NORMANHURST BOYS HIGH SCHOOL

MATHEMATICS EXTENSION 1

2019 Year 12 Course Trial Examination
Friday, 16 August 2019

General instructions

- Working time – 2 hours.
  (plus 5 minutes reading time)
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- NESA approved calculators may be used.
- Attempt all questions.
- At the conclusion of the examination, bundle the booklets used in the correct order behind this paper and hand to examination supervisors.

SECTION I

- Mark your answers on the answer grid provided

SECTION II

- Commence each new question on a new booklet. Write on both sides of the paper.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.

NESA STUDENT #: .... **# BOOKLETS USED:** ....

Class: (please ✓)

- ○ 12MAT.1 – Mr Lam
- ○ 12MAT.2 – Mrs Bhamra
- ○ 12MAT.3 – Mrs Gan
- ○ 12MAT.4 – Ms Park
- ○ 12MAT.5 – Mr Sekaran

Marker's use only

QUESTION	1-10	11	12	13	14	Total	%
MARKS	/10	/15	/15	/15	/15	/70	

Section I

10 marks
Attempt Questions 1 to 10
Allow approximately 10 minutes for this section

Mark your answers on the answer grid provided.

Questions **Marks**

1. Which expression is equal to $\int \sin^2 2x \, dx$? **1**

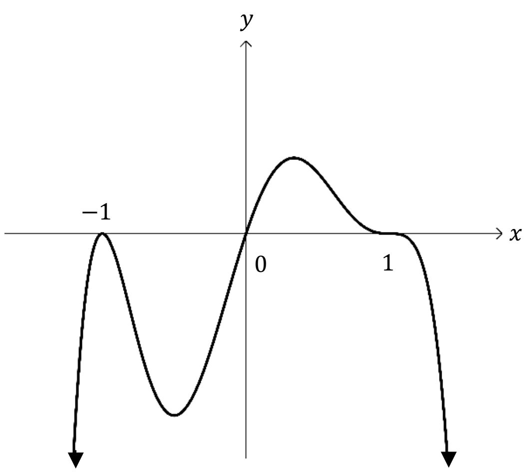
(A) $\frac{1}{8}(4x + \sin 4x) + c$

(B) $\frac{1}{8}(4x - \sin 4x) + c$

(C) $-\frac{\cos^3 2x}{6} + c$

(D) $\frac{\sin^3 2x}{6} + c$

2. Which of the following could be the equation of the graph given below? **1**



(A) $y = -\frac{x}{2}(1-x)^3(1-x)^2$

(B) $y = -\frac{x}{2}(1-x)^2(x+1)^3$

(C) $y = 3x(1-x)^3(x+1)^2$

(D) $y = 4x(1-x)^2(x-1)^3$

3. $x = 0.5$ is the first approximation to the positive root of the function $f(x) = xe^x - 1$. **1**

Which of the following is the second approximation using Newton's method, correct to two decimal places?

(A) $x = 0.56$

(B) $x = 0.57$

(C) $x = 0.58$

(D) $x = 0.59$

4. Which of the following is equal to $\lim_{x \to 0} \left(\frac{\sin 4x}{2x \cos 2x} \right)$? **1**

(A) $\frac{1}{2}$

(B) 0

(C) 4

(D) 2

5. The point P divides the interval from $A(2, -5)$ to $B(-4, 1)$, internally in the ratio $1 : 2$. **1**

What are the coordinates of P?

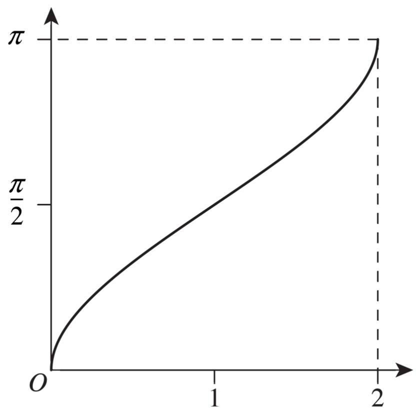
(A) $(-\frac{14}{3}, \frac{5}{3})$

(B) $(-\frac{14}{3}, -3)$

(C) $(0, -3)$

(D) $(-8, 11)$

6. Which function best describes the graph below? **1**



(A) $y = \cos^{-1} x$

(B) $y = 1 - \cos^{-1} x$

(C) $y = \cos^{-1}(x - 1)$

(D) $y = \cos^{-1}(1 - x)$

7. Which of the following is equal to $\frac{d}{dx}(\cos^{-1}\frac{x}{4})$? **1**

(A) $-\frac{1}{\sqrt{4-x^2}}$

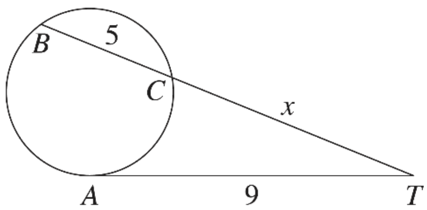
(B) $-\frac{1}{2\sqrt{4-x}}$

(C) $-\frac{4}{\sqrt{16-x^2}}$

(D) $-\frac{1}{\sqrt{16-x^2}}$

8. The line AT is the tangent to the circle at A and the line BT is a secant meeting the circle at B and C. $AT = 9$, $BC = 5$ and $CT = x$. **1**

Which of the following equations is correct?



(A) $x^2 + 5x - 81 = 0$

(B) $x^2 + 5x + 81 = 0$

(C) $x^2 - 5x - 81 = 0$

(D) $x^2 + 5x - 9 = 0$

9. If $y = \cos(\ln x)$, which is equal to $\frac{d^2y}{dx^2}$? **1**

(A) $\frac{\cos(\ln x)+\sin(\ln x)}{x^2}$

(B) $\frac{x^2\cos(\ln x)+\sin(\ln x)}{x^2}$

(C) $\frac{\sin(\ln x)-x^2\cos(\ln x)}{x^2}$

(D) $\frac{\sin(\ln x)-\cos(\ln x)}{x^2}$

10. Which inequality has the same solution as $|x+3| + |x-4| = 7$? **1**

(A) $|2x-1| \geq 7$

(B) $x^2 - x - 12 \leq 0$

(C) $\frac{7}{3-x} \geq 1$

(D) $\frac{1}{x-4} - \frac{1}{x+3} \leq 0$

Section II

60 marks
Attempt Questions 11 to 14
Allow approximately 1 hour and 50 minutes for this section.

Write your answers in the writing booklets supplied. Additional writing booklets are available.
Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) **Marks**

(a) Find all the asymptotes of $y = \frac{5x}{x^2+x-12}$. **2**

(b) Solve $\frac{1}{2x} \geq -\frac{1}{x-2}$. **4**

(c) The roots of $x^2 - 2x - 1 = 0$ are $\tan\alpha$ and $\tan\beta$. **3**
Given that α and β are acute, find the exact value of $\alpha + \beta$.

(d) Find the size of the acute angle between the curves $f(x) = (x+2)^2 + 4$ and $g(x) = x^2 - 4$ at their point of intersection. Give your answer to the nearest degree. **3**

(e) The region enclosed by the graph of $y = \frac{1}{\sqrt{1+9x^2}}$, the line $x = 0$, the line $x = \frac{1}{3}$ and the x-axis is rotated about the x-axis to form a solid of revolution. **3**

Find the exact volume of the solid formed.

Question 12 (15 marks) Use a SEPARATE Writing Booklet.

(a) A student tosses two regular dice with faces numbered 1 to 6. He records the maximum of the two uppermost faces as a score.

i. Find the probability that he records the score 1 in a single throw of the two dice. **1**

ii. Find the probability that he records the score 6 in a single throw of the two dice. **1**

(b) Consider the polynomial $P(x) = x^3 + bx^2 + cx - 18$. **3**

Given that $(x + 2)$ is a factor of $P(x)$ and -8 is the remainder when $P(x)$ is divided by $(x + 1)$, find the values of b and c.

(c) Evaluate, by using the substitution $u^2 = 1 + x$: **3**

$$\int_0^3 \frac{x(x+2)}{\sqrt{1+x}}\,dx$$

(d) i. Express $\cos x - \sqrt{3}\sin x$ in the form $R\cos(x + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. **2**

iii. Hence or otherwise, find the general solution of the equation $\cos x - \sqrt{3}\sin x = \sqrt{2}$. **2**

(e) The rate at which a body cools in air is assumed to be proportional to the difference between its temperature T and the constant room temperature P of the surrounding air.
This can be expressed as:

$$T = P + Ae^{kt}$$

where t is time in hours and k is a constant.

A cup of tea cools from 85°C to 25°C in 90 minutes. The air temperature P in the room is 20°C.

i. Show that $k = \frac{2}{3}\log_e \frac{1}{13}$. **2**

ii. Find the temperature of the cup of tea after a further 30 minutes. **1**
Give your answer correct to 2 decimal places.

Question 13 (15 marks) Use a SEPARATE Writing Booklet.

(a) By letting $t = \tan\theta$, prove that: **2**

$$\frac{\tan 2\theta + \cot\theta}{\tan 2\theta - \tan\theta} = \cot^2 x$$

(b) The velocity, v ms^{-1}, of a particle moving along a straight line is shown in the graph below.

Initially, the particle is at rest at the origin and returns to the origin at $t = 7$.

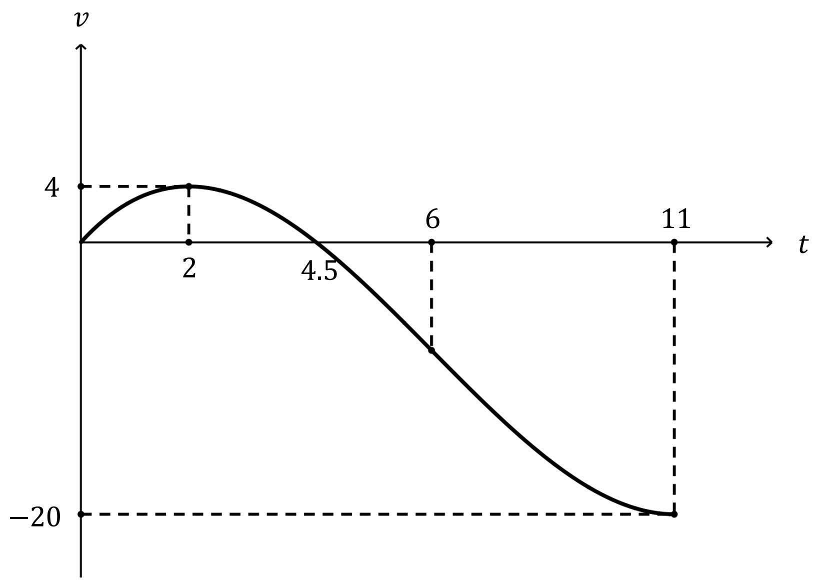


Diagram not drawn to scale.

i. When is the speed of the particle the greatest? **1**

ii. How many times does the particle change direction? **1**

iii. When is the acceleration of the particle the greatest after it starts moving? **1**

iv. Sketch the graph of the displacement function $x(t)$ for $0 \le t \le 7$, showing all important features. **2**

(c) Consider the function $f(x) = \sqrt{2x - 1} + 1$.

i. Find the equation of the inverse function $f^{-1}(x)$, stating the domain. **2**

ii. Find the x-coordinates of the points where the graphs $y = f(x)$ and $y = f^{-1}(x)$ intersect. **2**

(d) Liquid flows into a container made in the shape of a hollow, inverted right circular cone with radius 16 cm and height 24 cm, as shown in the diagram.

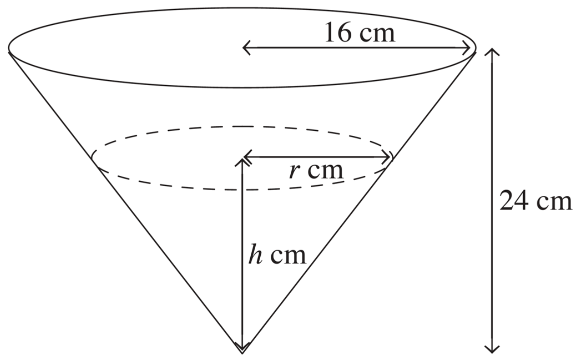


Diagram not drawn to scale.

At time t seconds, the height of the liquid is h cm, the surface of the liquid has radius r cm and the volume of the liquid is V cm^3.

i. Show that the $V = \frac{4\pi h^3}{27}$. **2**

ii. Liquid flows into the container at a rate of 8 cm^{3}s^{-1}. **2**

Show that $\frac{dh}{dt} = \frac{1}{2\pi}$ when $h = 6$.

Question 14 (15 marks) Use a SEPARATE Writing Booklet.

(a) Use mathematical induction to prove that, for every positive integer n: **3**

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{1}{4}n^2(n+1)^2$$

(b) Consider the following acute-angled triangle ABC.

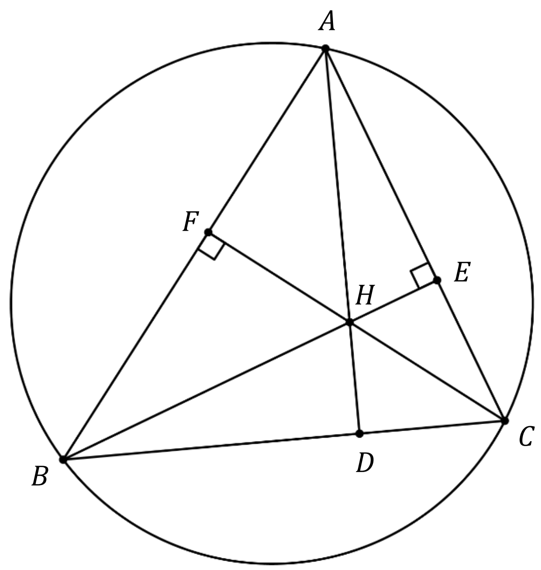


Diagram not drawn to scale.

Points E and F lie on AC and AB respectively, such that BE is perpendicular to AC and CF is perpendicular to AB. The line segments BE and CF meet at H.
The line segment AH produced meets BC at D.

Copy or trace the diagram into your writing booklet.

i. Give a reason for why $CEFB$ is a cyclic quadrilateral. **1**

ii. By considering cyclic quadrilaterals, show that $\angle CAD = \angle EFH = \angle EBC$, giving full reasons. **3**

iii. Hence, prove that AD is perpendicular to BC. **2**

(c) The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola with equation $x^2 = 4ay$, where $p \neq 0, q \neq 0$ and $p \neq q$. The parabola has a focus $S(0, a)$.

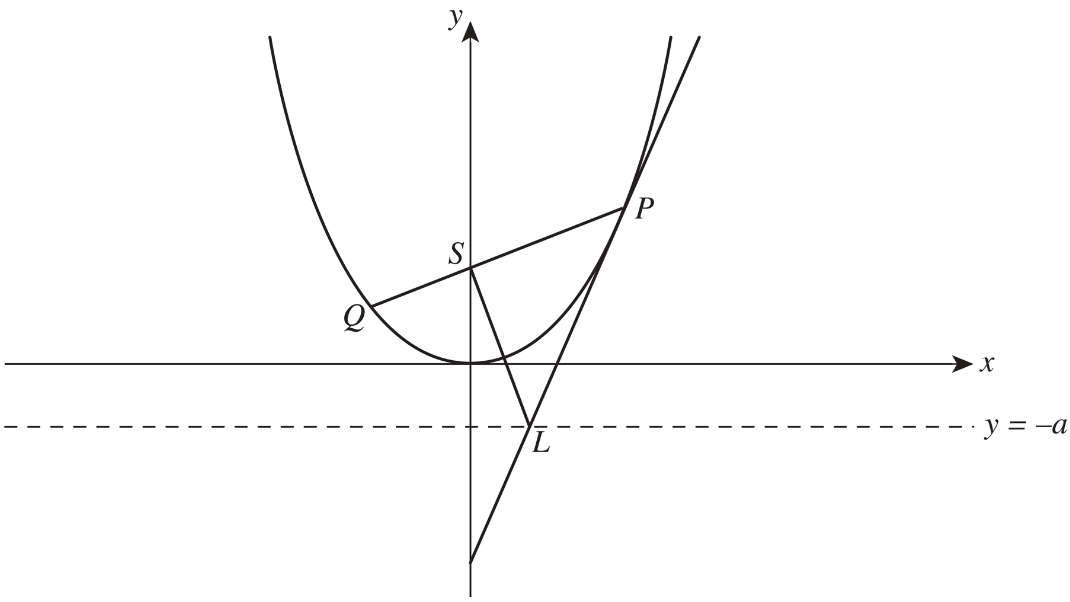


Diagram not drawn to scale.

Let L be the intersection of the tangent at P and the directrix, and PQ is a focal chord.

i. Show that $pq = -1$. **1**

ii. Explain why $PS = a(p^2 + 1)$. **1**

iii. Show that $PS \times QS = LS^2$. **4**

End of examination.

Q1) $\int \sin^2 2x \, dx$

$= \int \frac{1}{2}(1-\cos 4x)\, dx$

$= \frac{1}{2}\left(x - \frac{1}{4}\sin 4x\right) + c$

$= \frac{1}{8}(4x - \sin 4x) + c$ B

Q2) C

Q3) $f(x) = xe^x - 1$

$f'(x) = xe^x + e^x$

$= e^x(x+1)$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$= 0.5 - \frac{f(0.5)}{f'(0.5)}$

$= 0.5 - \frac{0.5e^{0.5} - 1}{e^{0.5}(0.5+1)}$

$= 0.5710\ldots$

$\doteqdot 0.57$ B

Q4) $\frac{\sin 4x}{2x\cos 2x} = \frac{\cancel{2}\sin 2x \cancel{\cos 2x}}{\cancel{2}x\cancel{\cos 2x}}$

$= \frac{\sin 2x}{x}$

$= 2 \cdot \frac{\sin 2x}{2x}$

$\therefore \lim_{x \to 0} \frac{\sin 4x}{2x\cos 2x} = 2 \lim_{x\to 0} \frac{\sin 2x}{2x}$

$= 2 \cdot 1$

$= 2$ D

Q5) $A(2,-5)$ $B(-4,1)$

$1:2$

$P = \left(\frac{1\times -4 + 2\times 2}{1+2}, \frac{1\times 1 + 2\times -5}{1+2}\right)$

$= (0,-3)$ C

Q6) The graph shows $y = \cos^{-1} x$
reflected in the y-axis
shifted right by 1.

D

Q7) $\frac{d}{dx} \cos^{-1} \frac{1}{4} x$

$$= -\frac{1}{\sqrt{1-\left(\frac{x}{4}\right)^2}} \cdot \frac{1}{4}$$

$$= -\frac{1}{4\sqrt{\frac{16-x^2}{16}}}$$

$$= -\frac{1}{\sqrt{16-x^2}}$$

D

Q8) $AT^2 = CT \times BT$

$9^2 = x \cdot (x+5)$

$0 = x^2 + 5x - 81$

A

Q9) $y = \cos(\ln x)$

$y' = -\frac{1}{x} \sin(\ln x)$

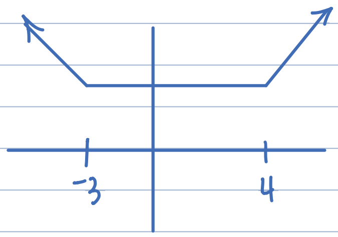
$= -x^{-1} \sin(\ln x)$

$y'' = 1x^{-2} \sin(\ln x) + -x^{-1} \cdot \frac{1}{x} \cos(\ln x)$

$$= \frac{\sin(\ln x) - \cos(\ln x)}{x^2}$$

D

Q10) $|x+3| + |x-4| = \begin{cases} -2x+1, & x < -3 \\ 7, & -3 \le x \le 4 \\ 2x-1, & x > 4 \end{cases}$



$x^2 - x - 12 \le 0$

$(x+3)(x-4) \le 0$

$-3 \le x \le 4$

B

Q11) a) $y = \dfrac{5x}{x^2+x-12}$

$= \dfrac{5x}{(x-3)(x+4)}$

asymptotes: $x = 3, -4$ ✓

$y = 0$ ✓

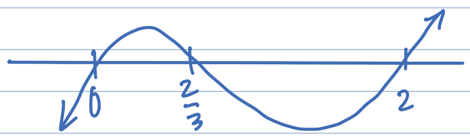
b) $\dfrac{1}{2x} \geq -\dfrac{1}{x-2}$, $x \neq 0, 2$

$x(x-2)^2 \geq -2x^2(x-2)$ ✓ multiply by $2x^2(x-2)^2$

$x(x-2)^2 + 2x^2(x-2) \geq 0$

$x(x-2)[x-2+2x] \geq 0$

$x(x-2)(3x-2) \geq 0$ ✓



$\therefore\ 0 < x \leq \frac{2}{3}$, $x > 2$ ✓

✓(excl. $x = 0, 2$)

c) $x^2 - 2x - 1 = 0$

$\tan\alpha + \tan\beta = \dfrac{-(-2)}{1}$

$= 2$

$\tan\alpha \cdot \tan\beta = \dfrac{-1}{1}$

$= -1$ ✓

$\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

$= \dfrac{2}{1-(-1)}$

$= 1$ ✓

$\alpha + \beta = \frac{\pi}{4}$ ✓

d) $f(x) = g(x)$

$$(x+2)^2 + 4 = x^2 - 4$$

$$\cancel{x^2} + 4x + 4 + 4 = \cancel{x^2} - 4$$

$$4x = -12$$

$$x = -3$$ ✓

$$f'(x) = 2(x+2)$$

$$m_1 = 2(-3+2)$$

$$= -2$$

$$g'(x) = 2x$$

$$m_2 = 2(-3)$$

$$= -6$$ ✓

$$\tan\theta = \left| \frac{-2 - -6}{1 + -2(-6)} \right|$$

$$= \frac{4}{13}$$

$$\theta = \tan^{-1}\left(\frac{4}{13}\right)$$

$$= 17.10\ldots$$

$$\doteqdot 17^\circ$$ ✓

e) $V = \pi \int y^2 \, dx$

$$= \pi \int_0^{\frac{1}{3}} \frac{1}{1+9x^2} \, dx$$ ✓

$$= \pi \int_0^{\frac{1}{3}} \frac{1}{1+(3x)^2} \, dx$$

$$= \pi \left[\frac{1}{3} \tan^{-1}(3x) \right]_0^{\frac{1}{3}}$$ ✓

$$= \frac{\pi}{3} \left(\tan^{-1}\left(3 \cdot \frac{1}{3}\right) - \tan^{-1}(0) \right)$$

$$= \frac{\pi}{3} \left(\frac{\pi}{4} - 0 \right)$$

$$= \frac{\pi^2}{12} \text{ units}^3$$ ✓

Q(2) a)

	1	2	3	4	5	6
1	·	·	·	·	·	·
2	·	·	·	·	·	·
3	·	·	·	·	·	·
4	·	·	·	·	·	·
5	·	·	·	·	·	·
6	·	·	·	·	·	·

i) $P(1,1) = \frac{1}{36}$ ✓

ii) $P(6,6) + P(6,\text{other}) + P(\text{other},6) = \frac{11}{36}$ ✓

b) $P(x) = x^3 + bx^2 + cx - 18$

$$P(-2) = (-2)^3 + b(-2)^2 + c(-2) - 18 = 0$$
$$-8 + 4b - 2c - 18 = 0$$
$$2b - c = 13 \quad \text{— ①}$$
✓

$$P(-1) = (-1)^3 + b(-1)^2 + c(-1) - 18 = -8$$
$$-1 + b - c - 18 = -8$$
$$b - c = 11 \quad \text{— ②}$$
✓

① − ②: $b = 2$

$\hookrightarrow 2 - c = 11$

$c = -9$
✓

c) $u^2 = 1+x$

$$\left.\begin{aligned} x &= u^2-1 \\ dx &= 2u\,du \\ x=3 &\rightarrow u=2 \\ x=0 &\rightarrow u=1 \end{aligned}\right\} \checkmark$$

$$\int_0^3 \frac{x(x+2)}{\sqrt{1+x}}\,dx = \int_1^2 \frac{(u^2-1)(u^2+1)}{\cancel{\sqrt{u^2}}}\,2\cancel{u}\,du \quad \checkmark$$

$$= 2\int_1^2 u^4-1\,du$$

$$= 2\left[\frac{u^5}{5} - u\right]_1^2$$

$$= 2\left(\frac{2^5}{5} - 2 - \left(\frac{1^5}{5} - 1\right)\right)$$

$$= 2\cdot\frac{26}{5}$$

$$= \frac{52}{5} \quad \checkmark$$

d) i) $\cos x - \sqrt{3}\sin x \equiv R\cos(x+\alpha)$

$= R\cos x\cos\alpha - R\sin x\sin\alpha$

$\left.\begin{array}{l} 1 = R\cos\alpha \\ \sqrt{3} = R\sin\alpha \end{array}\right\}$ $\tan\alpha = \sqrt{3}$

$\alpha = \frac{\pi}{3}$ ✓

$R^2 = 1+3$

$R = 2$ ✓

$\therefore \cos x - \sqrt{3}\sin x = 2\cos(x+\frac{\pi}{3})$

ii) $\cos x - \sqrt{3}\sin x = \sqrt{2}$

$2\cos(x+\frac{\pi}{3}) = \sqrt{2}$

$\cos(x+\frac{\pi}{3}) = \frac{1}{\sqrt{2}}$ ✓

$x+\frac{\pi}{3} = 2\pi n \pm \frac{\pi}{4}$, $n \in \mathbb{Z}$

$x = 2\pi n \pm \frac{\pi}{4} - \frac{\pi}{3}$ ✓

e) i) $T = P + Ae^{kt}$

$t=0 \rightarrow 85 = 20 + Ae^{0}$

$A = 65$ ✓

$\left.\begin{array}{l} t=1.5 \rightarrow 25 = 20+65e^{k(1.5)} \\ \frac{1}{13} = e^{1.5k} \\ \log_e \frac{1}{13} = 1.5k \\ k = \frac{2}{3}\log_e \frac{1}{13} \end{array}\right\}$ ✓

ii) $t=2 \rightarrow T = 20+65e^{k(2)}$

$= 22.1264\ldots$

$\doteqdot 22.13°$ ✓

Q(3) a) $LHS = \dfrac{\tan 2\theta + \cot\theta}{\tan 2\theta - \tan\theta}$

$= \left(\dfrac{2t}{1-t^2} + \dfrac{1}{t}\right) \div \left(\dfrac{2t}{1-t^2} - t\right)$, $\tan\theta = t$

$= \dfrac{2t^2 + 1 - t^2}{t(1-t^2)} \div \dfrac{2t - t(1-t^2)}{1-t^2}$ ✓

$= \dfrac{t^2+1}{t\cancel{(1-t^2)}} \times \dfrac{\cancel{1-t^2}}{t+t^3}$

$= \dfrac{\cancel{t^2+1}}{t} \times \dfrac{1}{t\cancel{(1+t^2)}}$ ✓

$= \dfrac{1}{t^2}$

$= \cot^2\theta$

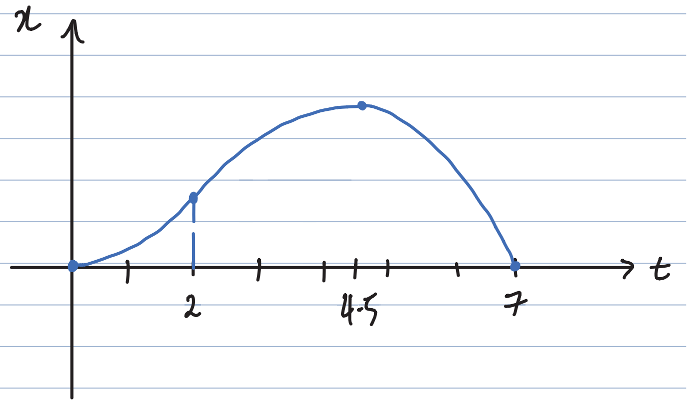
$= RHS$

b) i) $t = 11$ (20m/s)

ii) once (at $t = 4.5$)

iii) $t = 6$

iv)



✓ shape

✓ turning pts $x = 0, 4.5$
and x-int $x = 0, 7$
pt of inflexion $x = 2$

c) i) $y = \sqrt{2x-1} + 1$

$x = \sqrt{2y-1} + 1$

$(x-1)^2 = 2y - 1$ ✓

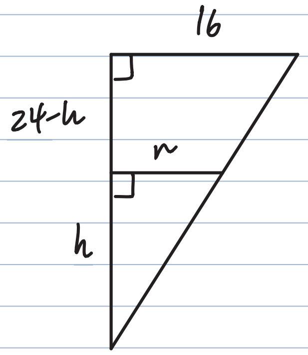
$y = \frac{1}{2}\left((x-1)^2 + 1\right)$, $x \geq 1$ ✓

ii) $\sqrt{2x-1}+1=x$

$$2x-1=(x-1)^2$$
$$=x^2-2x+1$$
$$0=x^2-4x+2$$ ✓

$$x=\frac{4\pm\sqrt{4^2-4(1)(2)}}{2(1)}$$
$$=\frac{4\pm\sqrt{8}}{2}$$
$$=2\pm\sqrt{2}$$
$$=2+\sqrt{2}\quad(x>1)$$ ✓

d) i)



$$\frac{r}{h}=\frac{16}{24}$$
$$r=\frac{2h}{3}$$ ✓

$$V=\frac{1}{3}\pi r^2h$$
$$=\frac{1}{3}\pi\left(\frac{2h}{3}\right)^2\cdot h$$
$$=\frac{4\pi}{27}h^3$$ ✓

ii) $\frac{dV}{dt}=8$

$$\frac{dV}{dh}=\frac{4\pi}{9}h^2$$ ✓

$$\frac{dV}{dt}=\frac{dV}{dh}\cdot\frac{dh}{dt}$$
$$8=\frac{4\pi}{9}h^2\cdot\frac{dh}{dt}$$
$$\frac{dh}{dt}=\frac{18}{\pi h^2}$$
$$h=6\rightarrow\frac{dh}{dt}=\frac{18}{36\pi}$$
$$=\frac{1}{2\pi}$$ ✓

Q(4) a) $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{1}{4} n^2 (n+1)^2$

prove true for $n=1$.

$LHS = 1^3$

$RHS = \frac{1}{4}(1)^2(1+1)^2$

$= 1$

$= LHS$

$\therefore$ The result is true for $n=1$ ✓

Assume true for $n=k$.

ie. $1^3 + 2^3 + 3^3 + \cdots + k^3 = \frac{1}{4} k^2 (k+1)^2$

prove true for $n=k+1$.

ie. $1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 = \frac{1}{4}(k+1)^2(k+2)^2$

$LHS = 1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3$

$= \frac{1}{4} k^2 (k+1)^2 + (k+1)^3$

$= \frac{1}{4}(k+1)^2 \left[k^2 + 4(k+1) \right]$

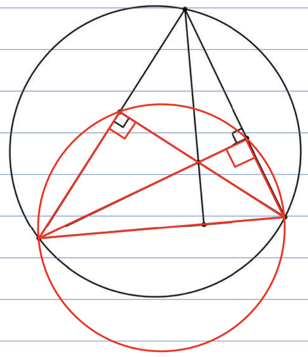
$= \frac{1}{4}(k+1)^2 (k+2)^2$

$= RHS$

$\therefore$ The result is true for $n=k+1$ ✓

Hence the statement is true for all positive integers n by mathematical induction. ✓

b) i)



$\angle CEB = 180° - \angle AEH$ ($\angle$s on a straight line)

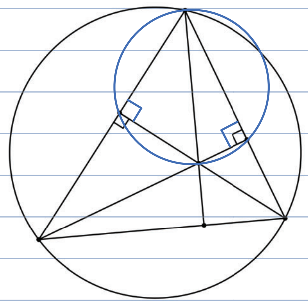
$= 90°$

$= \angle CFB$ (given)

$\therefore CEFB$ is a cyclic quad.

(interval BC subtends equal $\angle$s on the same side so $CEFB$ are concyclic)

ii)



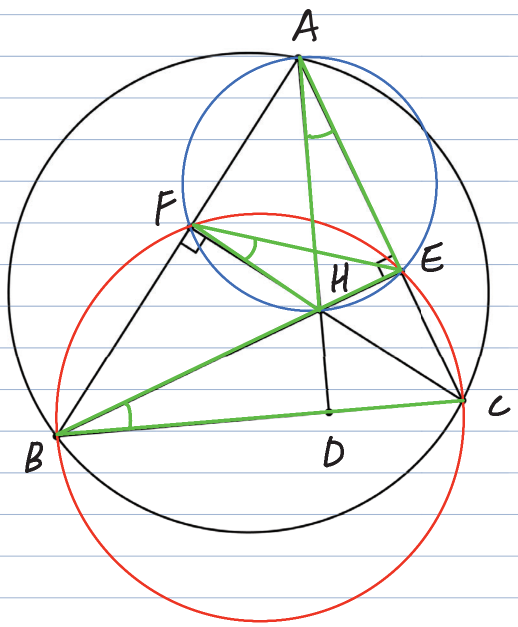
$\angle AFH = 180° - \angle BFC$ ($\angle$s on a straight line)

$= 90°$

$= \angle AEH$ (given)

$\therefore EAFH$ is a cyclic quadrilateral

(opposite $\angle$s of a cyclic quad. are supplementary)



$\angle CAD = \angle EFH$ ($\angle$s in the same segment)

and

$\angle EFH = \angle EBC$ (similarly)

$\therefore \angle CAD = \angle EFH = \angle EBC$

iii) $\angle ADC = 180° - \angle CAD - \angle DCA$ ($\angle$ sum of ΔADC)

$= 180° - \angle CAD - (90° - \angle EBC)$

$= 90° - \angle CAD + \angle EBC$ ($\angle CAD = \angle EBC$ proved in part ii)

$= 90°$

$\therefore AD \perp BC$

c) i) $m_{PQ} = \dfrac{ap^2 - aq^2}{2ap - 2aq}$

$= \dfrac{(p+q)(p-q)}{2(p-q)}$

$= \dfrac{p+q}{2}$

chord PQ : $y - \cancel{ap^2} = \dfrac{p+q}{2}(x - 2ap)$

$= \dfrac{p+q}{2}x - ap(\cancel{p}+q)$

$y = \dfrac{p+q}{2}x - apq$

$(0,a) \rightarrow$ chord PQ : $a = 0 - apq$

$pq = -1$

Alternatively,

$m_{PS} = \dfrac{ap^2 - a}{2ap - 0}$ $\quad$ $m_{SQ} = \dfrac{aq^2 - a}{2aq - 0}$

$= \dfrac{p^2 - 1}{2p}$ $\quad$ $= \dfrac{q^2 - 1}{2q}$

$m_{PS} = m_{SQ}$

$\dfrac{p^2 - 1}{2p} = \dfrac{q^2 - 1}{2q}$

$q(p^2 - 1) = p(q^2 - 1)$

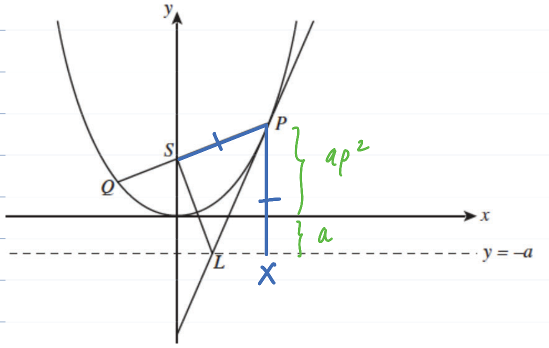
$p - q = pq^2 - qp^2$

$= pq(q - p)$

$pq = -1$

or some other valid method

ii)



$PS = PX$ (locus definition of parabola)

$= ap^2 + a$

$= a(p^2+1)$

iii) tangent p : $y = px - ap^2$

directrix : $y = -a$

$-a = px - ap^2$

$px = ap^2 - a$

$x = ap - \frac{a}{p}$

$\therefore L\left(ap - \frac{a}{p}, -a\right)$ ✓

$LS^2 = (2a)^2 + \left(ap - \frac{a}{p}\right)^2$

$= a^2\left(4 + \left(p - \frac{1}{p}\right)^2\right)$ ✓

$= a^2\left(4 + p^2 - 2 + \frac{1}{p^2}\right)$

$= a^2\left(p + \frac{1}{p}\right)^2$

$PS \times QS = a(p^2+1) \cdot a(q^2+1)$ ✓

$= a^2(p^2+1)(q^2+1)$

$= a^2(p^2+1)\left(\left(-\frac{1}{p}\right)^2 + 1\right)$

$= a^2(p^2+1)\left(\frac{1}{p^2} + 1\right)$

$= a^2\left(1 + p^2 + \frac{1}{p^2} + 1\right)$

$= a^2\left(p + \frac{1}{p}\right)^2$ ✓

$= LS^2$