NEWINGTON COLLEGE

HSC Trial Examination 2000

12 MATHEMATICS

3 UNIT ADDITIONAL

3/4 UNIT COMMON

Time allowed - 2 hours

(plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES:

All questions are of equal value.

All questions may be attempted.

In every question, show all necessary working.

Marks may not be awarded for careless or badly arranged work.

The marks allocated to each part of a question have been indicated on the LHS of each question.

Approved silent calculators may be used.

A table of standard integrals is provided for your convenience.

The answers to the seven questions in this paper are to be returned in separate bundles clearly marked Question 1, Question 2 etc. Start each question on a new page.

Each bundle must show the candidate's computer number.

The questions are not necessarily arranged in order of difficulty. Candidates are advised to read the whole paper carefully at the start of the examination.

Unless otherwise stated candidates should leave their answers in simplest exact form.

QUESTION ONE *(12 Marks)*

1 (a) The polynomial $P(x) = x^3 + ax^2 + 2x - 4$ has a remainder of -7 on division by $x+2$. Find the value of a.

2 (b) Find, to the nearest degree, the acute angle between the lines $x - y = 2$ and $3x + y = 5$.

3 (c) (i) Write $\cos 2x$ in terms of $\sin^2 x$.

(ii) Without the use of a calculator, evaluate $\cos\left(2\sin^{-1}\frac{\sqrt{3}}{2}\right)$.

3 (d) Solve for x: $\frac{1}{x-3} \geq 1$

3 (e) Evaluate $\int_0^1 x\sqrt{1-x}\,dx$ using the substitution $u = 1 - x$.

QUESTION 2 *(12 Marks)* Start a new page

3 (a) Prove the identity $\frac{\cos x - \cos 2x}{\sin 2x + \sin x} = \operatorname{cosec} x - \cot x$.

3 (b) The equation $2\sin x - 10x + 5 = 0$ has a root near $x = \frac{1}{2}$. Use Newton's method once to obtain a better approximation to the root, correct to two decimal places.

4 (c) Find all angles θ, where $0 \leq \theta \leq 2\pi$, for which $\sqrt{3}\cos\theta - \sin\theta = 1$.

2 (d) Sketch $y = 2\sin^{-1}\frac{x}{3}$.

Question 3 on page 2.....

QUESTION THREE *(12 Marks)* **Start a new page**

9 (a) $P(2ap, ap^2)$ is a point on the parabola $x^2 = 4ay$.

(i) Show that the equation of the normal to the parabola at the point P is $x + py = 2ap + ap^3$.

(ii) If the normal at P cuts the y-axis at Q show that the coordinates of Q are $(0, 2a + ap^2)$.

(iii) Show that the coordinates of R which divided the interval PQ externally in the ration 2:1 are $(-2ap, 4a + ap^2)$.

(iv) Find the cartesian equation of the locus of R.

(v) Show that if the normal at P passes through a given point (h, k) then p must be a root of the equation $ap^3 + (2a - k)p - h = 0$.

(vi) Hence state the maximum number of normals to the parabola $x^2 = 4ay$ which can pass through any given point.

3 (b) A spherical snowball is melting so that its volume decreases at a rate proportional to its surface area, that is $\frac{dV}{dt} = -kS$ where V is the volume at time t, S is the surface area at time t and k is a positive constant. Using the results $V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2$ show that:

(i) $\frac{dV}{dr} = S$,

(ii) the radius of the snowball is decreasing at a constant rate.

Question 4 on page 3....

QUESTION FOUR *(12 Marks)* **Start a new page**

8 (a) A particle is oscillating in simple harmonic motion about a fixed point. Its displacement x cm from the point $x = 0$ at time t seconds is given by $x = 2\cos 3t + 4$.

(i) Explain why $2 \le x \le 6$.

(ii) Hence write down the amplitude and the centre of the motion.

(iii) Find $\ddot{x}$ as a function of t.

(iv) Hence show that $\ddot{x} = -9(x - 4)$.

(v) Show that $v^2 = -9x^2 + 72x - 108$.

(vi) Hence, or otherwise find the greatest speed of the particle.

4 (b) (i) Show that the function $T = R + Ae^{-kt}$ is a solution of the differential equation $\dfrac{dT}{dt} = -k(T - R)$.

(ii) A metal baking dish is removed from an oven at 200^o C. If the dish takes one minute to cool to 170^o C and the room temperature is 20^o C find the time, correct to the nearest minute, that it takes for the baking dish to cool to 50^o C. (Assume that the baking dish cools at a rate proportional to the difference between the temperature of the baking dish and the temperature of the surrounding air.)

QUESTION FIVE *(12 Marks)* **Start a new page**

5 (a) Suppose that $(3 + 2x)^{15} = \sum_{k=0}^{15} t_k x^k$.

(i) Use the binomial theorem to write down an expression for t_k.

(ii) Show that $\dfrac{t_{k+1}}{t_k} = \dfrac{30 - 2k}{3k + 3}$.

(iii) Hence find the greatest coefficient in the expansion of $(3 + 2x)^{15}$.

Question 5 continued on page 4....

4 (b) Consider the geometric series $S = 1 + (1+x) + (1+x)^2 + (1+x)^3 + ... + (1+x)^n$.

(i) Show that $S = \dfrac{(1+x)^{n+1} - 1}{x}$.

(ii) Hence show that:

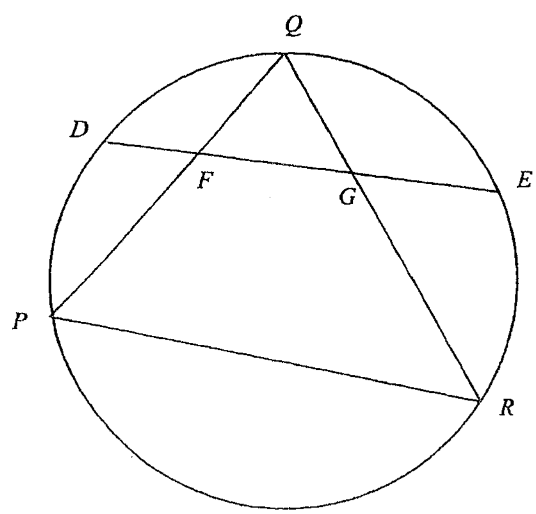
$$S = {}^{n+1}C_1 + {}^{n+1}C_2\, x + {}^{n+1}C_3\, x^2 + ... + {}^{n+1}C_{r+1}\, x^r + ... + {}^{n+1}C_{n+1} x^n$$

(iii) By considering terms in x^r from each expression for S, prove that ${}^nC_r + {}^{n-1}C_r + {}^{n-2}C_r + ... + {}^rC_r = {}^{n+1}C_{r+1}$.

3 (c) If one root of the equation $x^3 - px^2 + qx - r = 0$ is equal to the product of the other two, show that $(q+r)^2 = r(p+1)^2$.

QUESTION SIX *(12 Marks)* Start a new page

4 (a) ΔPQR is inscribed in a circle. D and E are the midpoints of the arcs PQ and QR respectively. The line DE intersects PQ and QR at F and G respectively.



(i) Let angle $QDE = \alpha$. Explain why angle $RQE = \alpha$.

(ii) Prove that ΔQFG is isosceles.

4 (b) If $x = \tan\theta + \sec\theta$, use the t-formulae to show that $\dfrac{x^2 - 1}{x^2 + 1} = \sin\theta$.

Question 6 continued on page 5....

4 (c) (i) Prove that $\frac{d}{dx}\left(\tan^{-1} x + \tan^{-1} \frac{1}{x}\right) = 0$.

(ii) Hence sketch the curve $y = \tan^{-1} x + \tan^{-1} \frac{1}{x}$.

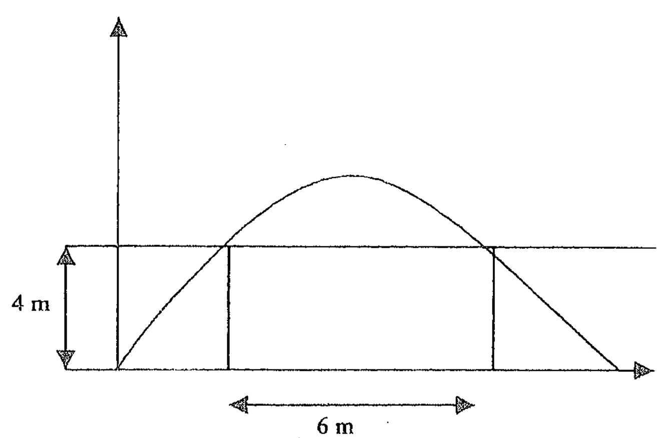
QUESTION SEVEN *(12 Marks)* **Start a new page**

5 (a) A particle is projected from a point O with an initial velocity of v m/s and with an angle of projection α, where $0^o \le \alpha \le 90^o$ and where g m/s^2 is the acceleration due to gravity. Under these circumstances the equations for the horizontal and vertical displacements at time t are given by $x = v\cos\alpha . t$ and $y = -\frac{1}{2}gt^2 + v\sin\alpha . t$ respectively.

(i) Prove that $y = x\tan\alpha - \frac{gx^2}{2v^2}\sec^2\alpha$.

(ii) If R is the range of the projectile on the horizontal plane, prove that $y = x\left(1 - \frac{x}{R}\right)\tan\alpha$.

(iii) If $\alpha = 45^o$ and the particle just clears two walls 6 m apart, both at a height of 4 m, find R.



Question 7 continued on page 6 ...

7 (b) (i) Show that $\sin x + \cos x = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$.

(ii) Prove that $\frac{d}{dx}(e^x \sin x) = \sqrt{2}e^x \sin\left(x + \frac{\pi}{4}\right)$.

(iii) Prove by mathematical induction that if $y = e^x \sin x$, then

$$\frac{d^n y}{dx^n} = \left(\sqrt{2}\right)^n e^x \sin\left(x + \frac{n\pi}{4}\right)$$ where n is a positive integer.

END OF PAPER

NEWINGTON

HSC TRIAL 2000 SOLUTIONS.

QUESTION ONE

(a) $-8 + 4a - 4 - 4 = -7$

$4a = 9$

$a = \frac{9}{4}$ ✓

(b) $\tan\theta = \left|\frac{1+3}{1-3}\right| = 2$ ✓ $\theta = 63^\circ 26'$ ✓

(c) (i) $\cos 2x = 1 - 2\sin^2 x$ ✓

(ii) Let $\sin^{-1}\frac{\sqrt{3}}{2} = x$

$\therefore \sin x = \frac{\sqrt{3}}{2}, \quad 0 < x < \frac{\pi}{2}$

$\cos\left(2\sin^{-1}\frac{\sqrt{3}}{2}\right) = \cos 2x$

$= 1 - 2\sin^2 x$

$= 1 - 2\left(\frac{3}{4}\right)$ ✓

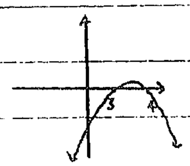
$= -\frac{1}{2}$ ✓

(d) $\frac{1}{x-3} \geq 1$

$x - 3 \geq (x-3)^2, \quad x \neq 3$ ✓

$(x-3)(1-x+3) \geq 0, \quad x \neq 3$

$(x-3)(4-x) \geq 0, \quad x \neq 3$ ✓



$3 < x \leq 4$ ✓

(e) let $u = 1 - x$ when $x = 1$ $u = 0$ when $x = 0$, $u = 1$

$\frac{du}{dx} = -1$

$\int_0^1 x\sqrt{1-x}\,dx = \int_1^0 -(1-u)\sqrt{u}\,du$ ✓

$= \int_0^1 u^{\frac{1}{2}} - u^{\frac{3}{2}}\,du$

$= \left[\frac{2}{3}u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}}\right]_0^1$ ✓

$= \frac{2}{3} - \frac{2}{5}$

$= \frac{4}{15}$ ✓

QUESTION TWO

(a) LHS $= \frac{\cos x - \cos 2x}{\sin 2x + \sin x}$

$= \frac{\cos x - 2\cos^2 x + 1}{2\sin x\cos x + \sin x}$ ✓

$= \frac{(1 + 2\cos x)(1 - \cos x)}{\sin x(2\cos x + 1)}$ ✓

$= \frac{1 - \cos x}{\sin x}$

$= \frac{1}{\sin x} - \frac{\cos x}{\sin x}$ ✓

(b) $f(x) = 2\sin x - 10x + 5$

$f'(x) = 2\cos x - 10$ ✓

If $x_1 = 0.5$, $x_2 = 0.5 - \frac{f(0.5)}{f'(0.5)}$

$= 0.5 - \frac{0.958851}{-8.24483}$ ✓

$= 0.62$ to 2 dec. pls. ✓

(c) let $\sqrt{3}\cos\theta - \sin\theta = r\cos(\theta + \alpha)$

$= r\cos\theta\cos\alpha - r\sin\theta\sin\alpha$

$\therefore r\cos\alpha = \sqrt{3}, \quad r\sin\alpha = 1$

$\therefore r^2 = 4$

when $r = 2$ ✓, $\alpha = \frac{\pi}{6}$ ✓

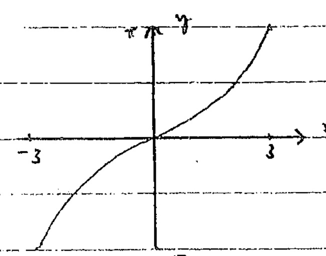
solve $2\cos\left(\theta + \frac{\pi}{6}\right) = 1$

$\cos\left(\theta + \frac{\pi}{6}\right) = \frac{1}{2}$

$\therefore \theta + \frac{\pi}{6} = \frac{\pi}{3}$ or $\frac{5\pi}{3}$ ✓

$\therefore \theta = \frac{\pi}{6}$ or $\frac{3\pi}{2}$ ✓

(d)



✓✓

QUESTION THREE

(a) (i) $y = \frac{x^2}{4a}$ $\quad \frac{dy}{dx} = \frac{x}{2a} = p$ at $P(2ap, ap^2)$

$\therefore$ gradient of normal is $-\frac{1}{p}$ ✓

eqn. of normal: $y - ap^2 = -\frac{1}{p}(x - 2ap)$

$py - ap^3 = -x + 2ap$

$x + py = 2ap + ap^3$ ✓

(ii) When $x = 0$ $\quad y = \frac{2ap + ap^3}{p} = 2a + ap^2$ ✓

$\therefore Q(0, 2a + ap^2)$.

(iii) ratio of $-2 : 1$

$R\left(\frac{2ap}{-1}, \frac{-2(2a + ap^2) + ap^2}{-1}\right)$ ✓

$= R(-2ap, 4a + ap^2)$ ✓

(v) sub (h, k) into $x + py = 2ap + ap^3$

$h + pk = 2ap + ap^3$ ✓

$ap^3 + (2a - k)p - h = 0$

(vi) 3 (cubic can have at most three distinct solns.) ✓

) (i) $V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2 = S$ ✓

(ii) $\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$ ✓

$-kS = S \cdot \frac{dr}{dt}$

$\therefore \frac{dr}{dt} = -k$ ✓

i.e. radius is decreasing at a constant rate.

QUESTION FOUR

(a) (i) $-1 \le \cos 3t \le 1$

$\therefore -2 \le 2\cos 3t \le 2$

$\therefore 2 \le 2\cos 3t + 4 \le 6$ ✓

(ii) centre 4, amplitude 2 ✓

(iii) $x = 2\cos 3t + 4$

$\dot{x} = -6\sin 3t$

$\ddot{x} = -18\cos 3t$ ✓

(iv) $\ddot{x} = -18\cos 3t$

but $\cos 3t = \frac{x - 4}{2}$

$\therefore \ddot{x} = -18\left(\frac{x-4}{2}\right) = -9(x - 4)$ ✓

(v) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -9x + 36$

$\frac{1}{2}v^2 = \frac{-9x^2}{2} + 36x + c$ ✓

when $t = 0$, $x = 6$, $v = 0$

$0 = \frac{-9 \times 36}{2} + 36 \times 6 + c$

$0 = -162 + 216 + c$

$\therefore c = -54$ ✓

$\frac{1}{2}v^2 = \frac{-9x^2}{2} + 36x - 54$

$v^2 = -9x^2 + 72x - 108$ ✓

(vi) 6 cm/s ✓

(b) (i) $T = R + Ae^{-kt}$

$\frac{dT}{dt} = -k \cdot Ae^{-kt}$

$\therefore \frac{dT}{dt} = -k(T - R)$ ✓

(ii) $T = 20 + Ae^{-kt}$

$t = 0$, $T = 200°c$ $\therefore A = 180$ ✓

$t = 1$, $T = 170°c$

$170 = 20 + 180e^{-k}$

$150 = 180e^{-k}$

$\frac{5}{6} = e^{-k}$

$e^{k} = \frac{6}{5}$ $\therefore k = \ln\frac{6}{5}$ ✓

when $T = 50°c$: $50 = 20 + 180e^{-\ln\frac{6}{5} \times t}$

$t \doteqdot 10$ minutes. ✓

QUESTION FIVE

(a) $(3 + 2x)^{15} = \sum_{k=0}^{15} t_k x^k = t_0 x^0 + t_1 x^1 + \dots + t_{15}$

(i) $t_k = {}^{15}C_k\, 3^{15-k}\, 2^k$ ✓

(ii) $t_{k+1} = {}^{15}C_{k+1}\, 3^{14-k}\, 2^{k+1}$

$\therefore \frac{t_{k+1}}{t_k} = \frac{15!}{(k+1)!\,(14-k)!} \cdot \frac{k!\,(15-k)!}{15!} \cdot \frac{3^{14-k}}{3^{15-k}} \cdot \frac{2^{k+1}}{2^k}$ ✓

$= \frac{15-k}{k+1} \cdot \frac{2}{3}$

$= \frac{30 - 2k}{3k + 3}$ ✓

(iii) $t_{k+1} > t_k$ when $30 - 2k > 3k + 3$

$27 > 5k$

$\frac{27}{5} > k$

i.e. when $k = 5, 4, 3, \dots$

i.e. $t_6 > t_5 > t_4 \dots$ ✓

$t_{k+1} < t_k$ when $\frac{27}{5} < k$ i.e. when $k = 6, 7, 8,$

i.e. $t_8 < t_7 < t_6$

$\therefore$ t_6 is the greatest coefficient

i.e. greatest coefficient is ${}^{15}C_6\, 3^9\, 2^6$ ✓

(b) (i) $a = 1$, $r = 1 + x$, $n + 1$ terms

$S = \frac{1\left((1+x)^{n+1} - 1\right)}{(1+x) - 1} = \frac{(1+x)^{n+1} - 1}{x}$ ✓

(ii) $S = \frac{1}{x}\left[{}^{n+1}C_0 + {}^{n+1}C_1 x + {}^{n+1}C_2 x^2 + \dots + {}^{n+1}C_{n+1} x^{n+1} - 1\right]$

$= {}^{n+1}C_1 + {}^{n+1}C_2 x + \dots + {}^{n+1}C_{n+1} x^n$. ✓

(iii) From (ii) coefficient of x^r is ${}^{n+1}C_{r+1}$ ✓

From (5) $= 1+(1+x)+(1+x)^2+\cdots+\underbrace{(1+x)^r+\cdots+(1+x)^n}_{\text{contain terms in } x^r}$

coefficient of x^r is ${}^{r}C_r + {}^{r+1}C_r + \cdots + {}^{n-1}C_r + {}^{n}C_r$ ✓

$\therefore\ {}^{n+1}C_{r+1} = {}^{n}C_r + {}^{n-1}C_r + {}^{n-2}C_r + \cdots + {}^{r}C_r$.

c) $x^3 - px^2 + qx - r = 0$

let roots be α, β and $\alpha\beta$

then
$$\alpha+\beta+\alpha\beta = p$$
$$\alpha\beta + \alpha^2\beta + \alpha\beta^2 = q$$
$$\alpha^2\beta^2 = r$$
✓

$$\begin{aligned} LHS &= (q+r)^2 \\ &= \left[\alpha\beta(\alpha+\beta+\alpha\beta+1)\right]^2 \\ &= (\alpha\beta)^2(\alpha+\beta+\alpha\beta+1)^2 \\ &= r(p+1)^2. \end{aligned}$$
✓ ✓

QUESTION SIX

a)(i) $\angle QDE = \alpha$

$\therefore \angle RQE = \alpha$ (angle subtended by equal arc) ✓

(ii) let $\angle QED = \beta$ ✓

$\therefore \angle DQF = \beta$ (angle subtended by equal arc)

$\therefore \angle QFG = \angle QGF = \alpha+\beta$ (exterior $\angle$ of Δ) ✓✓

$\therefore \Delta QFG$ is isosceles (base angles equal)

(b)
$$\begin{aligned} x &= \tan\theta + \sec\theta \\ &= \frac{2t}{1-t^2} + \frac{1+t^2}{1-t^2} \\ &= \frac{(1+t)^2}{(1-t)(1+t)} \\ &= \frac{1+t}{1-t} \end{aligned}$$
✓

$\therefore t = \frac{x-1}{x+1}$ ✓

$$\sin\theta = \frac{2t}{1+t^2} = \frac{\frac{2(x-1)}{x+1}}{1+\frac{(x-1)^2}{(x+1)^2}}$$
✓

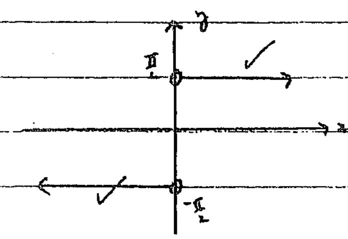
$$= \frac{2(x^2-1)}{2(x^2+1)} = \frac{x^2-1}{x^2+1}$$
✓

(c)(i)
$$\begin{aligned} &\frac{d}{dx}\left(\tan^{-1}x + \tan^{-1}\frac{1}{x}\right) \\ &= \frac{1}{1+x^2} + \frac{1}{1+\frac{1}{x^2}}\cdot -\frac{1}{x^2} \\ &= \frac{1}{1+x^2} - \frac{1}{x^2+1} \\ &= 0 \end{aligned}$$
✓ ✓

(ii) $\tan^{-1}x + \tan^{-1}\frac{1}{x} = $ constant

when $x>0$, $\tan^{-1}x + \tan^{-1}\frac{1}{x} = \frac{\pi}{2}$

$x<0$, $\tan^{-1}x + \tan^{-1}\frac{1}{x} = -\frac{\pi}{2}$



QUESTION SEVEN

(a) $x = V\cos\alpha \cdot t \qquad y = -\frac{1}{2}gt^2 + V\sin\alpha \cdot t$

$t = \frac{x}{V\cos\alpha}$

$$\begin{aligned} y &= -\tfrac{1}{2}g\left(\frac{x}{V\cos\alpha}\right)^2 + V\sin\alpha\left(\frac{x}{V\cos\alpha}\right) \\ &= \frac{-gx^2}{2V^2}\sec^2\alpha + x\tan\alpha \end{aligned}$$
✓

(ii) when $y=0$, $x=R$.

$$\begin{aligned} 0 &= \frac{-gR^2}{2V^2}\sec^2\alpha + R\tan\alpha \\ \frac{gR^2}{2V^2}\sec^2\alpha &= R\tan\alpha \\ \sec^2\alpha &= \frac{2V^2\tan\alpha}{gR} \\ \therefore y &= \frac{-gx^2}{2V^2}\left(\frac{2V^2\tan\alpha}{gR}\right) + x\tan\alpha \\ &= \frac{-x^2\tan\alpha}{R} + x\tan\alpha \\ &= x\tan\alpha\left(1-\frac{x}{R}\right). \end{aligned}$$
✓ ✓

(iii) when $y=4$, $x = \frac{R-6}{2}$, $\tan\alpha = 1$

$$\begin{aligned} \therefore 4 &= \left(\frac{R-6}{2}\right)\left(1-\frac{R-6}{2R}\right) \\ \frac{8}{R-6} &= \frac{R+6}{2R} \\ 16R &= R^2-36 \\ R^2-16R-36 &= 0 \\ (R-18)(R+2) &= 0. \end{aligned}$$
✓

) (i) RHS $= \sqrt{2}\sin(x+\frac{\pi}{4})$

$= \sqrt{2}\left(\sin x\cos\frac{\pi}{4} + \cos x\sin\frac{\pi}{4}\right)$

$= \sqrt{2}\left(\sin x.\frac{1}{\sqrt{2}} + \cos x.\frac{1}{\sqrt{2}}\right)$

$= \sin x + \cos x$

$=$ LHS ✓

(ii) $\frac{d}{dx}(e^x\sin x) = e^x\cos x + e^x\sin x$

$= e^x(\sin x + \cos x)$

$= \sqrt{2}e^x\sin(x+\frac{\pi}{4})$ from (i) ✓

(iii) $y = e^x\sin x$, $\frac{d^ny}{dx^n} = (\sqrt{2})^n e^x\sin(x+\frac{n\pi}{4})$

i) when $n=1$, $\frac{dy}{dx} = \sqrt{2}e^x\sin(x+\frac{\pi}{4})$ which is true from (ii)

i.e. statement is true when $n=1$. ✓

let statement be true when $n=k$

i.e. $\frac{d^ky}{dx^k} = (\sqrt{2})^k e^x\sin(x+\frac{k\pi}{4})$

prove the statement is true when $n=k+1$

i.e. prove that $\frac{d^{k+1}y}{dx^{k+1}} = (\sqrt{2})^{k+1}e^x\sin(x+\frac{(k+1)\pi}{4})$

LHS $= \frac{d^{k+1}y}{dx^{k+1}}$

$= \frac{d}{dx}\left(\frac{d^ky}{dx^k}\right)$

$= \frac{d}{dx}\left((\sqrt{2})^k e^x\sin(x+\frac{k\pi}{4})\right)$ by the induction hypothesis

$= (\sqrt{2})^k e^x\sin(x+\frac{k\pi}{4}) + (\sqrt{2})^k e^x\cos(x+\frac{k\pi}{4})$

✓✓ $= (\sqrt{2})^k e^x\left[\sin(x+\frac{k\pi}{4}) + \cos(x+\frac{k\pi}{4})\right]$

$= (\sqrt{2})^k e^x\left[\sqrt{2}\sin\left(x+\frac{k\pi}{4}+\frac{\pi}{4}\right)\right]$ from (i)

$= (\sqrt{2})^k e^x\sqrt{2}\sin\left(x+\frac{(k+1)\pi}{4}\right)$

$= (\sqrt{2})^{k+1} e^x\sin\left(x+\frac{(k+1)\pi}{4}\right)$

$=$ RHS.

) Steps A and B and the axiom of mathematical duction, the statement is true for all ✓ ositive integers n.