**Question 1** *(12 marks)* **Marks**

a) Solve $\dfrac{3}{x-1} < 4$. **3**

b) Find $\lim\limits_{x \to 0} \dfrac{\sin x}{5x}$. **1**

c) Given $\sin\theta + \cos\theta = A$ and $\sin\theta\cos\theta = B$. Express $\sin^3\theta + \cos^3\theta$ in terms of $A$ and $B$ **2**

d) Find $\displaystyle\int \frac{\cos\theta}{5-\sin\theta}\,d\theta$ **2**

e) The two parabolas $y = x^2$ and $y = (x-2)^2$ intersect at the point (1,1). Find the angle between the tangents to the curves at $x = 1$. Give your answer to the nearest degree. **4**

**Question 2** *(12 marks)* Start this question in a new booklet.

a) Use the substitution $u = 9 - x^2$ to find $\displaystyle\int_0^1 6x\sqrt{9-x^2}\,dx$. **4**

b) Given $\dfrac{d}{dx}(x\sin x) = \sin x + x\cos x$, find $\displaystyle\int x\cos x\,dx$. **2**

c) (i) State the domain and range of the function $y = 3\sin^{-1}\left(\dfrac{x}{2}\right)$ **2**

(ii) Draw a sketch of the function, carefully labeling the extremities of both the range and domain. **1**

d) The letters of the word *CALCULUS* are arranged in a row. **3**

(i) How many different arrangements are there?

(ii) If one of these arrangements is selected at random, find the probability that it begins with "*U*" and ends with a "*U*".

**Question 3** *(12 marks)* Start this question in a new booklet. **Marks**

a) Consider the parabola $x^2 = 4ay$ where $a>0$, and suppose the tangents at $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ intersect at the point $T$.

(i) Given the equation of the tangent from $P$ is $y = px - ap^2$, show that the point T has coordinates, $\left(a(p+q),\, apq\right)$ **2**

(ii) Given $S$ is the focus, show that $SP = a(p^2+1)$. **2**

(iii) If $P$ and $Q$ move along the parabola such that $SP + SQ = 4a$, find the equation of the locus of point $T$. **2**

b) The function $f(x) = x - e^{-x}$ has one root between 0 and 1.

(i) Show that the root lies between 0.4 and 0.6. **1**

(ii) Use one application of Newton's Method with an initial approximation of $x = 0.5$ to find a second approximation. **2**

c) A polynomial $f(x)$ leaves a remainder of $-10$ when divided by $(x - 1)$ and 20 when divided by $(x + 2)$. If $f(x) = x^n + ax - 6$, show that $x + 1$ is a factor. **3**

**Question 4** *(12 marks)* Start this question in a new booklet.

a) The speed $v$ cm/s of a particle with simple harmonic motion in a straight line is given by $v^2 = 6 + 4x - 2x^2$, where $x$ cm is the magnitude of the displacement from a fixed point O.

(i) Show that $\ddot{x} = -2(x-1)$. **2**

(ii) Find the centre of the motion. **1**

(iii) Find the period of the motion. **1**

(iv) Find the amplitude of the motion. **2**

**Question 4 continued** **Marks**

b) Find, as a rational number, the coefficient of $x$ in the expansion of $\left(x^2+\frac{1}{2x}\right)^8$. **3**

c) Colour blindness affects 5% of all men. What is the probability that any random sample of 20 men should contain:

(i) no colour blind men. **1**

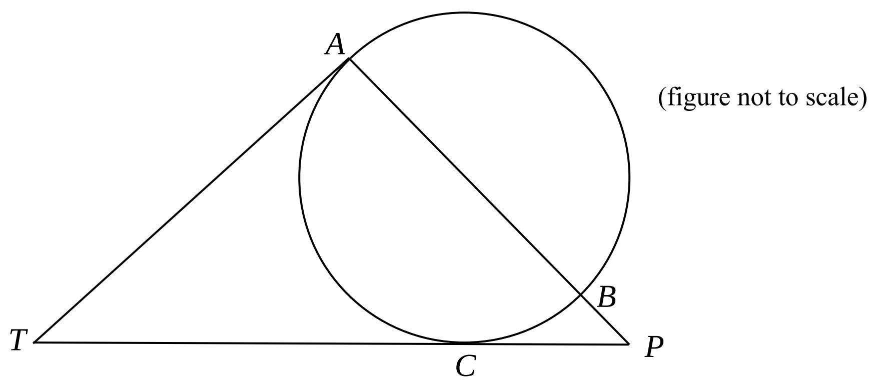
(ii) exactly 4 colour blind men. **2**

**Question 5** *(12 marks)* Start this question in a new booklet.

a) (i) Show that $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^2 x\,dx = \frac{\pi}{8} - \frac{1}{4}$. **2**

(ii) Hence or otherwise find the volume generated when $y = 1 - \cos x$ is rotated about the $x$ axis between $x = \frac{\pi}{4}$ and $x = \frac{\pi}{2}$. **2**

b)




*AB* is a diameter of a circle *ABC*. The tangents at *A* and *C* meet at *T*. The lines *TC* and *AB* are produced to meet at *P*.

Copy the diagram into your examination booklet. Join AC and CB.

(i) Prove that $\angle CAT = 90^0 - \angle BCP$. **2**

(ii) Hence, or otherwise prove that $\angle ATC = 2\angle BCP$. **2**

**Question 5 continued** **Marks**

c) At time *t* the temperature $T^0$ of a body in a room of constant temperature 20° is decreasing according to the equation $\frac{dT}{dt} = -k(T-20)$ for some constant *k*>0.

(i) Verify that $T = 20 + Ae^{-kt}$, where A is a constant, is a solution of the equation. **1**

(ii) The initial temperature of the body is $90^0$ and it falls to $70^0$ after 10 minutes. Find the temperature of the body after a further 5 minutes. ( Answer to the nearest degree.) **3**

**Question 6** *(12 marks)* Start this question in a new booklet.

a) Consider the function $f(x) = \frac{e^x}{1+e^x}$.

(i) Show that *f(x)* is increasing for all values of *x*. **2**

(ii) Explain why *f(x)* has an inverse function. **1**

(iii) Find the inverse function $y = f^{-1}(x)$. **2**

b) A yacht is sailing due north off the west coast of a small island with an extinct volcanic peak of height 1200m.
At 0900 hours the bearing of the peak is $010^0$ and the angle of elevation $5^0$. At 1030 hours the bearing of the peak is $120^0$ and the angle of elevation $20^0$.

(i) Draw a diagram to illustrate the information above. **1**

(ii) Calculate their speed in km/h. **3**

c) (i) Express $\sin x + \sqrt{3}\cos x$ in the form $R\sin(x+\alpha)$. **1**

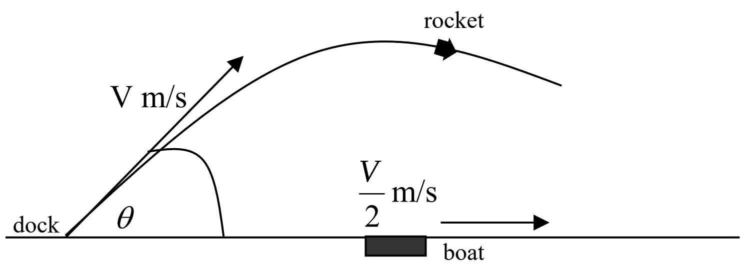
(ii) Hence solve $\sin x + \sqrt{3}\cos x = 1$ for $0 \le x \le 2\pi$ **2**

**Question 7** *(12 marks)* Start this question in a new booklet. **Marks**

a) A boat leaves a dock at a speed of $\frac{V}{2}$ $m/s$ and travels in a straight line. A rocket is fired simultaneously from the same starting point and in the same direction, with initial speed $V$ $m/s$ and angle of elevation $\theta$, where $0^0 < \theta < 90^0$. Also assume the height of the dock and the boat is zero metres.

Given the rocket's trajectory is defined by the equations:

$x = Vt\cos\theta$ and $y = -\frac{1}{2}gt^2 + Vt\sin\theta$




(i) show that the time of flight of the rocket is $\frac{2V\sin\theta}{g}$ seconds. **2**

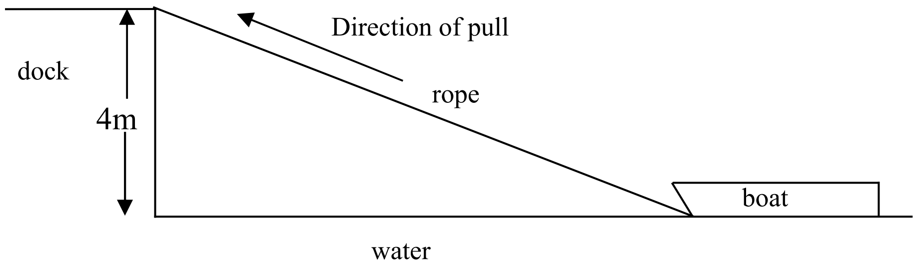
(ii) show that the range of the rocket is $\frac{2V^2\sin\theta\cos\theta}{g}$ metres. **1**

(iii) determine the angle of elevation of the rocket needed for the rocket to hit the boat **2**

b) Use mathematical induction to prove that, for every positive integer $n$, $9^n - 3$ is divisible by 6. **3**

**Question 7 continued** **Marks**

c) A boat is pulled towards a dock by a rope through a ring on the dock which is 4 metres above the boat. If the rope is pulled in at 2 m/s, how fast is the boat travelling when the rope is 10 metres long? **4**




# End of paper.

Ext 1 Year 12 Trial

(1) (a) $\frac{3}{x-1} \times (x-1)^2 < 4(x-1)^2$

$3(x-1) < 4(x-1)^2$

$4(x-1)^2 - 3(x-1) > 0$

$(x-1)(4x-7) > 0$

$x < 1$ or $x > \frac{7}{4}$

(b) $\frac{1}{5}$

(c) $\sin^3\theta + \cos^3\theta$

$= (\sin\theta + \cos\theta)(\sin^2\theta - \sin\theta\cos\theta + \cos^2\theta)$

$= A(1 - B)$

or

$(\sin\theta + \cos\theta)^3 = \sin^3\theta + 3\sin^2\theta\cos\theta + 3\sin\theta\cos^2\theta + \cos^3\theta$

$\therefore \sin^3\theta + \cos^3\theta = (\sin\theta + \cos\theta)^3 + 3\sin\theta\cos\theta(\sin\theta + \cos\theta)$

$= A^3 - 3BA$

(d) $-\ln(5 - \sin\theta) + C$

(e) Let $\theta$ = acute angle between lines

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$m_1 = 2x$, at $x = 1$
$= 2$

$m_2 = 2(x-2)$ at $x = 1$
$= -2$

$\tan\theta = \left|\frac{2 - -2}{1 + 2 \cdot -2}\right|$

$= \left|-\frac{4}{3}\right|$

$\theta = 53^\circ$ (nearest degree)

2(a) $\int_0^1 6x\sqrt{9-x^2}\,dx$

$= -3\int_0^1 \sqrt{9-x^2}\,(-2x\,dx)$

$= -3\int_9^8 u^{1/2}\,du$

$= -3\left[\frac{2}{3}u^{3/2}\right]_9^8$

$= -3\left(\frac{2}{3}\sqrt{512} - 18\right)$

$= -2\sqrt{512} + 54$

$= -32\sqrt{2} + 54$ 8.745166004

$u = 9 - x^2$

$\therefore du = -2x\,.dx$

if $x = 1$, $u = 8$

$x = 0$, $u = 9$

(b) $\frac{d}{dx}(x\sin x) = \sin x + x\cos x$

$\therefore x\cos x = \frac{d}{dx}(x\sin x) - \sin x$

$\therefore \int x\cos x\,dx = \int\left[\frac{d}{dx}(x\sin x) - \sin x\right]dx$

$= x\sin x - \int \sin x\,dx$

$= x\sin x + \cos x + C$

(c)(i) $y = 3\sin^{-1}\left(\frac{x}{2}\right)$

$\therefore \frac{1}{3}y = \sin^{-1}\left(\frac{x}{2}\right)$

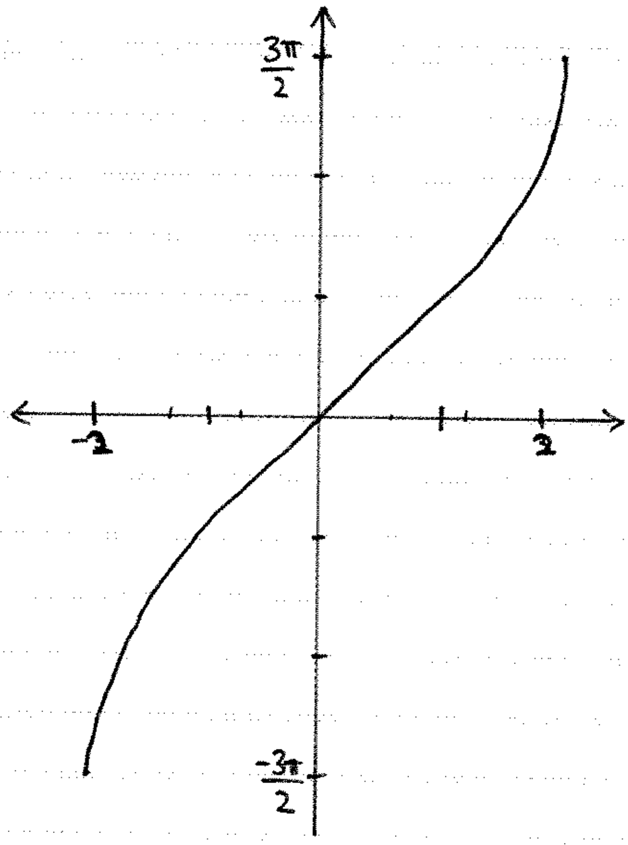
$\therefore \sin\frac{1}{3}y = \frac{x}{2}$

$\therefore 2\sin\frac{1}{3}y = x$

$\therefore -2 \le x \le 2$

$-\frac{3\pi}{2} \le y \le \frac{3\pi}{2}$

(ii)




✓

(d)(i) Arrangements $= \dfrac{8!}{2!2!2!}$

$= 5040$ ✓

(ii) Probability $= \dfrac{\frac{6!}{2!2!}}{5040}$ ✓

$= \dfrac{180}{5040}$

$= \dfrac{1}{28}$ ✓ for simpl. from factorial form

OR Prob = (Prob 1st letter is a U) × (Prob. last letter is a U)

$= \dfrac{2}{8} \times \dfrac{1}{7}$ ✓

$= \dfrac{1}{28}$ ✓

3 (a)(i) $y = px - ap^2$ —— ①

$y = qx - aq^2$ —— ②

Solve ①, ② simultaneously

$x = a(p+q)$ , $y = apq$

(ii) $S(0, a)$ , $P(2ap, ap^2)$

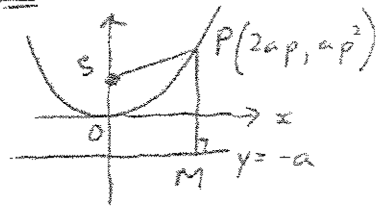
$$SP = \sqrt{(2ap-0)^2 + (ap^2-a)^2}$$

$$= \sqrt{4a^2p^2 + a^2p^4 - 2a^2p^2 + a^2}$$

$$= \sqrt{a^2(p^2+1)^2}$$

$$= a(p^2+1)$$

or




$SP = PM$ (by definition)

$= ap^2 - (-a)$

$= ap^2 + a = a(p^2+1)$

(iii) $SP = a(p^2+1)$, $SQ = a(q^2+1)$

$a(p^2+1) + a(q^2+1) = 4a$

$p^2 + q^2 = 2$ —— ③

$x = a(p+q)$, $y = apq$

$$x^2 = a^2(p^2 + q^2 + 2pq)$$

$$= a^2\left(2 + \frac{2apq}{a}\right) \quad \text{from ③}$$

$$= a^2\left(2 + \frac{2y}{a}\right)$$

$$= 2a(a + y)$$

(b)(i) $f(0.4) < 0$, $f(0.6) > 0$

or

$f(0.4) \cdot f(0.6) < 0$

$\therefore$ root exists between 0.4 + 0.6

b (ii) $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

$= x_1 - \dfrac{x_1 - e^{-x_1}}{1 + e^{-x_1}}$

$x_1 = 0.5$

$\therefore x_2 \approx 0.57$

(c) $f(1) = -10$, $f(2) = 20$

$\therefore 1^n + a - 6 = -10$

$a = -5$

$(-2)^n + 10 - 6 = 20$

$(-2)^n = 16$

$n = 4$

$\therefore f(x) = x^4 - 5x - 6$

$f(-1) = 1 + 5 - 6$

$= 0$

$\therefore x+1$ is a factor.

Question 4

a) i) $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d}{dx}\left(\frac{1}{2}(6+4x-2x^2)\right)$ ✓

$= \frac{d}{dx}\left(3+2x-x^2\right)$ ✓

$= 2-2x$

$= -2(x-1)$

ii) clearly centre is $x=1$.

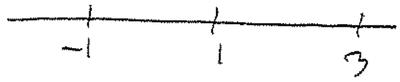
$V=0$ at end points

$\therefore\ 0 = 6+4x-2x^2$

$0 = x^2-2x-3$

$0 = (x-3)(x+1)$ ✓

$x = -1$ or $3$




✓

$\therefore$ centre of motion is $x=1$

iii) $n^2 = 2$

$n = \sqrt{2}$

Period $= \frac{2\pi}{n}$

$= \frac{2\pi}{\sqrt{2}}$

$= \sqrt{2}\,\pi$ ✓

iv) From part ii

amplitude $= 2$ ✓

Question 4

b) $\left(x^2+\frac{1}{2x}\right)^8 = \sum_{r=0}^{8}(x^2)^r\left(\frac{1}{2x}\right)^{8-r}\,{}^8C_r$ ✓

$\therefore\ (x^2)^r\left(\frac{1}{2x}\right)^{8-r} = \_\ x^1$

$2r + -1(8-r) = 1$

$2r-8+r = 1$

$3r = 9$

$r = 3$ ✓

Coefficent $= {}^8C_3\left(\frac{1}{2}\right)^{8-3}$

$= {}^8C_3\left(\frac{1}{2}\right)^5$

$= 1.75$ ✓

c) i) $(0.95)^{20} \doteqdot 0.36$ $\quad .358$ ✓

ii) ${}^{20}C_4\,(0.05)^4(0.95)^{16} \doteqdot 0.013$ ✓ ✓

Y12 Ext1 Trial '07

Q5(a)(i) $\int_{\pi/4}^{\pi/2} \cos^2 dx = \frac{1}{2}\int_{\pi/4}^{\pi/2} \cos 2x + 1 \; dx$

$= \frac{1}{2}\left[\frac{1}{2}\sin 2x + x\right]_{\pi/4}^{\pi/2}$

$= \frac{1}{2}\left[\left(\frac{1}{2}\sin\pi + \frac{\pi}{2}\right) - \left(\frac{1}{2}\sin\frac{\pi}{2} + \frac{\pi}{4}\right)\right]$

$= \frac{1}{2}\left(\frac{\pi}{2} - \frac{1}{2} - \frac{\pi}{4}\right)$ (2) generally well done.

$= \frac{\pi}{8} - \frac{1}{4}$

(ii) $Vol = \pi\int_{\pi/4}^{\pi/2} (1-\cos x)^2 \, dx$

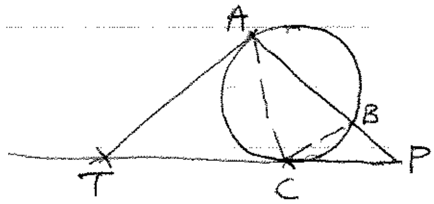
$= \pi \int_{\pi/4}^{\pi/2} 1 - 2\cos x + \cos^2 x \; dx$

$= \pi\left(\left[x - 2\sin x\right]_{\pi/4}^{\pi/2} + \frac{\pi}{8} - \frac{1}{4}\right)$ some did not use part (i).

$= \pi\left(\left(\frac{\pi}{2} - 2\right) - \left(\frac{\pi}{4} - \sqrt{2}\right) + \frac{\pi}{8} - \frac{1}{4}\right)$

$= \pi\left(\frac{3\pi}{8} - 2\frac{1}{4} + \sqrt{2}\right) \; u^3$. (2) Many messed up substitution.

Q5(b)




(i) $B\hat{A}C = B\hat{C}P$ (alt segment theorem) Many answers lacked sequence or had poor explanations.

$B\hat{A}T = 90°$ (angle between tangent and diameter).

$\therefore C\hat{A}T = B\hat{A}T - B\hat{A}C$ (2)

$= 90° - B\hat{C}P$.

(ii) Since $AT = CT$ (tangents from external point to a circle are equal length).

$\therefore \triangle ATC$ is isosceles.

then $A\hat{C}T = C\hat{A}T = 90 - B\hat{C}P$ (base < isos $\triangle$)

$\therefore A\hat{T}C = 180 - 2(90 - B\hat{C}P)$ (< sum $\triangle$)

$= 2B\hat{C}P$ // (2)

5(c)(i) $\frac{dT}{dt} = -kAe^{-kt}$

$= -k(T-20)$ (since $Ae^{-kt} = T-20$) (1)

(ii) $t=0$ $T=90$ then $A = 70$.

$T = 20 + 70e^{-kt}$ well done.

when $T=70$ $t=10$

$70 = 20 + 70e^{-10k} \Rightarrow k = -\frac{1}{10}\ln\frac{5}{7}$.

When $T=15$ $T = 20 + 70e^{\frac{1}{10}\ln\frac{5}{7}\times 15}$ (3)

$= 62°$ (nearest degree)

## Question 6

a) i) $f'(x) = \dfrac{e^x(1+e^x) - e^x \cdot e^x}{(1+e^x)^2}$

$= \dfrac{e^x + e^{2x} - e^{2x}}{(1+e^x)^2}$ ✓

$= \dfrac{e^x}{(1+e^x)^2}$ ✓

$> 0$ ~~for~~ all values of $x$

$\therefore f(x)$ allways has a positive gradient

$\therefore f(x)$ is increasing for all values of $x$

ii) Since $f(x)$ is always increasing for each y value there can only be one x value. ✓

iii) $x = \dfrac{e^y}{1+e^y}$ ✓

$(1+e^y)x = e^y$

$x + xe^y = e^y$

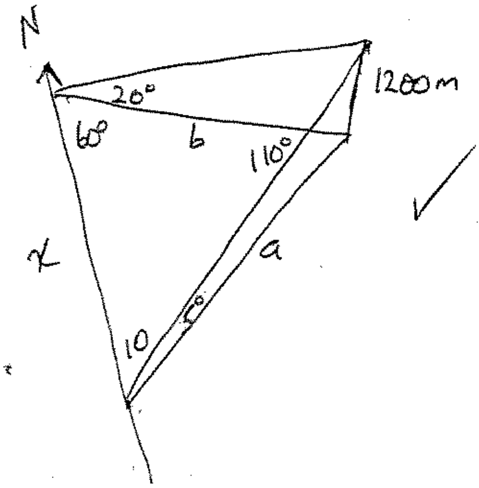
$xe^y - e^y = -x$

$e^y(x-1) = -x$

$e^y = \dfrac{-x}{x-1}$

$y = \ln\left(\dfrac{-x}{x-1}\right)$ ✓

b)




✓

$\tan 5 = \dfrac{1200}{a}$ $\quad$ $\tan 20 = \dfrac{1200}{b}$

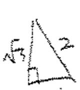
$a = \dfrac{1200}{\tan 5}$ $\quad$ $b = \dfrac{1200}{\tan 20}$ ✓

$x^2 = \left(\dfrac{1200}{\tan 5}\right)^2 + \left(\dfrac{1200}{\tan 20}\right)^2 - 2\left(\dfrac{1200}{\tan 5}\right)\left(\dfrac{1200}{\tan 20}\right)\cos 110$

$x = 15163.56578\text{m}$ ✓

Speed $= \dfrac{15.163}{1\frac{1}{2}} \doteqdot 10.1$ km/h ✓

c) $\sin x + \sqrt{3}\cos x$




$2\left(\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x\right)$

$2(\cos A \sin x + \sin A \cos x)$

$2\left(\sin\left(x + \frac{\pi}{3}\right)\right)$ ✓

$\sin\left(x + \frac{\pi}{3}\right) = \frac{1}{2}$ ✓

$x + \frac{\pi}{3} = \frac{13\pi}{6}$ or $\frac{5\pi}{6}$

$x = \frac{11\pi}{6}$ or $\frac{\pi}{2}$ ✓

7. (a) (i) $y=0$ when rocket hits ground

$\therefore -\frac{1}{2}gt^2 + Vt\sin\theta = 0$

$\therefore t\left(-\frac{1}{2}gt + V\sin\theta\right) = 0$

$\therefore$ either $\cancel{t=0}$ or $\frac{1}{2}gt = V\sin\theta$

$$t = \frac{2V\sin\theta}{g}$$

(ii) when $t = \frac{2V\sin\theta}{g}$ :

$$x = \frac{2V^2\sin\theta\cos\theta}{g} \qquad (1)$$

(iii) for the boat: $x = \frac{Vt}{2}$

$$\therefore x = \frac{2V^2\sin\theta}{2g} \qquad (2)$$

Equating (1) and (2):

$$\frac{\cancel{2V^2}\cancel{\sin\theta}\cos\theta}{\cancel{g}} = \frac{\cancel{2V^2}\cancel{\sin\theta}}{2\cancel{g}}$$

$\therefore 2\cos\theta = 1$

$\therefore \cos\theta = 1/2$

$\therefore \theta = \frac{\pi}{3}$ (must be in first quadrant)

(b) To prove: $9^n - 3$ is divisible by 6 for all positive integers $n$

For $n=1$: $9^1 - 3 = 6$

$\therefore$ true for $n=1$

Assume true for $n=k$

$\therefore 9^k - 3 = 6M$ where $M$ is an integer

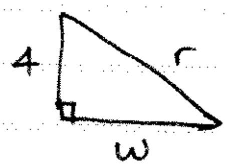
Prove true for $n=k+1$

$$\begin{aligned} 9^{k+1} - 3 &= 9.9^k - 3 \\ &= 9(6M+3) - 3 \\ &= 54M + 27 - 3 \\ &= 54M + 24 \\ &= 6(9M+4) \end{aligned}$$

$\therefore$ true for $n=k+1$

Since it is true for $n=1$, and when we assume that it is true for $n=k$ then it follows that it is also true for $n=k+1$, then it must be true for all integers greater than or equal to 1.

(c)




$$r^2 = 16 + w^2$$
$$r = \sqrt{16+w^2}$$ ✓

$$\therefore \frac{dr}{dw} = \frac{1}{2} \times 2w \times \left(16+w^2\right)^{-1/2}$$
$$= \frac{w}{\sqrt{16+w^2}}$$

$$\frac{dr}{dt} = 2$$

$$\frac{dw}{dt} = \frac{dw}{dr} \times \frac{dr}{dt}$$ ✓
$$= \frac{\sqrt{16+w^2}}{w} \times 2$$

when $r = 10$, $100 = 16 + w^2$
$$\therefore w = \sqrt{84}$$ ✓
$$\therefore \frac{dw}{dt} = \frac{\sqrt{16+84}}{\sqrt{84}} \times 2$$
$$= \frac{20}{\sqrt{84}}$$ ✓

2.18