2Y\hat{Z}R$ ✓

Question 3.

(a)(i). $f(x) = x^3 - 6x + 1 = 0$. $\qquad f'(x) = 3x^2 - 6$
$f(2) = 8 - 12 + 1 < 0$. ✓
$f(3) = 27 - 18 + 1 > 0$. ✓
$\therefore$ change in sign means there is a root between 2 & 3.

(ii). $x_1 = 2.5$
$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$ ✓
$= 2.5 - \frac{1.625}{12.75}$
$= 2.3725$ (2dp) ✓

(b). 3 Boys, 5 Girls.
(i). $8! = 40320$. ✓
(ii). $7! = 5040$. ✓
(iii). P(Two Boys not to sit Together) = All − P(Can sit together)
$= 7! - 6! \times 2$ ✓
$= 3600$.

(c). $(4+5x)^{17}$
$T_{r+1} = \binom{n}{r}(a)^{n-r}(b)^r$
$= \frac{\binom{17}{r}(4)^{17-r}(5)^r}{\binom{17}{r-1}(4)^{17-r+1}(5)^{r-1}}$
$T_r =$ ✓
$= \frac{\frac{17!}{r!(17-r)!} \times 5}{\frac{17!}{(r-1)!(17-r+1)!} \times 4} = \frac{\frac{5}{r}}{\frac{4}{(17-r+1)}}$ ✓
$= \frac{5(17-r+1)}{4r} = \frac{5(18-r)}{4r}$.

(ii). $\frac{5(18-r)}{4r} = 1$

$5(18-r) = 4r$

$90 - 5r = 4r$

$90 = 9r$ ✓

$r = 10.$

$\therefore T_{10}$ & T_{11}. ✓

Question 4.

(i). $T = A + Ce^{kt}$

$T - A = Ce^{kt}$

$\frac{dT}{dt} = kCe^{kt}$

$= k(T-A)$ ✓

$\therefore$ Satisfies Newton's Law of Cooling.

(ii). $A = 20.$

At $t=0$, $T = -5.$

$-5 = 20 + Ce^{0}.$

$-25 = C.$

At $t = 3$, $T = 10.$

$10 = 20 - 25e^{3k}$

$10 = 25e^{3k}$

$\frac{2}{5} = e^{3k}$ ✓

$\ln(2/5) = 3k$

$k = \frac{\ln 2/5}{3}.$ ✓

To Reach 0.

$0 = 20 - 25e^{\frac{\ln 2/5}{3}t}$

$\frac{+20}{25} = e^{\frac{\ln 2/5}{3}t}.$

$\ln(4/5) = \frac{\ln 2/5}{3}t.$ ✓

$\frac{3\ln(4/5)}{\ln(2/5)} = t$

$t = 12.32$ hrs. ✓

5). (i) $x^2 = -4ay$

$y = -\frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{-2x}{4a}$

$= \frac{-x}{2a}$

at $x = 2aq$ ✓

$m = \frac{-2aq}{4a}$ [sic]

$= -q.$

Eqn: $y + aq^2 = -q(x - 2aq)$

$y + aq^2 = -qx + 2aq^2$

$y + qx = aq^2$

(ii). ~~mn~~

R: when $x = 0$.

$y = aq^2$

$R: (0, aq^2)$ ✓

(iii). $\overset{m\ n}{3:1}$ $Q(2aq, -aq^2)$ $R(0, aq^2)$

External

$x = \frac{mx_2 - nx_1}{m-n}$ $y = \frac{my_2 - ny_1}{m-n}$

$= \frac{0 - 2aq}{2}$ $= \frac{3aq^2 + aq^2}{2}$ ✓

$= -aq$ $= 2aq^2.$

$\therefore S(-aq, 2aq^2)$ ✓

(iv). $x = -aq$ $y = 2aq^2$

$-\frac{x}{a} = q$

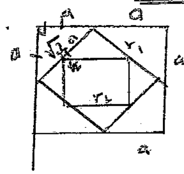
$\therefore y = 2a\left(\frac{x^2}{a^2}\right)$

$= \frac{2ax^2}{a^2}$ ✓

$y = \frac{2x^2}{a}$ ✓

$ay = 2x^2$

(c).



$r_1 = \sqrt{2a^2}$
$= \sqrt{2}a$

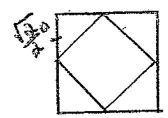
Square (Rectangle 1) = $4a^2$

Square of side $4a$ not $2a$, need to adjust your answers!

Inner Square $= 4a^2 - 4(\frac{1}{2}a^2)$

$= 4a^2 - 2a^2$

$= 2a^2$



$r_2 = \sqrt{2 \times \frac{1}{\sqrt{2}}a^2}$

$= \sqrt{\sqrt{2}a^2}$

$= 2^{1/4}a.$

Next Square $= 2a^2 - 4(\frac{1}{2} \times \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}a^2)$

$= 2a^2 - 2(\frac{1}{2})a^2$

$= 2a^2 - a^2$

$= a^2.$

$\therefore$ Sum Total Area $= 4a^2 + 2a^2 + a^2 + \frac{1}{2}a^2 \ldots$

$= a^2[4 + 2 + 1 + 1/2 + \ldots]$

Limiting Sum $= \frac{a}{1-r}.$

$= a^2\left[\frac{4}{1 - 1/2}\right]$

$= a^2[8]$ ✓

$= 8a^2.$

Question 5.

(a). $\frac{1}{1 \times 5} + \frac{1}{5 \times 9} + \ldots \frac{1}{(4n-3)(4n+1)} = \frac{n}{4n+1}$

Let $n = 1$.

$\frac{1}{(4-3)(4+1)} = \frac{1}{5}.$ ✓

$LHS = \frac{1}{1 \times 5}$

$= RHS.$

Assume true for $n = k$

$\frac{1}{5} + \frac{1}{5 \times 9} + \ldots - \frac{1}{(4k-3)(4k+1)} = \frac{k}{4k+1}$

Prove true for $n = k+1$

$LHS = \frac{k}{4k+1} + \frac{1}{(4(k+1)-3)(4(k+1)+1)}$

$= \frac{k}{4k+1} + \frac{1}{(4k+1)(4k+5)}$

$= \frac{k(4k+5) + 1}{(4k+1)(4k+5)}$

$= \frac{4k^2 + 5k + 1}{(4k+1)(4k+5)}$

$= \frac{(4k+1)(4k+1)}{(4k+1)(4k+5)}$ ✓ $\frac{k+1}{4k+5}$ = RHS.

$RHS = \frac{k+1}{4(k+1)+1}$

$= \frac{k+1}{4k+5}.$

$\therefore$ If true for $n = k$, then true for $n = k+1$, by the Principle of Maths Induction for all positive integers of n.

(b). SHM.

$V = \sqrt{20}$, at $x = 1$

$V^2 = 20$, at $x = 2$.

$$V^2 = n^2(a^2 - x^2)$$

$$20 = n^2(a^2 - 1) \quad ①$$

$$16 = n^2(a^2 - 4) \quad ②$$

$$\frac{①}{②}$$

$$= \frac{5}{4} = \frac{a^2 - 1}{a^2 - 4}.$$

$$5(a^2 - 4) = 4(a^2 - 1)$$

$$5a^2 - 20 = 4a^2 - 4$$

$$a^2 = 16$$

$$\underline{a = 4}.$$

sub into ①.

$$20 = n^2(16 - 1)$$

$$20 = 15n^2$$

$$\frac{4}{3} = n^2$$

$$\underline{\underline{n = \frac{2}{\sqrt{3}}}}$$

(c). $\int_0^{3/4} \frac{dx}{\sqrt{9 - 4x^2}}$

$$= \frac{1}{2}\int_0^{3/4} \frac{dx}{\sqrt{9/4 - x^2}}$$

$$= \left[\frac{1}{2}\sin^{-1}\left(\frac{2x}{\sqrt{3}}\right)\right]_0^{3/4} = \frac{1}{2}\left[\sin^{-1}\left(\frac{2}{\sqrt{3}} \times \frac{3}{4}\right)\right]$$

$$\cancel{= \frac{1}{2}\sin^{-1}(1)} \qquad = \frac{1}{2}\left[\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right]$$

$$\cancel{= \frac{\pi}{4}.} \qquad = \frac{1}{2} \times \frac{\pi}{3} = \frac{\pi}{6}$$

Question 6.

(a) $\frac{dr}{dt} = 0.4$

$$V = \tfrac{4}{3}\pi r^3$$

$$\frac{dV}{dr} = 4\pi r^2$$

$$\frac{dV}{dt} = \frac{dr}{dt} \times \frac{dV}{dr}$$

$$= 0.4 \times 4\pi r^2$$

$$= 1.6\pi r^2$$

at $r = 3.4$.

$$\underline{\underline{\frac{dV}{dt} = 18.496\pi}}$$

(b)(i) $f(x) = \frac{x^2 - 4}{x}$ $\quad y = x - \frac{4}{x}$

$x \neq 0$.

(ii). $f(-x) = \frac{(-x)^2 - 4}{(-x)}$

$$= -\left[\frac{x^2 - 4}{x}\right]$$

$$= -f(x)$$

$\therefore$ Odd. $f(x)$

(iii). As $x \to \infty$ $\quad$ As $x \to \pm\infty$

$f(x) \to \infty$ $\quad y \to \pm x$

~~As $x \to -\infty$~~

~~$f(x) \to -\infty$~~

$u = x^2 - 4 \quad v = x$

$u' = 2x \quad v' = 1$

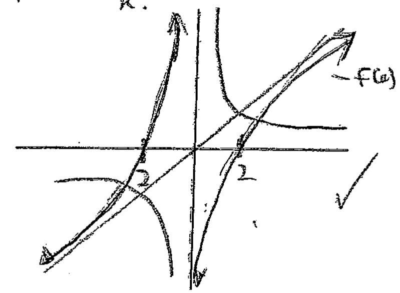
(iv). $f'(x) = \frac{vu' - uv'}{v^2}$

$$= \frac{2x^2 - x^2 + 4}{x^2}$$

$$= \frac{x^2 + 4}{x^2} > 0.$$

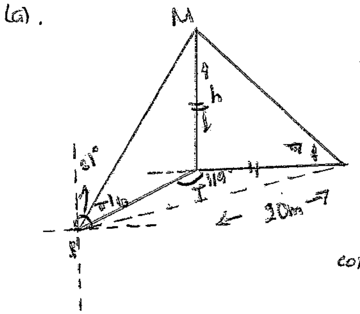
$x^2 + 4 > 0$. All real x, $x \neq 0$.

(v). $f(x) = \frac{x^2 - 4}{x}$.



Question 7.

(a).



A $\quad \tan \pi/6 = \frac{h}{BT}$ $\qquad \tan \pi/4 = \frac{h}{AT}$

$AT = h$.

$$\cot \pi/6 = \frac{BT}{h}.$$

$$\cot\frac{\pi}{6} = \frac{1}{\tan\frac{\pi}{6}} \qquad \sqrt{3} \quad \cancel{\frac{1}{\sqrt{3}}} = \frac{BT}{h}$$

$$= \frac{1}{\frac{1}{\sqrt{3}}} \qquad \frac{h}{\sqrt{3}} = BT$$

$$BA^2 = BT^2 + TA^2 - 2\,BT.TA.\cos B\hat{T}A$$

$$400 = \frac{h^2}{3} + h^2 - \frac{2h^2}{\sqrt{3}}\cos 119°$$ Correct approach!

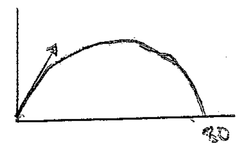
$$= h^2\left[\frac{1}{\sqrt{3}} + 1 + 0.4848\right]$$

$$400 = h^2[1.0925]$$

$$h^2 = 366.119$$

$\underline{\underline{h = 14.}}$ (2 sig figs).

(b). $V = 40\text{m/s}$.



$$\ddot{x} = 0.$$

$$\dot{x} = c_1$$

at $t = 0$, $\dot{x} = V\cos\alpha$

$$\therefore \dot{x} = V\cos\alpha$$

$$x = Vt\cos\alpha + c_2$$

at $t = 0$, $x = 0$.

$$\underline{\underline{x = Vt\cos\alpha}}$$

$$\ddot{y} = -g$$

$$\dot{y} = -gt + c_3$$

At $t = 0$, $\dot{y} = V\sin\alpha$

$$\dot{y} = -gt + V\sin\alpha$$

$$y = \frac{-gt^2}{2} + Vt\sin\alpha + c_4$$

At $t = 0$, $y = 0$.

$$\underline{\underline{y = \frac{-gt^2}{2} + Vt\sin\alpha.}}$$

$$x = Vt\cos\alpha$$

$$\frac{x\sec\alpha}{V} = t$$

$$y = \frac{-gt^2}{2} + Vt\sin\alpha \qquad \text{since } g = 10$$

$$y = -5t^2 + Vt\sin\alpha$$

$$y = -5\left(\frac{x^2\sec^2\alpha}{V^2}\right) + V\left(\frac{x\sin\alpha}{V\cos\alpha}\right)$$

$$y = \frac{-5x^2\sec^2\alpha}{V^2} + x\tan\alpha$$

$y = -\frac{5x^2}{V^2}\sec^2\alpha + x\tan\alpha.$

when $y=0$, $x=80$.

$$0 = -\frac{5x^2}{V^2}\sec^2\alpha + x\tan\alpha$$

$$0 = -\frac{32000}{V^2}\sec^2\alpha + 80\tan\alpha$$

$$V = 40.$$

$$0 = -20\sec^2\alpha + 80\tan\alpha$$

$$4\tan\alpha = \sec^2\alpha$$

$$4\tan\alpha = (1+\tan^2\alpha)$$

$$\tan^2\alpha - 4\tan\alpha + 1 = 0.$$

$$\tan\alpha = \frac{4 \pm \sqrt{16-4}}{2}.$$

$$= \frac{4 \pm \sqrt{12}}{2}$$

$$\tan\alpha = 2 \pm \sqrt{3}$$

$$\alpha = 75° \text{ or } 15$$

$$= \frac{5\pi}{12} \text{ or } \frac{\pi}{12}$$

or.

$$\text{Range} = \frac{V^2\sin 2\alpha}{g}$$

$$80 = \frac{1600\sin 2\alpha}{10}$$

$$\frac{1}{2} = \sin 2\alpha$$

$$2\alpha = \pi/6,\ \frac{5\pi}{6}$$

$$\alpha = \pi/12,\ \frac{5\pi}{12}.$$