BOSTES Number: ____________________________

CLASS (Please circle): 12M1 12M2 12M3 12M4 12M5

NORMANHURST BOYS HIGH SCHOOL
NEW SOUTH WALES

2015
HIGHER SCHOOL CERTIFICATE
TRIAL EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time - 5 minutes
- Working time - 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In Questions 11-14, show relevant mathematical reasoning and/or calculations
- Begin each question on a separate writing booklet

Total marks - 70

Section I Pages 2-4

10 marks

- Attempt Questions 1-10
- Answer on the Multiple Choice answer sheet provided
- Allow about 15 minutes for this section

Section II Pages 5-9

60 marks

- Attempt Questions 11-14
- Allow about 1 hour 45 minutes for this section

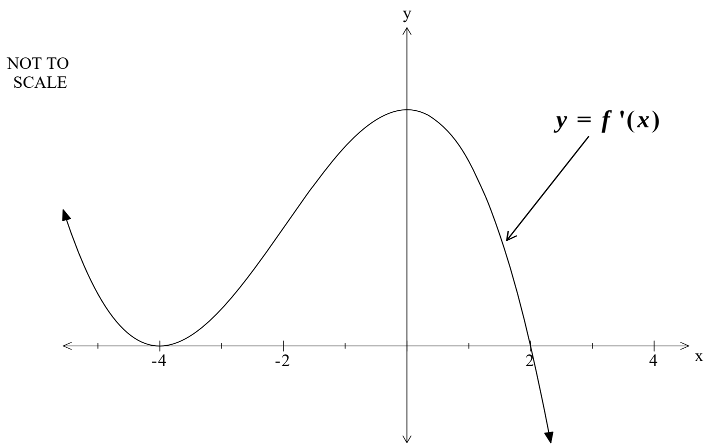
Students are advised that this is a school-based examination only and cannot in any way guarantee the content or format of future Higher School Certificate Examinations.

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1-10

1



The diagram above represents a sketch of the gradient function of the curve $y = f(x)$.
Which of the following is a true statement? The curve $y = f(x)$ has

(A) a minimum turning point occurs at $x = -4$

(B) a horizontal point of inflexion occurs at $x = 2$

(C) a horizontal point of inflexion occurs at $x = -4$

(D) a maximum turning point occurs at $x = 0$.

2 Solve $\dfrac{3}{x(2x-1)} \geq 1$

(A) $x \leq -1,\ 0 < x < \dfrac{1}{2}$ and $x \geq \dfrac{3}{2}$

(B) $-1 \leq x < 0$ and $\dfrac{1}{2} < x \leq \dfrac{3}{2}$

(C) $-\dfrac{3}{2} \leq x \leq -\dfrac{1}{2}$ and $0 < x \leq 1$

(D) $x \leq -\dfrac{3}{2},\ -\dfrac{1}{2} \leq x < 0$ and $x \geq 1$

3 If $sinx = \frac{3}{5}, \; \frac{\pi}{2} \leq x \leq \pi,$ evaluate $tan2x.$

(A) $-\frac{7}{24}$

(B) $-\frac{24}{7}$

(C) $\frac{7}{24}$

(D) $\frac{24}{7}$

4 Find the acute angle between the tangents to the graphs $y = x$ and $y = x^3$ at the point (1,1).

(A) 27°

(B) 30°

(C) 45°

(D) 63°

5 The polynomial $P(x) = 2x^3 + kx^2 + 5x - 2$ has $(x + 2)$ as a factor.
Find the value of k.

(A) –7

(B) 7

(C) –12

(D) 12

6 Find $\frac{d}{dx}(\sqrt{1 - x^2} + x \sin^{-1} x)$

(A) $\frac{-2}{\sqrt{1-x^2}}$

(B) $\frac{-1}{\sqrt{1-x^2}}$

(C) $\cos^{-1} x$

(D) $\sin^{-1} x$

7 Find the domain and range of $y = 3\cos^{-1}\pi x$

(A) Domain: $0 \le x \le 3$ Range: $0 \le y \le \pi$

(B) Domain: $-1 \le x \le 1$ Range: $0 \le y \le \pi$

(C) Domain: $-\pi \le x \le \pi$ Range: $0 \le y \le 3\pi$

(D) Domain: $-\frac{1}{\pi} \le x \le \frac{1}{\pi}$ Range: $0 \le y \le 3\pi$

8 A stone is thrown at an angle of α to the horizontal. The position of the stone at time t seconds is given by $x = Vt\cos\alpha$ and $y = Vt\sin\alpha - \frac{1}{2}gt^2$ where $g\ m/s^2$ is the acceleration due to gravity and v m/s is the initial velocity of projection.

What is the maximum height reached by the stone?

(A) $\dfrac{V\sin\alpha}{g}$

(B) $\dfrac{g\sin\alpha}{V}$

(C) $\dfrac{V^2\sin^2\alpha}{2g}$

(D) $\dfrac{g\sin^2\alpha}{2V^2}$

9 The volume of a sphere of radius 8 mm is increasing at a constant rate of $50\ \text{mm}^3$.

Determine the rate of increase of the surface area of the sphere.

(A) $0.06\ \text{mm}^2/\text{s}$

(B) $0.60\ \text{mm}^2/\text{s}$

(C) $1.25\ \text{mm}^2/\text{s}$

(D) $12.50\ \text{mm}^2/\text{s}$

10 The acceleration of a particle is defined in terms of its position by $\ddot{x} = 2x^3 + 4x$. The particle is initially $2m$ to the right of the origin, travelling with velocity $6ms^{-1}$. Find the minimum speed of the particle.

(A) $6\ ms^{-1}$

(B) $16\ ms^{-1}$

(C) $20\ ms^{-1}$

(D) $36\ ms^{-1}$

Section II

60 marks
Attempt Questions 11 – 14
Allow about 1 hour and 45 minutes for this section

Answer each question in a SEPARATE writing booklet.

Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Evaluate $\int_1^2 \frac{dx}{x}$. **1**

(ii) Use Simpson's rule with 3 function values to approximate $\int_1^2 \frac{dx}{x}$. **1**

(iii) Use your results to parts (i) and (ii) to obtain an approximation for e . Give your answer correct to 3 decimal places. **2**

(b) Using one application of Newton's method with $x = \frac{\pi}{2}$ as the first approximation, find the second approximation to the root of the equation $cosx = 2x - 3$. Correct answer to 3 decimal places. **2**

(c) The polynomial $P(x) = x^3 + bx^2 + cx + d$ has roots 0, 3 and –3.

(i) What are the values of b, c and d? **2**

(ii) Without using calculus, sketch the graph of $y = P(x)$. **1**

(iii) Hence or otherwise, solve the inequality $\frac{x^2 - 9}{x} > 0$ **1**

(d) Consider the function $f(x) = \tan^{-1}\frac{1}{x}$. Find the slope of the tangent to the curve where the function $y = f(x)$ cuts the y-axis. **2**

(e) Find the exact value of $\int_{-1}^{1} \sqrt{4 - x^2}\, dx$, using the substitution $x = 2\sin\theta$. **3**

Question 12 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) The point $C(-28,19)$ divides the interval AB externally in the ratio $k:5$. Find the value of k if A is the point $(-4,3)$ and B is the point $(2,-1)$ **2**

(b) (i) Show that $\sin(x+\frac{\pi}{4})=\frac{\sin x+\cos x}{\sqrt{2}}$ **1**

(ii) Hence or otherwise, solve $\frac{\sin x+\cos x}{\sqrt{2}}=\frac{\sqrt{3}}{2}$ for $0\le x\le 2\pi$. **2**

(c) $P(2ap,ap^2)$ and $Q(2aq,aq^2)$ are two points on the parabola $x^2=4ay$.

(i) Show that the equation of the normal to the parabola at P is given by $x+py=2ap+ap^3$. **1**

(ii) Find the co-ordinates of R, the point of intersection of the normals at P and Q, in terms of p and q. **2**

(iii) If $pq=-2$, find the cartesian equation of the locus of R. **2**

(d) Evaluate $\int_0^1 x^2\sqrt{1+3x^3}\,dx$ using the substitution $u=1+3x^3$. **2**

(e) After t years the number of animals, N, in a national park decreases according to the equation:

$$\frac{dN}{dt}=-0.09(N-100)$$

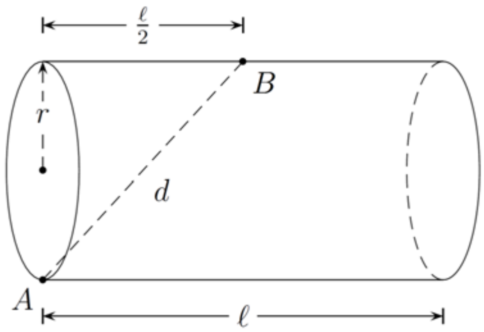
The initial number of animals in the national park is 400.

(i) Verify that $N=100+Ae^{-0.09t}$ is a solution of the above equation, where A is a constant. **1**

(ii) How many years does it take for the number of animals to reach 150? **2**

Question 13 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a)

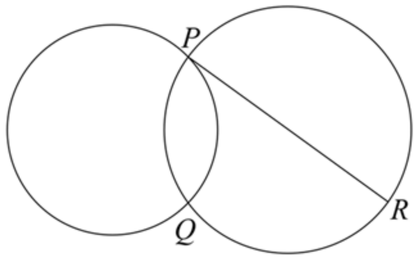


The diagram shows a cylindrical barrel of length *l* and radius *r*. The point A is at one end of the barrel, at the very bottom of the rim. The point B is at the very top of the barrel, half-way along its length. The length of AB is *d*.

(i) Show that the volume of the barrel is $V = \frac{\pi l}{4}\left(d^2 - \frac{l^2}{4}\right)$. **1**

(ii) Find *l* in terms of *d* if the barrel has maximum volume for the given *d*. **2**

(b) Two circles are intersecting at *P* and *Q*. The diameter of one of the circles is *PR*.



Copy this diagram into your writing booklet.

(i) Draw a straight line through *P*, parallel to *QR* to meet the circle *PRQ* at *S* and the other circle at *T*. Prove that *QS* is also diameter of the circle. **2**

(ii) Prove that the circles have equal radii if *TQ* is parallel to *PR*. **2**

Question 13 continues on page 9

Question 13 (continued)

(c) A rocket is fired from a pontoon on the sea. The rocket is aimed at a 60m high cliff, 240m from the pontoon. The angle of projection of the rocket is 45° and its initial velocity is $40\sqrt{2}\,ms^{-1}$.

(i) Taking the point of projection as the origin O, derive expressions for the horizontal component x and vertical component y of the position of the rocket at time t seconds. (Assume the acceleration due to gravity is $10ms^{-2}$) **2**

(ii) Show that the path of the rocket is given by the equation **1**

$$y = x - \frac{x^2}{320}$$

(iii) Find the time taken for the rocket to land on top of the cliff. **2**

(iv) Find the exact velocity of the rocket when it reaches this point. (Hint: velocity includes magnitude and direction) **3**

Question 14 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Use mathematical induction to prove that $3^n + 7^{n+1}$ is divisible by 4 **3**
for all integers $n \geq 1$.

(b) The velocity of a particle moving in a straight line is given by

$$v = 10 - x$$

where x metres is the displacement from a fixed point O and v is the velocity in metres per second. Initially the particle is at O.

(i) Show that the acceleration of the particle is given by **1**

$$\ddot{x} = x - 10$$

(ii) Express x in terms of time t. **3**

(iii) What is the limiting position of the particle? **1**

(c) A particle moves in a straight line and its displacement x metres from a fixed point O at any time t seconds is given by the equation

$$x = 4\cos^2 t - 1 .$$

(i) Prove that the particle is undergoing simple harmonic motion. **2**

(ii) State the period of the motion. **1**

(iii) Sketch the graph $x = 4\cos^2 t - 1$ for $0 \leq t \leq \pi$. **2**
Clearly show the times when the particle passes through O.

(iv) Find the time when the velocity of the particle is increasing most rapidly for $0 \leq t \leq \pi$. **2**

End of paper

STANDARD INTEGRALS

$$\int x^{n} dx = \frac{1}{n+1} x^{n+1}, \ n \neq -1; \ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \ x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}, \ a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \ a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \ a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \ a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \ a > 0, \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x, \ \ x > 0$

2015 Trial Extension 1 Solutions

1) (C)

2) $\frac{3}{x(2x-1)} \geq 1$

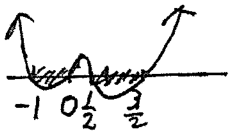
$x^2(2x-1)^2 \times \frac{3}{x(2x-1)} \geq x^2(2x-1)^2$

$3x(2x-1) \geq x^2(2x-1)^2$

$x(2x-1)[3 - x(2x-1)] \geq 0$

$x(2x-1)(2x^2-x-3) \leq 0$

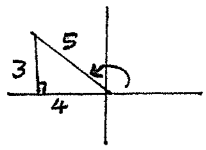
$x(2x-1)(2x-3)(x+1) \leq 0$



$\therefore -1 \leq x < 0$ & $\frac{1}{2} < x \leq \frac{3}{2}$

(B)

3) $\tan 2x = \frac{2\tan x}{1-\tan^2 x}$



$= \frac{2(-\frac{3}{4})}{1-(-\frac{3}{4})^2} = -\frac{24}{7}$

(B)

4) $y = x$, $m_1 = 1$

$y = x^3$

$y' = 3x^2$

$m_2 = 3$ at $(1,1)$

$\tan\theta = \left|\frac{1-3}{1+(1)(3)}\right| = \frac{1}{2}$

$\therefore \theta \doteqdot 27°$

(A)

5) $P(x) = 2x^3 + kx^2 + 5x - 2$

$P(-2) = 0$

$2(-2)^3 + k(-2)^2 + 5(-2) - 2 = 0$

$-16 + 4k - 12 = 0$

$k = 7$

(B)

6) $\frac{d}{dx}(\sqrt{1-x^2} + x\sin^{-1}x)$

$= \frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x) + x \cdot \frac{1}{\sqrt{1-x^2}} + \sin^{-1}x$

$= \sin^{-1}x$

(D)

7) $y = 3\cos^{-1}\pi x$

$D: -1 < \pi x < 1$

$-\frac{1}{\pi} < x < \frac{1}{\pi}$

$R: 0 \leq y \leq 3\pi$

(D)

8) $y = Vt\sin\alpha - \frac{gt^2}{2}$ — ①

$\dot{y} = V\sin\alpha - gt$

At max height, $\dot{y} = 0$

$V\sin\alpha - gt = 0$

$t = \frac{V\sin\alpha}{g}$

Sub into ①,

Max ht $= V\sin\alpha \times \frac{V\sin\alpha}{g} - \frac{g}{2}\left(\frac{V\sin\alpha}{g}\right)^2$

$= \frac{V^2\sin^2\alpha}{2g}$

(C)

9) $V = \frac{4}{3}\pi r^3$

$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$

$50 = 4\pi r^2 \times \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{25}{128\pi}$

$S = 4\pi r^2$

$\frac{dS}{dt} = \frac{dS}{dr} \times \frac{dr}{dt}$

$= 8\pi \times 8 \times \frac{25}{128\pi}$

$= 12.5 \text{ mm}^2/\text{s}$

(D)

10) $\ddot{x} = 2x^3 + 4x$

$\frac{d}{dx}\left(\frac{v^2}{2}\right) = 2x^3 + 4x$

$\frac{v^2}{2} = \frac{x^4}{2} + 2x^2 + C$

when $t = 0$, $x = 2$, $v = 6$

$\therefore \frac{36}{2} = \frac{16}{2} + 8 + C$

$C = 2$

$\therefore v^2 = (x^2+2)^2$

When $t = 0$, $x = 2$, $v = 6 > 0$

$\therefore v = x^2 + 2$

Hence $v \geq 6$ as the particle never changes direction and it is moving with increasing speed.

(A)

11) a) i) $\int_1^2 \frac{dx}{x} = [\ln x]_1^2$

$= \ln 2 - \ln 1$
$= \ln 2$ ✓ (1)

ii) $\int_1^2 \frac{dx}{x} \doteqdot \frac{0.5}{3}\left(1+4\times\frac{1}{1.5}+\frac{1}{2}\right)$

$= \frac{25}{36}$ ✓ (1)

iii) $\ln 2 \doteqdot \frac{25}{36}$

$2 \doteqdot e^{\frac{25}{36}}$ ✓

$e \doteqdot 2^{\frac{36}{25}}$

$= 2.713$ (3 dec pl) ✓ (2)

b) $f(x) = \cos x - 2x + 3$

$f'(x) = -\sin x - 2$

$f(\frac{\pi}{2}) = \cos\frac{\pi}{2} - 2(\frac{\pi}{2}) + 3$

$= 3 - \pi$

$f'(\frac{\pi}{2}) = -\sin\frac{\pi}{2} - 2$

$= -3$

$\therefore x_1 = \frac{\pi}{2} - \frac{3-\pi}{-3}$ ✓

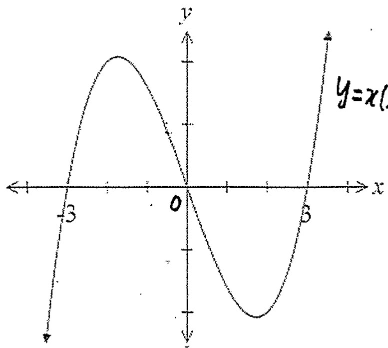
$= \frac{\pi}{6} + 1$

$= 1.524$ (3 dec pl.) ✓ (2)

c) i) $x^3 + bx^2 + cx + d = x(x-3)(x+3)$
$= x^3 - 9x$ ✓

$\therefore b = 0$
$c = -9$
$d = 0$ ✓ (2)

ii)



✓ (1)

iii) $x^2 \times \frac{x^2-9}{x} > 0 \times x^2$

$x(x^2-9) > 0$

From graph, $-3 < x < 0$ or $x > 3$ ✓ (1)

d) $y = \tan^{-1}\frac{1}{x}$

$y' = \frac{1}{1+(\frac{1}{x})^2}\left(-\frac{1}{x^2}\right)$ ✓

$= -\frac{1}{x^2+1}$

$= -1$ at $x = 0$ ✓ (2)

Discontinuity at $x = 0$ (1 mark)

e) $\int_{-1}^{1} \sqrt{4-x^2}\,dx$

$x = 2\sin\theta$

$\frac{dx}{d\theta} = 2\cos\theta$

$= \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \sqrt{4-4\sin^2\theta} \times 2\cos\theta\,d\theta$ ✓

$= 4\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \cos^2\theta\,d\theta$

$= 8\int_0^{\frac{\pi}{6}} \frac{1}{2}(\cos 2\theta + 1)\,d\theta$ ✓

$= 4\left[\frac{1}{2}\sin 2\theta + \theta\right]_0^{\frac{\pi}{6}}$

$= 4\left(\frac{1}{2}\sin\frac{\pi}{3} + \frac{\pi}{6} - 0 - 0\right)$

$= 4\left(\frac{1}{2}\times\frac{\sqrt{3}}{2} + \frac{\pi}{6}\right)$

$= \sqrt{3} + \frac{2\pi}{3}$ ✓ (3)

12) a) $-28 = \frac{2(-k)+(-4)(5)}{-k+5}$ ✓

$k = 4$ ✓ (2)

b) i) $LHS = \sin(x + \frac{\pi}{4})$

$= \sin x \cos\frac{\pi}{4} + \cos x \sin\frac{\pi}{4}$

$= (\sin x)\times\frac{1}{\sqrt{2}} + (\cos x)\times\frac{1}{\sqrt{2}}$

$= \frac{\sin x + \cos x}{\sqrt{2}}$ ✓

$= RHS$ (1)

ii) $\sin(x + \frac{\pi}{4}) = \frac{\sqrt{3}}{2}$

$x + \frac{\pi}{4} = \frac{\pi}{3}$ or $\frac{2\pi}{3}$ ✓

$\therefore x = \frac{\pi}{12}$ or $\frac{5\pi}{12}$ ✓ (2)

12c) i)

$y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{x}{2a}$

$= \frac{2ap}{2a}$

$= p$

$\therefore$ gradient of normal $= -\frac{1}{p}$

Equation of normal at P:

$y - ap^2 = -\frac{1}{p}(x - 2ap)$ ✓

$py - ap^3 = -x + 2ap$

$x + py = 2ap + ap^3$ — ①   (1)

ii) Equation of normal at Q:

$x + qy = 2aq + aq^3$ — ②

① − ②,

$y = \frac{2a(p-q) + a(p^3 - q^3)}{p-q}$

$= \frac{2a(p-q) + a(p-q)(p^2+pq+q^2)}{p-q}$

$= a(2 + p^2 + pq + q^2)$ ✓

Sub into ①,

$x = 2ap + ap^3 - ap(2+p^2+pq+q^2)$

$= -apq(p+q)$ ✓

$\therefore R = [-apq(p+q), a(2+p^2+pq+q^2)]$ (2)

iii) $x = -apq(p+q)$ — ①

$y = a(2+p^2+pq+q^2)$ — ②

$pq = -2$ — ③

Sub ③ into ①,

$x = -a(-2)(p+q)$

$p + q = \frac{x}{2a}$ — ④ ✓

From ②, $y = a[pq + (p+q)^2 - 2pq + 2]$

$= a[-2 + (\frac{x}{2a})^2 - 2(-2) + 2]$ using ③ & ④

$= \frac{x^2}{4a} + 4a$

$\therefore x^2 = 4a(y - 4a)$ ✓ is the locus of R   (2)

d) $\int_0^1 x^2\sqrt{1+3x^2}\,dx$ $u = 1+3x^3$, $\frac{du}{dx} = 9x^2$

$= \frac{1}{9}\int_1^4 \sqrt{u}\,du$ ✓

$= \frac{1}{9} \times \frac{2}{3}[u^{\frac{3}{2}}]_1^4$

$= \frac{2}{27}(8-1)$

$= \frac{14}{27}$ ✓   (2)

e) i) $N = 100 + Ae^{-0.09t}$ — ①

$\frac{dN}{dt} = -0.09Ae^{-0.09t}$ ✓

$= -0.09(N - 100)$ using ①   (1)

ii) $N = 100 + Ae^{-0.09t}$

when $t = 0$, $N = 400$

$400 = 100 + A$

$\therefore A = 300$

$N = 100 + 300e^{-0.09t}$

$150 = 100 + 300e^{-0.09t}$ ✓

$\frac{50}{300} = e^{-0.09t}$

$\ln 6^{-1} = -0.09t$

$t = \frac{\ln 6}{0.09}$

$\doteqdot 19.9$

It takes 20 years for the number to reach 150. ✓   (2)

13) a) i) $V = \pi r^2 l$

$d^2 = (2r)^2 + (\frac{l}{2})^2$

$= 4r^2 + \frac{l^2}{4}$

$r^2 = \frac{1}{4}(d^2 - \frac{l^2}{4})$

$\therefore V = \pi l [\frac{1}{4}(d^2 - \frac{l^2}{4})]$

$= \frac{\pi l}{4}(d^2 - \frac{l^2}{4})$ ✓ (1)

ii) $V = \frac{\pi l d^2}{4} - \frac{\pi}{16} l^3$

$\frac{dV}{dl} = \frac{\pi d^2}{4} - \frac{3\pi}{16} l^2$

$= 0$

when $\frac{\pi d^2}{4} = \frac{3\pi}{16} l^2$

$l^2 = \frac{\pi d^2}{4} \times \frac{16}{3\pi}$

$= \frac{4d^2}{3}$

$l = \frac{2}{\sqrt{3}} d$ ✓

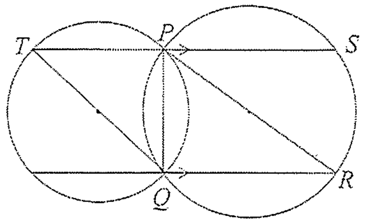
$\frac{d^2V}{dl^2} = -\frac{3\pi}{8} l$

$= -\frac{3\pi}{8} \times \frac{2d}{\sqrt{3}}$

< 0 ✓

$\therefore$ Max V when $l = \frac{2}{\sqrt{3}} d$ (2)

b) i)



$\angle PQR = 90°$ (Angle in a semicircle) ✓

$\angle SPQ + \angle PQR = 180°$ (cointerior angles, PS//QR)

$\therefore \angle SPQ = 90°$ ✓

Hence QS is a diameter (Angle in a semicircle equals 90°) (2)

ii) $\angle TPQ = \angle PQR$ (Alternate angles, TS//QR)

$= 90°$

$\therefore$ TQ is a diameter (Angle in a semicircle $= 90°$) ✓

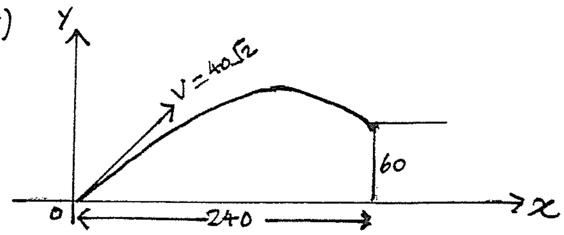
PRQT is a parallelogram (TQ//PR, TP//QR)

PR = QT (Opposite sides of parallelogram)

PR and QT are equal diameters ✓

$\therefore$ the two circles have equal radii. (2)

c)



i) $\ddot{y} = -10$

$\dot{y} = -10t + C_1$

When $t=0$, $\dot{y} = 40\sqrt{2} \sin 45°$

$= 40$

$\therefore C_1 = 40$

$\dot{y} = -10t + 40$

$y = -\frac{10t^2}{2} + 40t + C_2$

When $t=0$, $y=0$

$\therefore C_2 = 0$

$y = -5t^2 + 40t$ ✓

$\ddot{x} = 0$

$\dot{x} = 40\sqrt{2} \cos 45°$

$= 40$

$x = 40t + C_3$

When $t=0$, $x=0$

$\therefore C_3 = 0$

$x = 40t$ ✓ (2)

ii) $x = 40t$ —(1)

$y = -5t^2 + 40t$ —(2)

Sub (1) into (2),

$y = -5(\frac{x}{40})^2 + 40(\frac{x}{40})$ ✓

$y = x - \frac{x^2}{320}$ (1)

iii) Sub $y = 60$ into (2),

$60 = -5t^2 + 40t$ ✓

$t^2 - 8t + 12 = 0$

$(t-6)(t-2) = 0$

$t = 2$ or 6

$\therefore$ it takes 6 s to reach the top. ✓ (2)

Alternative Method

Sub $x = 240$ into (1),

$240 = 40t$ ✓

$t = 6$ ✓

iv) At the top of the cliff,

$\dot{x} = 40$

$\dot{y} = -10t + 40$

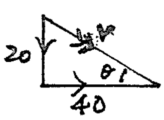
$= -10(6) + 40$

$= -20$ ✓

$v^2 = (-20)^2 + 40^2$

$= 2000$

$|v| = 20\sqrt{5}$ ✓



$\tan\theta = \frac{20}{40}$

$\theta = 26°34'$ ✓

The velocity is $20\sqrt{5}$ m/s, at an angle of $26°34'$ to the horizontal. (3)

14) a) $n=1$,

$3^n + 7^{n+1} = 3 + 7^2$

$= 52$ which is divisible by 4

$\therefore$ true for $n=1$

Assume true for $n=k$,

ie. $3^k + 7^{k+1} = 4P$ where P is an integer

Prove true for $n=k+1$,

ie. $3^{k+1} + 7^{k+2} = 4Q$, Q an integer ✓

$LHS = 3\times 3^k + 7^k \times 49$

$= 3(4P - 7\times 7^k) + 49\times 7^k$ using assumption ✓

$= 3\times 4P - 21\times 7^k + 49\times 7^k$

$= 3\times 4P + 28\times 7^k$

$= 4(3P + 7\times 7^k)$

$= 4Q$

Hence result is true for $n=k$, it is also true for $n=k+1$. ✓

By principal of mathematical induction, result is ~~true~~ for all integers $n \geq 1$. (3)

b) i) $\ddot{x} = \frac{d}{dx}\left(\frac{v^2}{2}\right)$

$= \frac{d}{dx}\frac{1}{2}(10-x)^2$

$= \frac{d}{dx}(50 - 10x + \frac{1}{2}x^2)$ ✓

$= x - 10$ (1)

ii) $\frac{dx}{dt} = 10 - x$

$\int \frac{1}{10-x}\,dx = \int dt$

$t = -\ln(10-x) + c$ ✓

when $t=0$, $x=0$

$0 = -\ln 10 + c$

$\therefore t = -\ln(10-x) + \ln 10$

$= \ln \frac{10}{10-x}$ ✓

$e^t = \frac{10}{10-x}$

$10 - x = 10e^{-t}$

$x = 10 - 10e^{-t}$ ✓ (3)

iii) 10 m on the RHS of O. ✓ (1)

c) i) $x = 4\cos^2 t - 1$

$= 2\cos 2t + 1$

$\dot{x} = -4\sin 2t$

$\ddot{x} = -4 \times 2\cos 2t$

$= -4(x-1)$

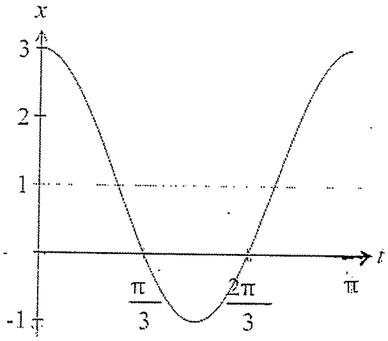
$= -2^2(x-1)$ ✓

$\therefore$ SHM about the point $x=1$ ✓ (2)

ii) Period $= \frac{2\pi}{2}$

$= \pi$ seconds (1)

iii)



Shape ✓

t intercepts ✓ (2)

iv) Velocity is increasing most rapidly when $\ddot{x}$ has the greatest positive value or x takes the least value, ie. $x=-1$ ✓

$x = 4\cos^2 t - 1$

$-1 = 4\cos^2 t - 1$

$\cos^2 t = 0$

$t = \frac{\pi}{2}$ ✓

OR From graph,

when $x = -1$

$t = \frac{1}{2}\left(\frac{\pi}{3} + \frac{2\pi}{3}\right)$

$= \frac{\pi}{2}$

$\therefore$ velocity is increasing most rapidly at $\frac{\pi}{2}$ seconds (2)