Newington College

2004

TRIAL HSC EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is proved at the back of this paper
- All necessary working should be shown in every question

Total marks: 84

- Attempt Questions 1 – 7
- All questions are of equal value

QUESTION 1 *(12 marks)* **Marks**

(a) Write down an indefinite integral of 3

(i) $\dfrac{1}{1+x^2}$

(ii) $\dfrac{1}{(1+x)^2}$

(iii) $\dfrac{x}{1+x^2}$

(b) Find the size of the acute angle between the lines with equations 3

$$x-2y+1=0 \quad \text{and} \quad 3x-y-2=0\ .$$

(c) Using the expansion of $\cos(A+B)$ or otherwise, find the exact value of 3

$$\cos\frac{5\pi}{12}$$

(d) Sketch the graph of $y=x-x^3$, indicating the x intercepts only and hence or otherwise solve the inequation $x-x^3<0$ 3

QUESTION 2 *(12 marks)* **Start a new booklet**

(a) (i) Find the domain and range of the function $y=2\sin^{-1}3x$. 2

(ii) Hence make a neat sketch of this function. 1

(b) By using the substitution $u=\cos x$, evaluate

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sqrt{4-\cos^2 x}}\,dx$$ 3

(c) If the parametric equations of a parabola are given by

$$x=\cos t \qquad y=\frac{1}{8}\cos 2t$$

find the co-ordinates of the vertex of the parabola. 3

Question 2 continued page 2.

Question 2 (continued) **Marks**

d) (i) How many teams of 3 women can be chosen from a group of 5 women? 1

(ii) If the names of 6 men and 5 women are placed in a hat and 3 names are then drawn from the hat simultaneously to form a team, what is the probability that the team is exclusively female? 2

QUESTION 3 *(12 marks)* **Start a new booklet**

(a) Find all solutions of the equation $\sin 2x = \sin x$ for $0 \le x \le 2\pi$. 3

(b) Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \sin^2 3x\,dx$ 3

(c) The polynomial $P(x) = ax^3 - 4x^2 + 3bx + 2$ has a factor of $(x-1)$ and leaves a remainder of 15 when divided by $(x+2)$. Find the values of a and b. 3

(d) Consider the equation $x - 3 + \log_e x = 0$

If x_0 is an approximation to a solution of this equation, show that Newton's Method gives a second approximation of 3

$$x_1 = \frac{x_0(4 - \log_e x_0)}{1 + x_0}$$

QUESTION 4 *(12 marks)* **Start a new booklet**

(a) Find $\dfrac{d}{dx}\left(x\cos^{-1} x - \sqrt{1 - x^2}\right)$ 2

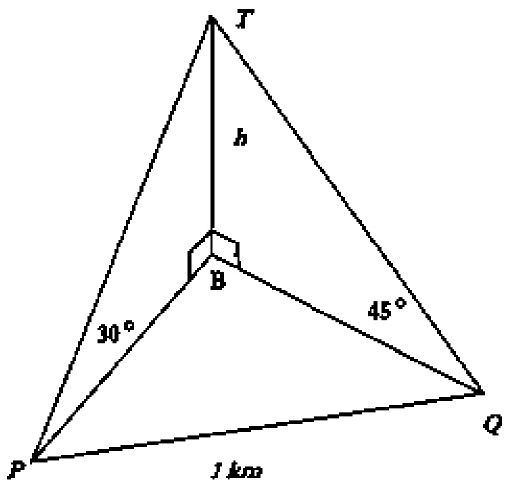
(b) If $2\sin^{-1} x = \cos^{-1} x$, find x when $0 \le x \le 1$ 2

(c) At a reception for the Queen, 6 dignitaries are to be seated on stage with the Queen seated in the middle of them. Two of these dignitaries are Mr Hill and Mr Hall. Before the reception they quarrel and insist that they be seated on different sides of the Queen. How many seating plans can be drawn up to satisfy their wish? 3

Question 4 continued page 3.

Question 4 (continued) **Marks**

(d)



The angle of elevation from a boat at P to a point T at the top of a vertical cliff is $30°$. The boat sails 1 *km* to a second point Q, from which the angle of elevation of T is $45°$. B is the point at the base of the cliff directly below T and h is the height of the cliff in metres. The bearings of B from P and Q are $50°$ and $290°$ respectively.

(i) Show that $\angle PBQ = 120°$ **1**

(ii) By finding expressions for PB and QB in terms of h, show that **4**

$$h = \frac{10\sqrt{10}}{\sqrt{4+\sqrt{3}}} \text{ metres.}$$

QUESTION 5 *(12 marks)* **Start a new booklet**

(a) (i) In the expansion of $\left(2x - \frac{3}{x}\right)^7$ show that there is no term in x^4. **3**

(ii) By finding the co-efficients of x^3 and x^5 in the above expansion, hence or otherwise find the co-efficient of x^5 in the expansion of **3**

$$\left(x^2+1\right)\left(2x - \frac{3}{x}\right)^7$$

(b) Solve $(n-2)! = 56(n-4)!$ **2**

Question 5 continued page 4.

Question 5 (continued) **Marks**

(c) (i) Express $\cos 2x - \sin 2x$ in the form $R\cos(2x+\alpha)$, where α is acute and $R > 0$. **2**

(iii) Hence or otherwise solve the equation **2**

$$\cos 2x - \sin 2x = 1 \quad \text{for} \quad 0 \le x \le \pi .$$

QUESTION 6 *(12 marks)* **Start a new booklet**

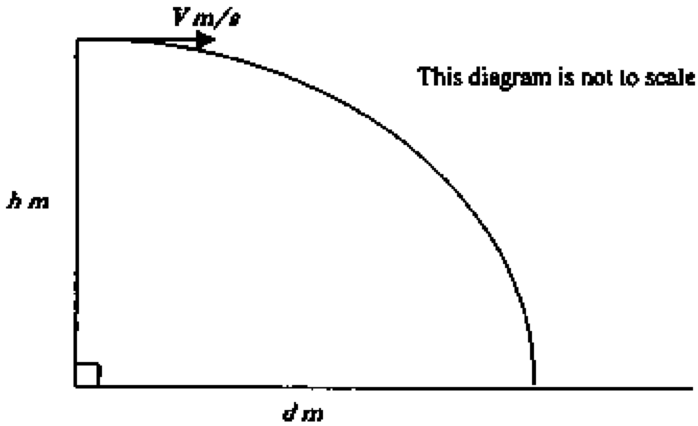
(a) (i) Show that the equation of the normal to the parabola $x^2 = 4y$ at the point $P(2p, p^2)$ is given by $x + py = p(p^2 + 2)$. **2**

(ii) If this normal cuts the axis of the parabola at Q, find the equation of the locus of the mid-point of PQ. **3**

(b) Solve for θ in the range $0° \le \theta \le 2\pi$

$$6\cot^2\theta - 4\cos^2\theta = 1$$ **3**

(c) A plane wants to drop a water bomb on spot fires to extinguish them before a bushfire begins. The plane is travelling horizontally at V *m/s* and at a height of h metres. A spot fire is directly ahead of the plane, *d metres* away. The water bomb is dropped at that instant



(i) Show that if the origin is taken as the point on the ground directly below the plane when the water bomb is released, that the equations of motion of the water bomb in flight are **2**

$$x = Vt \qquad \text{and} \qquad y = -5t^2 + h$$

Air resistance is to be neglected and the acceleration due to gravity can be assumed to be $10 m/s^2$.

(ii) Show that $5d^2 = V^2 h$ if the water bomb hits the spot fire. **2**

QUESTION 7 *(12 marks)* **Start a new booklet** **Marks**

(a) The journey between two mini rail stations A and B, where A is to the west of B can be modelled by the equation $v^2 = 2(8x - x^2 - 7)$ where v is the velocity in *km/h* and x is the displacement in *km* from the control office, which is mid-way between A and B.

(i) Show that the motion is simple harmonic. **2**

(ii) Find the distance between the two stations. **2**

(b) (i) A particle moves along a path described by the equation $y = \frac{1}{4}x^2$. Show that the perpendicular distance from any point on its path to the line with equation $3x - 4y + 4 = 0$ is **1**

$$\frac{1}{5}\left|x^2 - 3x - 4\right|$$

(ii) Sketch the curve $y = \frac{1}{5}\left|x^2 - 3x - 4\right|$, showing any intercepts and the vertex. **2**

(iii) Using this graph or otherwise, show that the distance from the particle to the line is $1\frac{1}{4}$ *units* on exactly three occasions. **2**

(c) The volume of a sphere is increasing at a rate of $5\ cm^3/s$. At what rate is the surface area increasing when the radius is $20\ cm$? **3**

END OF PAPER

Ext 1 Trial Solution

(1) (a) (i) $\tan^{-1}x + C$ (ii) $-\frac{1}{(1+x)} + C$ (iii) $\frac{1}{2}\log_e(1+x^2) + C$

(b) $2y = x+1$, $y = \frac{1}{2}x + \frac{1}{2}$, $\therefore m_1 = \frac{1}{2}$

$3x - y - 2 = 0$, $y = 3x - 2$, $\therefore m_2 = 3$

$\therefore \tan\theta = \frac{3 - \frac{1}{2}}{1 + 3.\frac{1}{2}} = \frac{\frac{5}{2}}{\frac{5}{2}} = 1$

$\therefore \theta = 45°$.

(c) $\cos\frac{5\pi}{12} = \cos 75° = \cos(45° + 30°) = \cos 45° . \cos 30° - \sin 45° . \sin 30°$

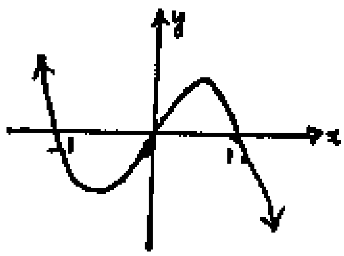
$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2}$

$= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}}$.

(d) $x - x^3 = 0$

$x(1 - x^2) = 0$

$x = 0, \pm 1$



$-1 < x < 0$, $x > 1$

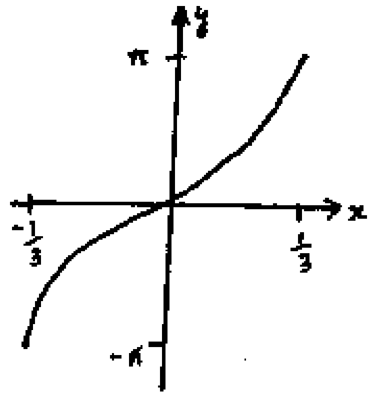
(2) (a) (i) $\frac{y}{2} = \sin^{-1}3x$

D: $-1 \le 3x \le 1$

$-\frac{1}{3} \le x \le \frac{1}{3}$

R: $-\frac{\pi}{2} \le \frac{y}{2} \le \frac{\pi}{2}$

$-\pi \le y \le \pi$



(b) $u = \cos x$

$du = -\sin x\, dx$

$-du = \sin x\, dx$

$x = \frac{\pi}{2}, u = 0$

$x = 0, u = 1$

$\therefore \int_1^0 \frac{-du}{\sqrt{4 - u^2}}$

$= -\left[\sin^{-1}\frac{u}{2}\right]_1^0$

$= -\left(\sin^{-1}0 - \sin^{-1}\frac{1}{2}\right) = \frac{\pi}{6}$

(c) $x = \cos t$

$y = \frac{1}{8}(2\cos^2 t - 1)$

$\therefore y = \frac{1}{8}(2x^2 - 1)$

$8y + 1 = 2x^2$

$x^2 = 4y + \frac{1}{2}$

$x^2 = 4\left(y + \frac{1}{8}\right)$

$\therefore$ Vertex is $\left(0, -\frac{1}{8}\right)$

(d) (i) ${}^5C_3 = 10$

(ii) $\frac{{}^5C_3}{{}^{11}C_3} = \frac{10}{165} = \frac{2}{33}$.

(3) (a)

$$2\sin x . \cos x - \sin x = 0$$
$$\sin x(2\cos x - 1) = 0$$
$$\sin x = 0 \qquad \cos x = \tfrac{1}{2}$$
$$\therefore x = 0, \pi, 2\pi, \frac{\pi}{3}, \frac{5\pi}{3}.$$

(b)

$$\frac{1}{2}\int_0^{\frac{\pi}{2}} (1 - \cos 6x)\, dx$$
$$= \frac{1}{2}\left[x - \frac{1}{6}\sin 6x\right]_0^{\frac{\pi}{2}}$$
$$= \frac{1}{2}\left(\frac{\pi}{2} - \frac{1}{6}\times 0 - 0 + 0\right)$$
$$= \frac{\pi}{4}.$$

(c)

$$P(1) = a - 4 + 3b + 2 = 0$$
$$a + 3b = 2$$
$$P(-2) = -8a - 16 - 6b + 2 = 15$$
$$8a + 6b = -29$$

$$\left.\begin{array}{l} 2a + 6b = 4 \\ 8a + 6b = -29 \end{array}\right\} \quad \begin{array}{l} 6a = -33 \\ a = -5\tfrac{1}{2} \end{array}$$

$$-44 + 6b = -29$$
$$6b = 15$$
$$b = 2\tfrac{1}{2}$$

$$\left.\begin{array}{l} a = -5\tfrac{1}{2} \\ b = 2\tfrac{1}{2} \end{array}\right\}$$

(d)

$$\left.\begin{array}{l} f(x) = x - 3 + \log_e x \\ f'(x) = 1 + \frac{1}{x} \end{array}\right\}$$

$$x_1 = x_0 - \frac{x_0 - 3 + \log_e x_0}{1 + \frac{1}{x_0}}$$
$$= \frac{x_0 + 1 - x_0 + 3 - \log_e x_0}{\frac{x_0 + 1}{x_0}}$$
$$= \frac{x_0(4 - \log_e x_0)}{x_0 + 1}.$$

(4) (a)

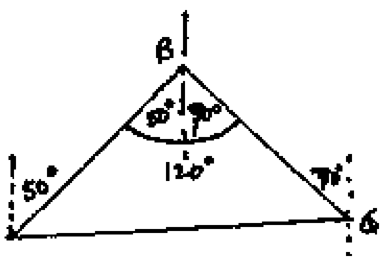
$$x . -\frac{1}{\sqrt{1-x^2}} + 1 . \cos^{-1} x - \frac{1}{2}(1-x^2)^{-\frac{1}{2}} . -2x$$
$$= -\frac{x}{\sqrt{1-x^2}} + \cos^{-1} x + \frac{x}{\sqrt{1-x^2}}$$
$$= \cos^{-1} x.$$

(b)

$$2\sin^{-1} x = \frac{\pi}{2} - \sin^{-1} x$$
$$3\sin^{-1} x = \frac{\pi}{2}$$
$$\sin^{-1} x = \frac{\pi}{6}$$
$$\therefore x = \tfrac{1}{2}$$

(c) $4! \times 3 \times 3 \times 2 = 432$ plans

(d) (i)



(ii) $PB = h\cot 30°$, $BQ = h\cot 45°$

$$\therefore 1000^2 = h^2\cot^2 30° + h^2\cot^2 45° - 2h\cot 30° h\cot 45° \cos 12\ldots$$
$$1000^2 = h^2(\cot^2 30° + \cot^2 45° - 2\cot 30°\cot 4\ldots \cos 12\ldots$$
$$\therefore h^2 = \frac{1000^2}{\cot^2 30° + \cot^2 45° - 2\cot 30°\cot 45° \cos 12\ldots}$$
$$\therefore h = \frac{10\sqrt{10}}{\ldots} = \frac{10\sqrt{10}}{\ldots}$$

(5) (a)(i) $T_{r+1} = {}^7C_r (2x)^{7-r} . \left(-\frac{3}{x}\right)^r$

$= (-1)^r . 3^r . 2^{7-r} . {}^7C_r \; x^{7-r} x^{-r}$

$= (-1)^r . 3^r . 2^{7-r} . {}^7C_r \; x^{7-2r}$

Let $7-2r = 4$

$\therefore 2r = 3$

$r = \frac{3}{2}$

$\therefore$ As r is not an integer, there is no term in x^4.

(ii) Co-eff of x^3 : $r = 2$

$3^2 2^5 \, {}^7C_2$

Co-eff of x^5 : $r = 1$

$-3 . 2^6 \, {}^7C_1$

$\therefore$ Co-eff of x^5 in full expansion

$3^2 2^5 . {}^7C_2 - 3.2^6 \, {}^7C_1$

$= 4704$

(b) $\frac{(n-2)!}{(n-4)!} = 56$

$(n-2)(n-3) = 56$

$n^2 - 5n + 6 = 56$

$n^2 - 5n - 50 = 0$

$(n-10)(n+5) = 0$

$n = 10, -5$

$\therefore n = 10$

(c)(i) $\cos 2x - \sin 2x = R\cos 2x \cos\alpha - R\sin 2x \sin\alpha$

$\therefore$ $R\cos\alpha = 1$, $R\sin\alpha = 1$ $\quad R = \sqrt{1^2+1^2} = \sqrt{2}$

$\tan\alpha = 1 \quad \therefore \alpha = \frac{\pi}{4}$

$\therefore \cos 2x - \sin 2x = \sqrt{2}\cos\left(2x + \frac{\pi}{4}\right)$

(ii) $\sqrt{2}\cos\left(2x+\frac{\pi}{4}\right) = 1$

$\cos\left(2x+\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$2x + \frac{\pi}{4} = \frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}$.

$2x = 0, \frac{3\pi}{2}, 2\pi$

$x = 0, \frac{3\pi}{4}, \pi$.

(6)(a)(i) $y = \frac{x^2}{4}$

$\frac{dy}{dx} = \frac{x}{2}$

$m_T = p$

$m_N = -\frac{1}{p}$

$\therefore$ normal is

$y - p^2 = -\frac{1}{p}(x - 2p)$

$py - p^3 = -x + 2p$

$x + py = p(p^2+2)$

(ii) when $x = 0$

$y = p^2 + 2$

ie $Q(0, p^2+2)$

Mid pt of PQ is

(p, p^2+1)

Parametrics of locus are

$x = p$, $y = p^2+1$

$\therefore$ Eqn is

$y = x^2 + 1$.

(b) $6\frac{\cos^2\theta}{\sin^2\theta} - 4\cos^2\theta = 1$

$6\frac{\cos^2\theta}{1-\cos^2\theta} - 4\cos^2\theta = 1$

$6\cos^2\theta - 4\cos^2\theta + 4\cos^4\theta = 1 - \cos^2\theta$

$4\cos^4\theta + 3\cos^2\theta - 1 = 0$

$(4\cos^2\theta - 1)(\cos^2\theta + 1) = 0$

$\cos^2\theta = \frac{1}{4}$ $\quad$ $\cos^2\theta = -1$

$\cos\theta = \pm\frac{1}{2}$ $\quad$ no sol

$\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$.

(c) (i) $t = 0$: $x = 0$, $y = h$; $\dot{x} = V$, $\dot{y} = 0$

$\ddot{x} = 0$

$\dot{x} = c$

$\therefore \dot{x} = V$

$x = Vt + c$

$c = 0$

$\therefore x = Vt$

$\ddot{y} = -10$

$\dot{y} = -10t + c$

$c = 0$

$\therefore \dot{y} = -10t$

$y = -5t^2 + c$

$c = h$

$\therefore y = -5t^2 + h$

(ii) when $x = d$, $y = 0$

$d = Vt \quad \therefore t = \frac{d}{V}$

$\therefore 0 = -5 . \frac{d^2}{V^2} + h$

$\frac{5d^2}{V^2} = h$

$\therefore 5d^2 = V^2 h$.

(7) (a) (i) $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$= \frac{d}{dx}\left(8x - x^2 - 7\right)$

$= 8 - 2x$

$\therefore \ddot{x} = -2(x-4)$

hence SHM

(ii) when $v = 0$

$8x - x^2 - 7 = 0$

$x^2 - 8x + 7 = 0$

$(x-7)(x-1) = 0$

$x = 7, 1$

$\therefore$ Stations are 6 km apart

(b) (i) Co-ords of all points on $y = \frac{1}{4}x^2$ are $(x, \frac{1}{4}x^2)$

$\therefore$ Perp distance

$= \left| \frac{3x - 4 \times \frac{1}{4}x^2 + 4}{\sqrt{3^2 + (-4)^2}} \right|$

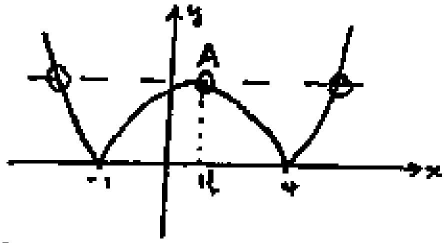
$= \frac{1}{5}\left|3x - x^2 + 4\right|$

$= \frac{1}{5}\left|x^2 - 3x - 4\right|$

(ii) $x^2 - 3x - 4 = 0$

$(x-4)(x+1) = 0$

$x = 4, -1$



(iii) Co-ords of A are $(1\frac{1}{2}, 1\frac{1}{4})$

$\therefore y = 1\frac{1}{4}$ meets $y = \frac{1}{5}\left|x^2 - 3x - 4\right|$ in exactly 3 places

(c) $\frac{dV}{dt} = 5$, $\frac{dS}{dt} = ?$, $r = 20$

$\frac{dS}{dt} = \frac{dS}{dr} \cdot \frac{dr}{dt} = \frac{dS}{dr} \cdot \frac{dr}{dV} \cdot \frac{dV}{dt}$

$= 8\pi r \cdot \frac{1}{4\pi r^2} \cdot 5$

$= \frac{2}{r} \cdot 5$

$\therefore \frac{dS}{dt} = \frac{10}{20} = 0.5$ cm²/sec