Student Number:................................ Teacher's Name:................................

North Sydney Boys High School

YEAR 12 Trial Higher School Certificate Examination

2002

Mathematics Extension 1

Time allowed – 2 hours (plus 5 minutes reading time)

General Instructions

- Attempt all questions on the writing paper supplied
- Write on one side of the paper only
- Start each question on a new page
- Write using black or blue pen
- Board approved calculators may be used
- A table of standard integrals is supplied
- All necessary working should be shown in every question

Total marks – **84**

- Attempt Questions 1–7
- All questions are of equal value

Question	Mark
1	
2	
3	
4	
5	
6	
7	
TOTAL	

Marks

QUESTION 1

(a) Differentiate: (i) $\sin^2 x$ **2**

(ii) $\sin^{-1}(2x)$ **2**

(b) Find the coordinates of the point P which divides the interval AB internally in the ratio 2 : 3 where A and B have coordinates (1, –3) and (6, 7) respectively. **2**

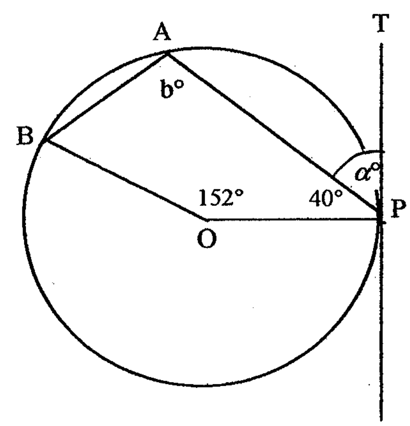
(c) Solve the inequality $\dfrac{2x+3}{x-4} > 1$ **2**

(d) $\int x\sqrt{x+1}\,dx$, using the substitution $u = 1 + x$ **3**

(e) Find $\lim\limits_{x \to 0} \dfrac{2\sin\frac{x}{2}}{x}$ **1**

QUESTION 2

(a) PT is a tangent to the circle centre 0. Find the sizes of the angles marked a and b giving reasons for your answers. **4**



(b) (i) Write down the expansion of $\tan(\alpha + \beta)$. **3**

(ii) Hence find the exact value of $\tan 75°$.

(c) Consider the function $f(x) = 3\cos^{-1}\left(\dfrac{x}{2}\right)$

(i) Evaluate $f(2)$. **1**

(ii) State the domain and range of $y = f(x)$. **2**

(iii) Draw the graph of $y = f(x)$. **2**

Marks

QUESTION 3

(a) Write $9 + 16 + 25 + \dots\dots\dots + n^2$ using $\sum$ notation **1**

(b) Solve $\sin 2x = \cos x$ for $0 \le x \le 2\pi$ **4**

(c) Find the indefinite integrals: **3**

(i) $\int \frac{dx}{x^2 + 4}$

(ii) $\int \sin^2 2x \, dx$

(d) Evaluate $\int_0^{\ln 3} \frac{e^x dx}{\sqrt{1+e^x}}$ using the substitution $u = e^x$. **4**

QUESTION 4

(a) The polynomial $P(x) = x^3 + ax^2 - 3ax$ has a factor $(x + 2)$. Find the value of a. **2**

(b) Express $\sqrt{3}\cos\theta + \sin\theta$ in the form $A\cos(\theta - \alpha)$. Hence solve the equation $\sqrt{3}\cos\theta + \sin\theta = 1$ for $-\pi \le \theta \le \pi$. **4**

(c) Differentiate $x\tan^{-1}x$ and hence evaluate $\int_0^1 \tan^{-1}x \, dx$. **4**

(d) Sketch $y = \sin(\cos^{-1}x)$ showing clearly the domain and range. **2**

Marks

QUESTION 5

(a) (i) Draw the graph of $y = e^{-x}$. By drawing another graph on the same set of axes, show that $f(x) = e^{-x} - x + 1$ has exactly one root. **2**

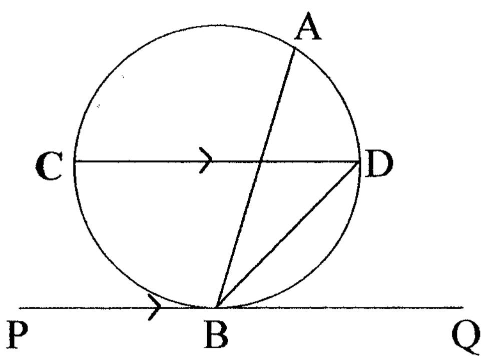
(ii) Let $x = 1$ be a first approximation to the root. Apply Newton's method once to obtain another approximation. Answer to 3 significant figures. **3**

(b) Prove that $\dfrac{cosec\beta - cot\beta}{cosec\beta + cot\beta} = tan^2\dfrac{\beta}{2}$. Hint: Let $tan\dfrac{\beta}{2} = t$ **3**

(c) AB and CD are two intersecting chords of a circle and CD is parallel to the tangent to the circle at B. **4**

Copy the diagram in your booklet.

Prove that AB bisects $\angle CAD$.



QUESTION 6

(a) $P(4p, 2p^2)$ is any point on the parabola $x^2 = 8y$. The tangent to the parabola meets the x-axis at M, and the y-axis at N.

(i) Show that the tangent at P is given by $y = px - 2p^2$. **2**

(ii) Find the co-ordinates of M and N. **2**

(iii) Find the equation of the locus of the midpoint of MN as P varies. **2**

(b) A spherical balloon leaks air such that the radius decreases at the rate of 5 mm/sec. Calculate the rate of change of the volume of the balloon when the radius is 100mm. **3**

(c) Evaluate $\sin^{-1}\left(\dfrac{1}{\sqrt{5}}\right) + \sin^{-1}\left(\dfrac{1}{\sqrt{10}}\right)$. **3**

Marks

QUESTION 7

(a) (i) Given that $S_n = 1\times 2 + 2\times 3 + 3\times 4 + \ldots\ldots + n(n+1)$,
show by mathematical induction,

$$S_n = \frac{n}{3}(n+1)(n+2), \text{ for all positive integers n.}$$ **4**

(ii) Evaluate $\lim\limits_{n\to\infty} \dfrac{1\times 2 + 2\times 3 + 3\times 4 + \ldots\ldots + n(n+1)}{n^3}$ **1**

(b) $P(x) = x^3 - 6x^2 + ax - 4$ where $a > 0$. Given that all the roots of $P(x) = 0$ are real and positive, and that one of the roots is the product of the other 2 roots, find the value of a. **3**

(c) Given $f(x) = 2\cos^{-1}\left(\frac{x}{\sqrt{2}}\right) - \sin^{-1}(1 - x^2)$. Show that $f'(x) = 0$. **4**

Hence sketch $f(x)$.

End of paper

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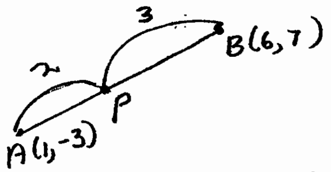
QUESTION 1

a) i) $\frac{d}{dx}(\sin^2 x) = 2\sin x \cos x$ (2)

ii) $\frac{d}{dx} \sin^{-1} 2x = \frac{2}{\sqrt{1-4x^2}}$ (2)

b)



$P\left(\frac{3\times 1 + 2\times 6}{2+3}, \frac{3\times -3 + 2\times 7}{2+3}\right)$ (2)

$P(3, 1)$

c) $\frac{2x+3}{x-4} > 1$

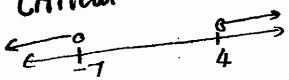
Let $\frac{2x+3}{x-4} = 1$

$2x+3 = x-4$

$x = -7$

asymptote $x = 4$

Critical values $x = -7, 4$



(2)

$x < -7$ OR $x > 4$

d) $\int x\sqrt{x+1}\, dx$ Let $u = 1+x$, $du = dx$

$= \int (u-1)\sqrt{u}\, du$

$= \int u^{3/2} - u^{1/2}\, du$

$= \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + c$ (3)

$= \frac{2}{5}(1+x)^{5/2} - \frac{2}{3}(1+x)^{3/2} + c$

e) $\lim \frac{2\sin\frac{x}{2}}{x} = \lim_{x\to 0} \frac{\sin\frac{x}{2}}{\frac{x}{2}}$ (1)

QUESTION 2

a) $40° + a° = 90°$ (radius ⊥ tangent)

$a° = 50°$

Reflex ∠ at O = 208°

∴ $b° = 104°$ (∠ at centre = 2×∠ at circumference) (4)

b) i) $\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

ii) $\tan(45°+30°) = \frac{\tan 45° + \tan 30°}{1 - \tan 45° \tan 30°}$

$= \frac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}}$ (3)

$= \frac{\sqrt{3}+1}{\sqrt{3}-1}$

c) $f(x) = 3\cos^{-1}\frac{x}{2}$

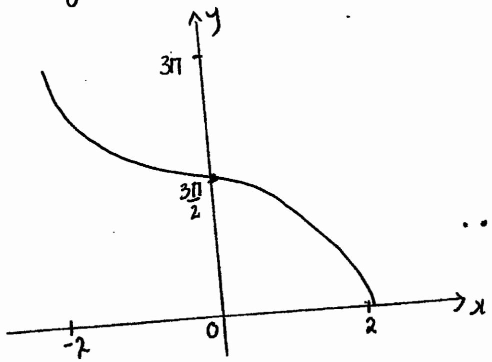
i) $f(2) = 3\cos^{-1} 1$ (1)

$= 0$

ii) Domain $-2 \le x \le 2$ (2)

Range $0 \le y \le 3\pi$

iii)



(2)

QUESTION 3.

a) $9 + 16 + 25 + \cdots + n^2 = \sum_{k=3}^{n} k^2$ (1)

b) $\sin 2x = \cos x$

$2\sin x \cos x = \cos x$

$\cos x(2\sin x - 1) = 0$

$\cos x = 0$ $\quad$ $\sin x = \frac{1}{2}$

$x = \frac{\pi}{2}, \frac{3\pi}{2}$ $\quad$ $x = \frac{\pi}{6}, \frac{5\pi}{6}$

c) i) $\int \frac{dx}{x^2+4} = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$ (1)

ii) $\int \sin^2 2x\, dx = \frac{1}{2}\int (1 - \cos 4x)\, dx$

$= \frac{1}{2}\left[x - \frac{\sin 4x}{4}\right] + c$

$= \frac{1}{2}x - \frac{1}{8}\sin 4x + c$ (2)

d) $\int_0^{\ln 3} \frac{e^x\, dx}{\sqrt{1+e^x}} = \int_1^3 \frac{du}{\sqrt{1+u}}$

$u = e^x$

$du = e^x dx$

When $x = 0$, $u = 1$

$x = \ln 3$, $u = 3$

$= \int_1^3 (1+u)^{-1/2}\, dx$

$= 2\left[(1+u)^{1/2}\right]_1^3$

$= 2\left[2 - \sqrt{2}\right]$

$= 4 - 2\sqrt{2}$ (4)

QUESTION 4

a) $P(x) = x^3 + ax^2 - 3ax$

If $x+2$ is factor $P(-2) = 0$

$\therefore -8 + 4a + 6a = 0$

$10a = 8$

$a = \frac{4}{5}$ (2)

b) $\sqrt{3}\cos\theta + \sin\theta = A\cos(\theta - \alpha)$

$A\cos(\theta - \alpha) = A\cos\theta\cos\alpha + A\sin\theta\sin\alpha$

$\therefore A\cos\alpha = \sqrt{3}$

$A\sin\alpha = 1$

$\therefore \tan\alpha = \frac{1}{\sqrt{3}}$

$\alpha = \frac{\pi}{6}$

$A^2 = 1 + 3$

$A = 2$

$\therefore \sqrt{3}\cos\theta + \sin\alpha = 2\cos\left(\theta - \frac{\pi}{6}\right)$

$2\cos\left(\theta - \frac{\pi}{6}\right) = 1$

$\cos\left(\theta - \frac{\pi}{6}\right) = \frac{1}{2}$

$\theta - \frac{\pi}{6} = -\frac{\pi}{3}, \frac{\pi}{3}$

$\theta = -\frac{\pi}{6}, \frac{\pi}{2}$ (4)

c) $y = x\tan^{-1}x$

$\frac{dy}{dx} = 1.\tan^{-1}x + x.\frac{1}{1+x^2}$

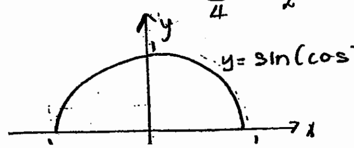
$\therefore \left[x\tan^{-1}x\right]_0^1 = \int_0^1 \tan^{-1}x\, dx + \int_0^1 \frac{x}{1+x^2}\, dx$

$\therefore \int_0^1 \tan^{-1}x\, dx = \left[x\tan^{-1}x\right]_0^1 - \int_0^1 \frac{x}{1+x^2}\, dx$

$= \left[x\tan^{-1}x - \frac{1}{2}\ln(1+x^2)\right]_0^1$

$= \frac{\pi}{4} - \frac{1}{2}\ln 2$ (4)

d)

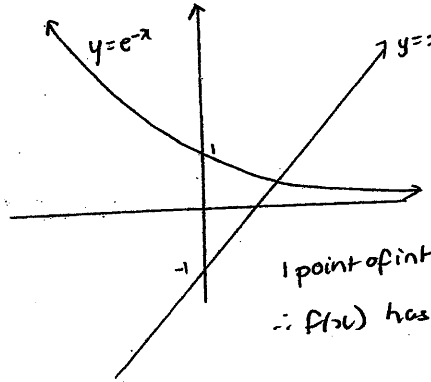


(2)

QUESTION 5

a) i) $f(x) = e^{-x} - x + 1 = 0$

$e^{-x} = x - 1$



(2)

1 point of intersection

$\therefore f(x)$ has one root

ii) $f(x) = e^{-x} - x + 1$, $f(1) = e^{-1} \doteqdot 0.3679$

$f'(x) = -e^{-x} - 1$, $f'(1) = -e^{-1} - 1 \doteqdot -1.3679$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$= 1 - \frac{e^{-1}}{-e^{-1} - 1}$

$\doteqdot 1.27$ (3)

) $\tan\frac{\beta}{2} = t$

L.H.S $= \frac{\operatorname{cosec}\beta - \cot\beta}{\operatorname{cosec}\beta + \cot\beta}$

$= \frac{\frac{1+t^2}{2t} - \frac{1-t^2}{2t}}{\frac{1+t^2}{2t} + \frac{1-t^2}{2t}}$

$= \frac{1 + t^2 - 1 + t^2}{1 + t^2 + 1 - t^2}$

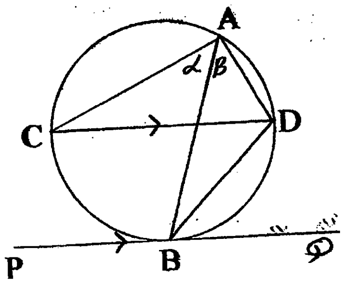
$= \frac{2t^2}{2}$

$= t^2$

$= \tan^2\frac{\beta}{2}$

= R.H.S (3)

c)



Let $\angle CAB = \alpha$, $\angle BAD = \beta$

$\beta = \angle DBQ$ (Alternate segment Th)

$\angle DBQ = \angle CDB$ (alternate ∠s, // lines)

$\angle CDB = \alpha$ (angles in same segment)

$\therefore \beta = \alpha$

$\therefore$ AB bisects $\angle CAD$ (4)

QUESTION 6

i) $x^2 = 8y$

$y = \frac{x^2}{8}$

$\frac{dy}{dx} = \frac{x}{4}$

at $P(4p, 2p^2)$, $m = p$

Eqn of tangent. $y - y_1 = m(x - x_1)$

$y - 2p^2 = p(x - 4p)$ (2)

$\therefore y = px - 2p^2$

) x-axis intercept : $y = 0$

$\therefore px = 2p^2$

$x = 2p$

$\therefore M(2p, 0)$

y-axis intercept $x = 0$

$\therefore y = -2p^2$ (1)

$N(0, -2p^2)$

iii) Midpoint of $MN = \left(\frac{2p+0}{2}, \frac{0-2p^2}{2}\right)$

i.e $(p, -p^2)$

$x = p$

$y = -p^2$

$\therefore y = -x^2$ is locus. (2)

) $V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2$

$\frac{dr}{dt} = -5$ mm/sec

$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$

$= 4\pi r^2 \times -5$

$= -20\pi r^2$

when $r = 100$ mm $= 10$ cm

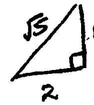
$\frac{dV}{dr} = -200000\pi$ mm³/sec

or decreasing at a rate of $200\,000\pi$ mm³/sec (3)

c) Let $\sin^{-1}\left(\frac{1}{\sqrt{5}}\right) + \sin^{-1}\left(\frac{1}{\sqrt{10}}\right) = x$

Let $\sin^{-1}\left(\frac{1}{\sqrt{5}}\right) = \alpha$

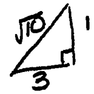
$\sin\alpha = \frac{1}{\sqrt{5}}$



$\therefore \cos\alpha = \frac{2}{\sqrt{5}}$

Let $\sin^{-1}\left(\frac{1}{\sqrt{10}}\right) = \beta$

$\sin\beta = \frac{1}{\sqrt{10}}$



$\cos\beta = \frac{3}{\sqrt{10}}$

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$= \frac{1}{\sqrt{5}}\frac{3}{\sqrt{10}} + \frac{2}{\sqrt{5}}\frac{1}{\sqrt{10}}$

$= \frac{5}{\sqrt{50}}$

$= \frac{1}{\sqrt{2}}$

$\therefore \alpha + \beta = \frac{\pi}{4}$

$\sin^{-1}\left(\frac{1}{\sqrt{5}}\right) + \sin^{-1}\left(\frac{1}{\sqrt{10}}\right) = \frac{\pi}{4}$ (3

16

QUESTION 7

a) i) $S_n = 1\times2 + 2\times3 + \cdots + n(n+1) = \frac{n}{3}(n+1)(n+2)$

① If $n=1$

$L.H.S = 1\times 2 = 2$

$R.H.S = \frac{1}{3}(1+1)(1+2) = 2$

$\therefore$ True for $n=1$

② Assume true for $n=k$

i.e. $1\times2 + 2\times3 + \cdots + k(k+1) = \frac{k}{3}(k+1)(k+2)$

③ If $n=k+1$

$S_{k+1} = 1\times2 + 2\times3 + \cdots + k(k+1) + (k+1)(k+2)$

$= \frac{k}{3}(k+1)(k+2) + (k+1)(k+2)$

$= \frac{(k+1)(k+2)}{3}(k+3)$

$\therefore$ True for $n=k+1$

④ Since result is true for $n=1$, it is true for next integer, $n=2$ (i.e. $n+1$) and so it is true for $n=3$ and so on. $\therefore$ True for all integers n.

(4)

ii) $\lim_{n\to\infty} \frac{1\times2 + 2\times3 + \cdots + n(n+1)}{n^3} = \lim_{n\to\infty} \frac{\frac{n}{3}(n+1)(n+2)}{n^3}$

$= \lim_{n\to\infty} \frac{\frac{1}{3}(1+\frac{1}{n})(1+\frac{2}{n})}{1}$

$= \frac{1}{3}$

(1)

b) $P(x) = x^3 - 6x^2 + ax - 4$

Let roots be $\alpha, \beta, \alpha\beta$

$\alpha + \beta + \alpha\beta = 6$ — (1)

$\alpha\beta + \alpha^2\beta + \alpha\beta^2 = a$ — (2)

$\alpha^2\beta^2 = 4$ — (3)

$\alpha\beta = \pm 2$

$\alpha\beta = 2$ (since roots are +ve).

$\therefore$ (1): $\alpha + \beta + 2 = 6$

$\alpha + \beta = 4$

(2): $\alpha\beta(1 + \alpha + \beta) = a$

$\therefore 2(1+4) = a$

$\therefore a = 10$

(3)

7 c) $f(x) = 2\cos^{-1}\left(\frac{x}{\sqrt{2}}\right) - \sin^{-1}(1-x^2)$

$f'(x) = -\frac{2}{\sqrt{2}} \frac{1}{\sqrt{1-\frac{x^2}{2}}} - \frac{-2x}{\sqrt{1-(1-x^2)^2}}$

$= \frac{-2}{\sqrt{2-x^2}} + \frac{2x}{\sqrt{1-(1-2x^2+x^4)}}$

$= \frac{-2}{\sqrt{2-x^2}} + \frac{2x}{\sqrt{2x^2-x^4}}$

$= \frac{-2}{\sqrt{2-x^2}} + \frac{2}{\sqrt{2-x^2}}$ if $x > 0$, $\sqrt{x^2} = |x| = x$

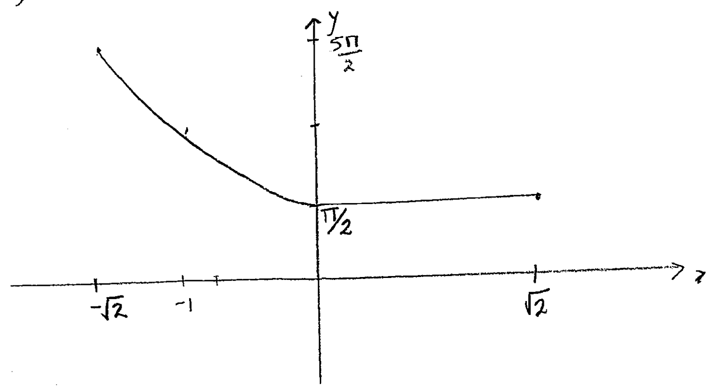
$= 0$ if $x > 0$ $\left(f'(x) = \frac{-4}{\sqrt{2-x^2}} \text{ if } x < 0\right)$

If $f'(x) = 0$, $f(x)$ is constant

Domain: $-1 \le \frac{x}{\sqrt{2}} \le 1$ & $-1 \le 1-x^2 \le 1$

$-\sqrt{2} \le x \le \sqrt{2}$ $\quad -\sqrt{2} \le x \le \sqrt{2}$

$f(0) = 2\times\cos(0) - \sin^{-1}(1) = \frac{\pi}{2}$



- $f'(x) = 0$
- GRAPH
  - CORRECT DOMAIN
  - $f(x)$ = constant $(x > 0)$
  - $f(x) = \frac{\pi}{2}$

(4)