Merewether High School

Trial Higher School Certificate Examination 1999

Mathematics 3U Additional, 4U Common Paper

Time Allowed: 2 hours, plus 5 minutes reading time.

Instructions:

- Marks may be deducted for untidy or poorly arranged work.
- Approved calculators may be used, and the table of standard integrals is provided.
- Show all necessary working - a correct answer without appropriate working may not receive full marks.
- Put your student's number on each page.
- START EACH QUESTION ON A NEW SHEET OF PAPER.

QUESTION 1 (12 marks)

(a) Solve for $0 \le x \le 2\pi$

$$\cos 2x - 3\sin x - 2 = 0$$ 3

(b) Find $\int x\sqrt{1-x}\,dx$, using the substitution $u = 1 - x$ 3

(c)

(i) Sketch the graph of the function

$$y = 2\tan^{-1}x$$ 2

(ii) What value does $2\tan^{-1}x$ approach as x increases indefinitely? 1

(iii) Find the exact equation of the tangent to the curve

$y = 2\tan^{-1}x$, at the point where $x = 1$. 3

QUESTION 2 (12 marks)

(a) Solve $x + \frac{1}{x} \geq 2$ 2

(b) Find:

(i) $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$ 2

(ii) $\int \frac{x.dx}{x^2+1}$ 2

(c) If α, β and γ are the roots of the equation

$$2x^3 + 6x - 3 = 0,$$

find the value of:

(i) $\alpha+\beta+\gamma$, $\alpha\beta+\alpha\gamma+\beta\gamma$ and $\alpha\beta\gamma$ 3

(ii) $\alpha^{-1} + \beta^{-1} + \gamma^{-1}$ 3

QUESTION 3 (12 marks)

(a) Find the derivative, with respect to x, of

(i) $\log_e(\cos^{-1}x)$ 2

(ii) $\sin^{-1}5x$ 2

(b)Use Mathematical Induction to prove that for all positive integral values of n:

$$1\times2 + 2\times3 + 3\times4 + \ldots + n\times(n+1) = \frac{n(n+1)(n+2)}{3}$$ 5

QUESTION 3 continued

(b) The area under the curve y = sinx between x = 0 and $x = \frac{\pi}{4}$ is rotated about the x-axis through one complete revolution. Find the volume of the solid formed, to one decimal place. 3

QUESTION 4 (12 marks)

(a) Find the acute angle between the straight lines $2y - x + 1 = 0$ and $y = 5x + 2$, giving the answer correct to the nearest degree. 2

(b) A is the point (-2,-1) and B is the point (1,5). Find the co-ordinates of the point Q, which divides the interval AB externally in the ratio 5:3 2

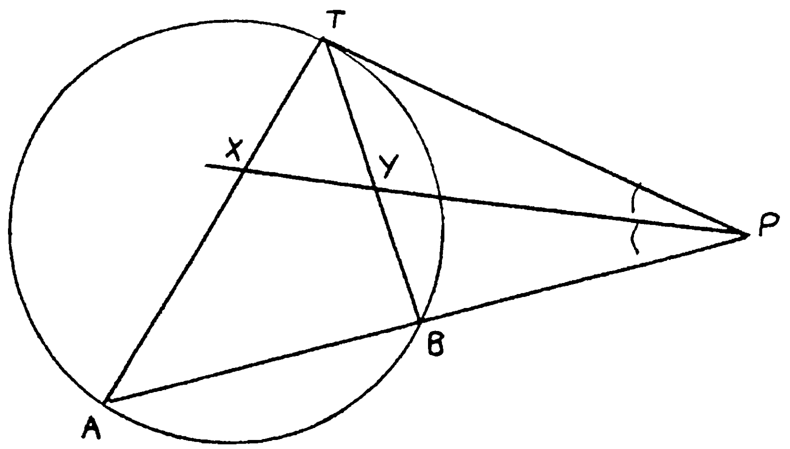
(c) (i) Express $7\cos\theta - \sin\theta$ in the form $R\cos(\theta + \alpha)$, where $R>0$ and $0° \le \alpha \le 90°$ 2

(ii) Hence solve $7\cos\theta - \sin\theta = 5$ for $0° \le \theta \le 360°$, giving answers to the nearest degree. 3

(c) Evaluate exactly $\int_0^1 (e^{-x} + \frac{1}{1+x} - \frac{1}{\sqrt{1-x^2}})dx$ 3

QUESTION 5 (12 marks)

(a)



The tangent at T on the circle meets a chord AB produced at P. The bisector of (TPA meets TA and TB at X and Y respectively.

(i) Give the reason why $\angle PTB = \angle TAB$ 1

(ii) Prove that TX = TY. 3

(iii) Prove that $\dfrac{TX}{XA} = \dfrac{TP}{PA}$ 2

(b) In how many ways can a train of nine carriages be arranged if four of the carriages (A,B,C, D):

(i) are to be at the rear in the order ABCD with D last? 1

(ii) must be kept together but in any order? 2

(c) (i) Expand $\tan(\alpha + \beta)$ 1

(ii) If tanA and tanB are the roots of the equation

$$3x^2 - 5x - 2 = 0,$$

find the value of tan(A + B). 2

QUESTION 6 (12 marks)

(a) The tangent at P $(2ap, ap^2)$ on the parabola $x^2 = 4ay$ meets the x-axis in T. The normal at P meets the y-axis in N.

(i) Find the co-ordinates of M, the midpoint of TN. The equations of the tangent and the normal need NOT be derived. 2

(ii) Show that the locus of M is the parabola

$$x^2 = \frac{a}{2}(y - a).$$ 3

(b) A circular oil slick lies on the surface of calm water. Its area is increasing at the rate of 12 m^2/min. At what rate is the radius increasing at the time at which the radius is 3 metres? 3

(c) Find $\frac{d}{dx}(\sqrt{1-x^2} + x\sin^{-1}x)$. Hence evaluate $\int_0^{\frac{1}{2}} \sin^{-1}x.dx$ correct to three significant figures. 4

QUESTION 7 (12 marks)

(a) A certain particle moves along the x-axis according to the law

$$t = 2x^2 - 5x + 3$$

where x is measured in centimetres and t in seconds. Initially the particle is 1.5 cm to the right of the origin O and moving away from O.

(i) Prove that the velocity, v cms^{-1}, is given by

$$v = \frac{1}{4x-5}$$ 2

(ii) Find an expression for the acceleration, a cms^{-2}, in terms of x. 2

(iii) Find the velocity of the particle when t = 6 seconds. 3

QUESTION 7 continued

(d) A particle is moving in Simple Harmonic Motion with acceleration

$$\ddot{x} = -4x \text{ mms}^{-2}$$

If the particle starts at the origin with a velocity of 8 mms^{-1}.

(i) State its period 1

(i) Find its displacement after $\frac{\pi}{3}$ seconds. 4

'99 3U Trial HSC – SOLUTIONS

1. a) $\cos 2x - 3\sin x - 2 = 0$

$(1 - 2\sin^2 x) - 3\sin x - 2 = 0$

$2\sin^2 x + 3\sin x + 1 = 0$

$(2\sin x + 1)(\sin x + 1) = 0$

$\sin x = -\frac{1}{2}, -1$

$\therefore x = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{3\pi}{2}$

b) $I = \int x\sqrt{1-x}\,.dx$

let $u = 1 - x \Rightarrow x = 1 - u$

$\therefore du = -1.dx$

$\therefore I = \int (1-u)\sqrt{u}\,.-du$

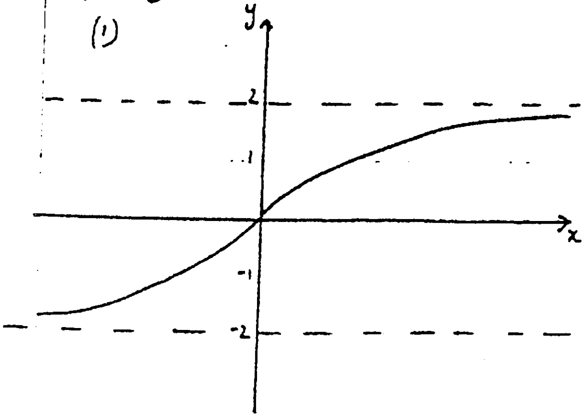
$= \int (u^{\frac{3}{2}} - u^{\frac{1}{2}})du$

$= \frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}} + c$

$= \frac{2(1-x)^{\frac{5}{2}}}{5} - \frac{2(1-x)^{\frac{3}{2}}}{3} + c$

c) $y = 2\tan^{-1}x$

(i)



(ii) as $x \to \infty$, $2\tan^{-1}x \to 2$

(iii) at $x = 1$

$y = 2\tan^{-1}1$

$= \frac{\pi}{2}$

also $\frac{dy}{dx} = \frac{2}{1+x^2}$

at $x = 1$, $\frac{dy}{dx} = \frac{2}{1+1} = 1$

eq'n of tangent:

$\frac{y - y_1}{x - x_1} = m$

$\frac{y - \frac{\pi}{2}}{x - 1} = 1$

$y - \frac{\pi}{2} = x - 1$

$\therefore y = x - 1 + \frac{\pi}{2}$ is eq'n of tange

2 a) $x + \frac{1}{x} \geq 2$

$x \neq 0$

Solve $x + \frac{1}{x} = 2$

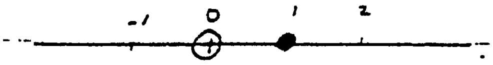
$x^2 + 1 = 2x$

$x^2 - 2x + 1 = 0$

$(x-1)^2 = 0$

$x = 1$

Graph + test



$x = -1$	$x = \frac{1}{2}$	$x = 2$
$-1 + \frac{1}{-1} \geq 2$	$\frac{1}{2} + \frac{1}{\frac{1}{2}} \geq 2$	$2 + \frac{1}{2} \geq 2$
False	True	True

$\therefore x > 0$ is the solution

b) (i) $I = \int_0^1 \frac{dx}{\sqrt{1-x^2}}$

$= [\sin^{-1}x]_0^1$

$= \sin^{-1}1 - \sin^{-1}0$

$= \frac{\pi}{2}$

(ii) $I = \int \frac{x}{x^2+1}.dx$

$= \frac{1}{2}\int \frac{2x}{x^2+1}dx$

$= \frac{1}{2}\ln(x^2+1) + c$

c) $2x^3 + 0x^2 + 6x - 3 = 0$

(i) $\alpha + \beta + \gamma = -\frac{b}{a} = -\frac{0}{2} = 0$

$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a} = \frac{6}{2} = 3$

$\alpha\beta\gamma = -\frac{d}{a} = -\frac{-3}{2} = 1\frac{1}{2}$

(ii) $Exp = \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

$= \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$

$= \frac{3}{1\frac{1}{2}}$

$= 2$

3. a) (i) $f(x) = \ln(\cos^{-1}x)$

$f'(x) = \frac{1}{\cos^{-1}x} \times \frac{-1}{\sqrt{1-x^2}}$

$= \frac{-1}{\cos^{-1}x . \sqrt{1-x^2}}$

(ii) $f(x) = \sin^{-1}5x$

$f'(x) = \frac{1}{\sqrt{1-25x^2}} \times 5$

$= \frac{5}{\sqrt{1-25x^2}}$

b) To prove $S(n)$:

$1\times 2 + 2\times 3 + 3\times 4 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$

Step 1 Test $S(1)$

$LHS = 1\times 2 = 2$

$RHS = \frac{1(1+1)(1+2)}{3} = \frac{6}{3} = 2$

$\therefore LHS = RHS$

$\therefore$ statement is true for $n = 1$.

Step 2: Assume $S(k)$ is true

ie $1\times 2 + 2\times 3 + \cdots + k(k+1) = \frac{k(k+1)(k+2)}{3}$

Step 3: Test $S(k+1)$

ie Is $1\times 2 + 2\times 3 + \cdots + (k+1)(k+2) = \frac{(k+1)(k+2)(k+3)}{3}$

$LHS = 1\times 2 + 2\times 3 + \cdots + k(k+1) + (k+1)(k+2)$

$= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2)$, from (2

$= \frac{k(k+1)(k+2) + 3(k+1)(k+2)}{3}$

$= \frac{(k+1)(k+2)[k+3]}{3}$

$= RHS$

$\therefore$ statement is true for $n = k+1$ if it is true for $n = k$.

Step 4 $S(1)$ is true

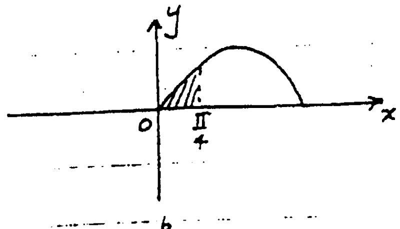
$\therefore S(1+1)$ is true ie $S(2)$ is true

$\therefore S(2+1)$ is true ie $S(3)$ is true

and so on.

$\therefore$ by the principle of Math. Induction $S(n)$ is true for all positive integral values of n.

b)



$V = \pi\int_a^b y^2\,dx$

$= \pi\int_0^{\frac{\pi}{4}} \sin^2 x\,dx$

$= \pi\int_0^{\frac{\pi}{4}} \frac{1}{2}(1-\cos 2x)\,dx$

$= \frac{\pi}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{4}}$

$= \frac{\pi}{2}\left[\frac{\pi}{4} - \frac{\sin\frac{\pi}{2}}{2} - 0 + \frac{\sin 0}{2}\right]$

$= \frac{\pi}{2}\left[\frac{\pi}{4} - \frac{1}{2}\right]$

$= 0.05$ units^3 (2 dec. pl.)

4. a) $y = \frac{1}{2}x - \frac{1}{2}$

2

$\therefore m_1 = \frac{1}{2}$

$y = 5x + 2$

$\therefore m_2 = 5$

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

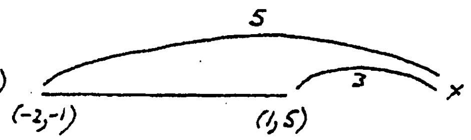
$= \left|\frac{\frac{1}{2} - 5}{1 + \frac{1}{2}\times 5}\right|$

$= 1\frac{2}{7}$

$\therefore \theta = 52°$ (nearest degree)

b)

2



$x = \frac{kx_2 + lx_1}{k+l}$

$= \frac{5\times 1 + -3\times -2}{5 + -3}$

$= 5\frac{1}{2}$

$y = \frac{ky_2 + ly_1}{k+l}$

$= \frac{5\times 5 + -3\times -1}{5 + -3}$

$= 14$

$\therefore P$ is $(5\frac{1}{2}, 14)$

c) (i) let $7\cos\theta - \sin\theta \equiv R\cos(\theta + \alpha)$

2

$\equiv R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$

$\therefore R\cos\alpha = 7$

$R\sin\alpha = 1$

$\therefore R^2\cos^2\alpha + R^2\sin^2\alpha = 49 + 1$

$R = \sqrt{50}$, $R > 0$

$= 5\sqrt{2}$

also $\frac{R\sin\alpha}{R\cos\alpha} = \frac{1}{7}$

$\tan\alpha = \frac{1}{7}$

$\alpha = 8°8'$

$\therefore 7\cos\theta - \sin\theta = 5\sqrt{2}\cos(\theta + 8°8')$

(ii) $7\cos\theta - \sin\theta = 5$

3

$5\sqrt{2}\cos(\theta + 8°8') = 5$

$\cos(\theta + 8°8') = \frac{1}{\sqrt{2}}$

$\therefore (\theta + 8°8') = 45°, 315°$

$\theta = 37°, 307°$ (nearest deg

d) $I = \int_0^1 \left(e^{-x} + \frac{1}{1+x} - \frac{1}{\sqrt{1-x^2}}\right)dx$

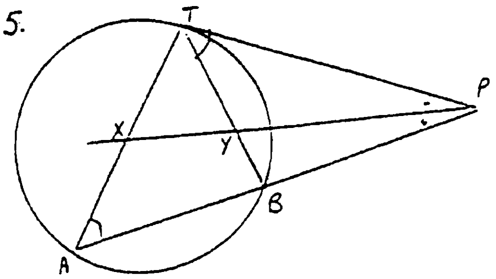
3

$= \left[-e^{-x} + \ln(1+x) - \sin^{-1}x\right]_0^1$

$= (-e^{-1} + \ln 2 - \sin^{-1}1) - (-e^{0} + \ln 1 - \sin^{-1}0$

$= -\frac{1}{e} + \ln 2 - \frac{\pi}{2} + 1$

5.



(i) The angle between a tangent and a chord of contact is equal to the angle in the alternate segment. 1

(ii) In Δ's AXP and TYP,

3 $\angle PTB = \angle TAB$ from (i)

$\angle TPY = \angle XPA$ (PX bisector of $\angle TPA$, data)

$\therefore \Delta AXP \ ||| \ \Delta TYP$ (2 prs corr. ∠'s equal)

$\therefore \angle AXP = \angle TYP$ (3rd ∠ sim Δ's)

$\angle TXY = \angle 180° - \angle AXP$ (∠'s st. line)

and $\angle TYX = \angle 180° - \angle TYP$ (∠'s st. line)

$\therefore \angle TXY = \angle TYX$

$\therefore \Delta TXY$ is isosceles (base ∠s equal)

$\therefore TX = TY$ (equal sides isos. Δ)

(iii) Since Δ's AXP & TPY are similar from (ii),

2 corr. sides are in same ratio

ie $\frac{TY}{XA} = \frac{TP}{PA}$

But TY = TX from (ii)

$\therefore \frac{TX}{XA} = \frac{TP}{PA}$

b)(i) ABCD is 1 arrangement.

∴ ABCD are at the end + 5 other carriages must be arrange at the front.

No of ways $= 5! \times 1$

$= 120$ 1

(ii) A, B, C and D can be arranged in 4! ways. 2

∴ there are 6 'objects' to be arrang.

No of ways $= 6! \times 4!$

$= 17280$

2880
$5! \times 4!$
Pay 1

c)(i) $\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$ 1

(ii) $3x^2 - 5x - 2 = 0$

$(x-2)(3x+1) = 0$ 2

$x = 2, -\frac{1}{3}$ 1

∴ let tan A be 2, tan B be $-\frac{1}{3}$

$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

$= \frac{2 + -\frac{1}{3}}{1 - 2 \times -\frac{1}{3}}$

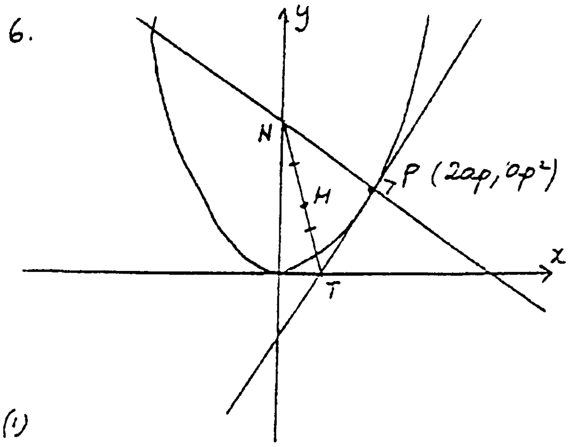
$= \frac{1\frac{2}{3}}{1\frac{2}{3}}$

$= 1$

Note same answer if tan A is $-\frac{1}{3}$, tan B is.

or $\tan A + \tan B = \frac{5}{3}$
$\tan A \tan B = -\frac{2}{3}$ } 1

6.



(i)

tangent at P is $y - px + ap^2 = 0$

let $y = 0$ ie $-px + ap^2 = 0$

$x = ap$

$\therefore$ T is $(ap, 0)$

normal at P is $x + py = ap^3 + 2ap$

let $x = 0$ ie $py = ap^3 + 2ap$

$y = ap^2 + 2a$

$\therefore$ N is $(0, ap^2 + 2a)$

$\therefore$ M is $\left(\frac{ap+0}{2}, \frac{0+ap^2+2a}{2}\right)$

ie $\left(\frac{ap}{2}, \frac{ap^2+2a}{2}\right)$

(ii) parametric equations of locus of M are

$x = \frac{ap}{2}$ (i)

$y = \frac{ap^2+2a}{2}$ (ii)

from (i) $p = \frac{2x}{a}$

sub. into (ii)

$y = \frac{a\left(\frac{2x}{a}\right)^2 + 2a}{2}$

$2y = \frac{4ax^2}{a^2} + 2a$

$2y = \frac{4x^2}{a} + 2a$

$2ay = 4x^2 + 2a^2$

$\therefore 4x^2 = 2ay - 2a^2$

$4x^2 = 2a(y - a)$

$x^2 = \frac{2a}{4}(y - a)$

$\therefore$ $x^2 = \frac{a}{2}(y - a)$ is Cartesian equation of the locus of M.

b) $A = \pi r^2$

$\therefore \frac{dA}{dr} = 2\pi r$

given that $\frac{dA}{dt} = 12$, find $\frac{dr}{dt}$

$\frac{dr}{dt} = \frac{dr}{dA} \times \frac{dA}{dt}$ by the Chain Rule

$= \frac{1}{2\pi r} \times 12$

at $r = 3$

$\frac{dr}{dt} = \frac{1}{2\pi(3)} \times 12$

$= \frac{2}{\pi}$

$\therefore$ radius is increasing at $\frac{2}{\pi}$ m/min

c) $Exp = \frac{d}{dx}\left(\sqrt{1-x^2} + x\sin^{-1}x\right)$

$= \frac{1}{2}(1-x^2)^{-\frac{1}{2}} \times -2x + \sin^{-1}x + x \times \frac{1}{\sqrt{1-x^2}}$

$= \frac{-x}{\sqrt{1-x^2}} + \sin^{-1}x + \frac{x}{\sqrt{1-x^2}}$

$= \sin^{-1}x$

$\therefore I = \int_0^{\frac{1}{2}} \sin^{-1}x \, dx$

$= \left[\sqrt{1-x^2} + x\sin^{-1}x\right]_0^{\frac{1}{2}}$

$= \left(\sqrt{\frac{3}{4}} + \frac{1}{2}\sin^{-1}\frac{1}{2}\right) - \left(\sqrt{1} + 0\sin^{-1}0\right)$

$= \frac{\sqrt{3}}{2} + \frac{1}{2} \times \frac{\pi}{6} - 1$

$= \frac{\sqrt{3}}{2} + \frac{\pi}{12} - 1$

$= 0.128$ to 3 sig. fig.

7\. a) $t = 2x^2 - 5x + 3$

(i) $\frac{dt}{dx} = 4x - 5$

$\therefore \frac{dx}{dt} = \frac{1}{4x-5}$

$\therefore v = \frac{1}{4x-5}$

(ii) $\frac{d^2x}{dt^2} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$= \frac{d}{dx}\left(\frac{1}{2} \cdot \frac{1}{(4x-5)^2}\right)$

$= \frac{1}{2} \times -2(4x-5)^{-3} \times 4$

$= \frac{-4}{(4x-5)^3}$

$\therefore a = \frac{-4}{(4x-5)^3}$

(iii) at $t = 6$, $6 = 2x^2 - 5x + 3$

$2x^2 - 5x - 3 = 0$

$(x-3)(2x+1) = 0$

$x = 3, -\frac{1}{2}$

But $\frac{1}{4x-5} \neq 0$

$\therefore v \neq 0$

$\therefore$ particle doesn't change direction

it starts at $x = 1.5$ + moves right

$\therefore$ at $t = 6$, $x = 3$

at $x = 3$ $\quad v = \frac{1}{4(3)-5}$

$= \frac{1}{7}$

$\therefore$ velocity is $\frac{1}{7}$ cm.s^{-1} when $t = 6$ s.

7\. d) $\ddot{x} = -4x$

$\therefore n = 2$

(i) period $= \frac{2\pi}{n}$

$= \frac{2\pi}{2}$

$= \pi$

(ii) $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$\therefore \frac{1}{2}v^2 = \int -4x\,dx$

$= -2x^2 + c$

at $x = 0$, $v = 8$

$32 = 0 + c$

$c = 32$

$\therefore \frac{1}{2}v^2 = -2x^2 + 32$

$v^2 = 64 - 4x^2$

$\therefore v = \pm\sqrt{64 - 4x^2}$

$= \pm 2\sqrt{16 - x^2}$

at $x = 0$, v is positive

$\therefore v = 2\sqrt{16 - x^2}$

$\frac{dx}{dt} = 2\sqrt{16 - x^2}$

$\frac{dt}{dx} = \frac{1}{2} \cdot \frac{1}{\sqrt{16 - x^2}}$

$\therefore t = \frac{1}{2}\sin^{-1}\frac{x}{4} + k$

at $t = 0$, $x = 0$

$\therefore 0 = 0 + k$

$\therefore t = \frac{1}{2}\sin^{-1}\frac{x}{4}$

$2t = \sin^{-1}\frac{x}{4}$

$\therefore \frac{x}{4} = \sin 2t$

$x = 4\sin 2t$

at $t = \frac{\pi}{3}$

$x = 4\sin\frac{2\pi}{3}$

$= 4 \times \frac{\sqrt{3}}{2} = 2\sqrt{3}$

$\therefore$ displacement is $2\sqrt{3}$ mm after $\frac{\pi}{3}$ s