NSBH EXN 1 2003

QUESTION 1

(a) Solve the inequality $\frac{1}{x} < \frac{1}{x+1}$. **3**

(b) (i) When a polynomial is divided by a quadratic, what is the most general form of the remainder? **1**

(ii) The remainder when $P(x)$ is divided by $(x-2)$ is 4. The remainder when $P(x)$ is divided by $(x-3)$ is 9. Find the remainder when $P(x)$ is divided by $(x-2)(x-3)$. **2**

(c) Use the substitution $u = 2 - x$, to evaluate $\int_{-1}^{2} x\sqrt{2-x}\,dx$. **4**

(d) Find $\int \sin^2 x\,dx$. **2**

QUESTION 2

(a) Find $\lim_{x\to 0} \frac{\sin\frac{x}{3}}{x}$ **1**

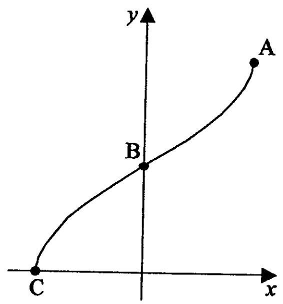
(b) If $f(x) = e^{x+2}$,

(i) Find the inverse function $f^{-1}(x)$. **2**

(ii) State the domain and range of $f^{-1}(x)$. **2**

(iii) On one diagram sketch the graphs of $f(x)$ and $f^{-1}(x)$. **2**

(c) The diagram below shows the graph of $y = \pi + 2\sin^{-1}3x$.



(i) Write down the co-ordinates of the endpoints A and C. **2**

(ii) Write down the co-ordinates of the point B. **1**

(iii) Find the equation of the tangent to the curve $y = \pi + 2\sin^{-1}3x$ at the point B. **2**

QUESTION 3

(a) The equation $x^3 - 2x^2 + 4x - 5 = 0$ has roots α, β, γ.

Find the values of: (i) $\alpha\beta\gamma$ **1**

(ii) $\alpha\beta + \beta\gamma + \gamma\alpha$ **1**

(iii) $\alpha^{-1} + \beta^{-1} + \gamma^{-1}$. **2**

(b) By using the expansion of $\tan(\alpha+\beta)$ find the value of k such that **3**

$$\tan^{-1}(k) + \tan^{-1}\left(\frac{2}{3}\right) = \frac{\pi}{4}$$

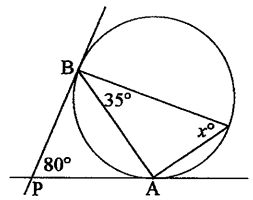
(c) (i) Express $\sqrt{3}\cos\theta - \sin\theta$ in the form $A\cos(\theta + \alpha)$ **2**

(ii) Solve the equation $\sqrt{3}\cos\theta - \sin\theta = 1$ for $0 \le x \le 2\pi$ **2**

(iii) What is the general solution of the equation? **1**

QUESTION 4

(a) **3**



NOT TO SCALE

PA and PB are tangents to the circle. Find the value of x giving reasons for your answer.

(b) A man standing 80 metres from the base of a high–rise building observes an external lift moving up outside the building at a constant rate of 7 metres per second.

(i) If θ radians is the angle of elevation of the lift from the observer, find an expression for $\frac{d\theta}{dt}$ in terms of θ. **3**

(ii) Evaluate $\frac{d\theta}{dt}$ at the instant when the lift is 30 metres above the observer's horizontal line of vision. Give your answer to 2 significant figures. **1**

(c) The speed v centimetres/second of a particle moving with simple harmonic motion in a straight line is given by $v^2 = 6 + 4x - 2x^2$, where x cm is the magnitude of the displacement from a fixed point O.

(i) Show that $\ddot{x} = -2(x-1)$. **2**

(ii) Find the centre of the motion. **1**

(iii) Find the period of the motion. **1**

(iv) Find the amplitude of the motion. **1**

UESTION 5

) (i) Differentiate $x\cos^{-1}x - \sqrt{1-x^2}$ **2**

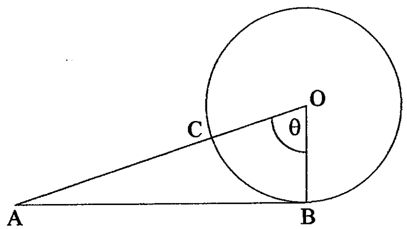
(ii) Hence evaluate $\int_0^1 \cos^{-1}x\,dx$ **2**

) Find:

(i) $\int \frac{x}{3+4x^2}\,dx$ **1**

(ii) $\int \frac{dx}{3+4x^2}$ **2**

)



In the above diagram, O is the centre of a circle and AB is a tangent to the circle, meeting it at point B. The line interval OA cuts the circumference of the circle at a point C.

(i) If the arc of the circle CB divides the triangle AOB into two portions of equal area and if the angle AOB is denoted by θ, show that $\tan\theta = 2\theta$. **2**

(ii) If $\theta = 1.2$ radians is an approximate solution to the equation in (i) above, use one application of Newton's Method to find a better approximation, correct to two decimal places. **3**

UESTION 6

) Use Mathematical Induction to show that $\cos(x + n\pi) = (-1)^n \cos x$ for all positive integers $n \geq 1$. **4**

) $P(2ap, ap^2)$ is a point on the parabola $x^2 = 4ay$.

(i) Show that the equation of the normal to the curve of the parabola at the point, P, is: $x + py = 2ap + ap^3$. **2**

(ii) Find the co-ordinates of the point Q where the normal at P meets the y axis. **2**

(iii) Determine the co-ordinates of the point R which divides PQ externally in the ratio 2 : 1. **2**

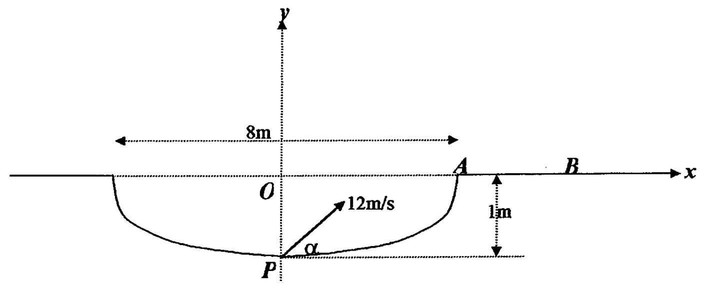
(iv) Find the cartesian equation of the locus of R and describe the locus in geometrical terms. **2**

QUESTION 7

(a) Prove that $\dfrac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta} = \tan\left(\dfrac{\theta}{2}\right)$ **4**

(b) A golf ball is lying at point P, at the middle of a sand bunker, which is surrounded by level ground. The point A is at the edge of the bunker, and line AB lies on level ground. The bunker is 8 metres wide and 1 metre deep.

The ball is hit towards A with an initial speed of 12 metres per second, and angle of elevation α. You may assume that the acceleration due to gravity is 10m/s^2.



(i) Show that the golf ball's trajectory at time t seconds after being hit is defined by the equations : **2**

$$x = 12\,t\cos\alpha \quad \text{and } y = -5t^2 + 12\,t\sin\alpha - 1$$

where x and y are the horizontal and vertical displacements in metres of the ball from the origin O shown in the diagram.

(ii) Given $\alpha = 30°$, how far from A will the ball land? **2**

(iii) Find the maximum height above the ground reached by the ball if $\alpha = 30°$. **1**

(iv) Find the range of values of α, to the nearest degree, at which the ball must be hit so that it will land to the right of A. **3**

2003 NSBHS TRIAL EXT 1

QUESTION 1

a) $\frac{1}{x} < \frac{1}{x+1}$

Boundary pts.
$x = 0, \; x = -1$

Let $\frac{1}{x} = \frac{1}{x+1}$
$x + 1 = x$
No solution

$\therefore -1 < x < 0$

OR

$x(x+1)^2 < x^2(x+1)$
$x^3 + 2x^2 + x < x^3 + x^2$
$x^2 + x < 0$
$x(x+1) < 0$
$-1 < x < 0$ (3)

b) i) $ax + b$, a and b constants

ii) $P(x) = (x-2)(x-3)Q(x) + ax + b$
$P(2) = 2a + b = 4$
$P(3) = 3a + b = 9$
$\therefore a = 5, \; b = -6$
$\therefore$ Remainder $5x - 6$ (3)

c) $\int_{-1}^{2} x\sqrt{2-x}\,dx$ $\quad u = 2 - x$, $du = -dx$, when $x = -1, u = 3$; $x = 2, u = 0$

$= -\int_3^0 (2-u)\sqrt{u}\,du$

$= \int_0^3 2u^{1/2} - u^{3/2}\,du$

$= \left[2 \cdot \frac{2}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right]_0^3$

$= 4\sqrt{3} - \frac{2}{5}9\sqrt{3}$ (4)

$= \frac{2\sqrt{3}}{5}$

d) $\int \sin^2 x\,dx = \frac{1}{2}\int(1 - \cos 2x)\,dx$
$= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right] + c$
$= \frac{x}{2} - \frac{1}{4}\sin 2x + c$ (2)

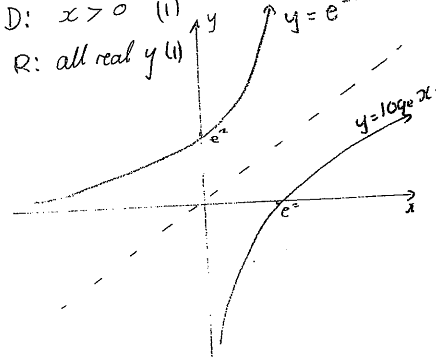
QUESTION 2

a) $\lim_{x \to 0} \frac{\sin\frac{x}{3}}{x} = \lim_{x \to 0} \frac{\sin\frac{x}{3}}{\frac{x}{3} \times 3}$ (1)
$= \frac{1}{3}$

b)(i) $f(x) = e^{x+2}$

for inverse: $x = e^{y+2}$
$\log_e x = y + 2$
$y = \log_e x - 2$
$f^{-1}(x) = \log_e x - 2$ (2)

ii) D: $x > 0$ (1)
R: all real y (1)



(2)

c) i) C: $y = 0$, $0 = \pi + 2\sin^{-1}3x$
$\sin^{-1}3x = -\frac{\pi}{2}$
$3x = -1$
$x = -\frac{1}{3}$
$\therefore C\left(-\frac{1}{3}, 0\right)$

A: $x = \frac{1}{3}$ $\quad \therefore y = \pi + 2 \cdot \sin^{-1}1 = 2\pi$
$\therefore A\left(\frac{1}{3}, 2\pi\right)$

ii) B $(0, \pi)$

iii) $y = \pi + 2\sin^{-1}3x$ $\quad$ when $x = 0$
$\frac{dy}{dx} = 2\frac{1}{\sqrt{1-9x^2}} \cdot 3$
$\frac{dy}{dx} = 6$
$\therefore$ Eqn: $y - \pi = 6x$
$6x - y + \pi = 0$

QUESTION 3

a) $x^3 - 2x^2 + 4x - 5 = 0$

i) $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} = 4$
$\alpha\beta\gamma = \frac{-d}{a} = 5$

ii) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\alpha\beta + \beta\gamma + \alpha\gamma}{\alpha\beta\gamma}$
$= \frac{4}{5}$ (4)

b) $\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

$\tan\left(\tan^{-1}k + \tan^{-1}\frac{2}{3}\right) = \frac{k + \frac{2}{3}}{1 - \frac{2}{3}k}$

$\tan\frac{\pi}{4} = 1$

$\therefore \frac{k + \frac{2}{3}}{1 - \frac{2}{3}k} = 1$

$k + \frac{2}{3} = 1 - \frac{2}{3}k$

$\therefore \frac{5}{3}k = \frac{1}{3}$

$\therefore k = \frac{1}{5}$ (3)

c)(i) $\sqrt{3}\cos\theta - \sin\theta = A\cos(\theta + \alpha)$
$= A\cos\theta\cos\alpha - A\sin\theta\sin\alpha$

$\therefore A\cos\alpha = \sqrt{3}$
$A\sin\alpha = 1$
$\tan\alpha = \frac{1}{\sqrt{3}}$, $\alpha = \frac{\pi}{6}$
$A = 2$
$\therefore \sqrt{3}\sin\theta - \sin\theta = 2\cos\left(\theta + \frac{\pi}{6}\right)$

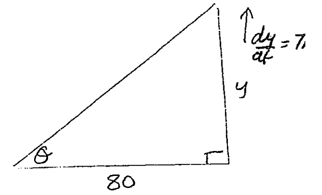
(ii) $2\cos\left(\theta + \frac{\pi}{6}\right) = 1$
$\cos\left(\theta + \frac{\pi}{6}\right) = \frac{1}{2}$
$\theta + \frac{\pi}{6} = \frac{\pi}{3}, \frac{5\pi}{3}$
$\theta = \frac{\pi}{6}, \frac{3\pi}{2}$

(iii) $\theta = \frac{\pi}{6} \pm 2n\pi, \; \frac{3\pi}{2} \pm 2n\pi$ (5)

QUESTION 4

4) a) PA = PB (tangents from external pt. are =)
$\therefore \angle PBA = \angle PAB$ (base $\angle$s of isos.
$\therefore \angle PAB = \frac{180 - 80}{2}$ (sum of $\angle$s of
$= 50°$
$\therefore x° = 50°$ (alternate segment

b)



$\tan\theta = \frac{y}{80}$
$y = 80\tan\theta$
$\frac{dy}{d\theta} = 80\sec^2\theta$

$\frac{d\theta}{dt} = \frac{d\theta}{dy} \cdot \frac{dy}{dt}$
$= \frac{1}{80\sec^2\theta} \cdot 7$
$= \frac{7\cos^2\theta}{80}$

When $y = 30$, $\tan\theta = \frac{3}{8}$
$\theta \doteqdot 0.3587$

$\frac{d\theta}{dt} \doteqdot 0.077$ rad/sec

c) $v^2 = 6 + 4x - 2x^2$

(i) $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$
$= \frac{d}{dx}(3 + 2x - x^2)$
$= 2 - 2x$
$= -2(x - 1)$

ii) Centre of motion $\ddot{x} = 0$
$x = 1$

iii) Period $= \frac{2\pi}{n} = \frac{2\pi}{\sqrt{2}}$

iv) $v = 0$
$x^2 - 2x - 3 = 0$
$(x-3)(x+1) = 0$
Extremities $x = 3, -1$, Centre $x = 1$
$\therefore$ AMPLITUDE IS 2cm

QUESTION 5

a) i) $\frac{d}{dx}\left(x\cos^{-1}x - \sqrt{1-x^2}\right)$ 1 for $\frac{d}{dx}(x\cos^{-1}x)$ 1 for $\frac{d}{dx}\left(\sqrt{1-x^2}\right)$ (2)

$= 1\cdot\cos^{-1}x - x\frac{1}{\sqrt{1-x^2}} - \frac{1}{2}\frac{1\times -2x}{\sqrt{1-x^2}}$

$= \cos^{-1}x$ 1

ii) $\int_0^1 \cos^{-1}x\,dx = \left[x\cos^{-1}x - \sqrt{1-x^2}\right]_0^1$

$= \cos^{-1}1 - (-1)$

$= 0+1$ 1 (2)

$= 1$

b) i) $\int\frac{x}{3+4x^2}\,dx = \frac{1}{8}\ln(3+4x^2)+C$ (1)

ii) $\int\frac{dx}{3+4x^2} = \int\frac{dx}{(\sqrt{3})^2+(2x)^2}$ 1 for $\tan^{-1}\left(\frac{2x}{\sqrt{3}}\right)$ 1 for correct answer

$= \frac{1}{2}\cdot\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x}{\sqrt{3}}\right)+C$ (2)

$= \frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{2x}{\sqrt{3}}\right)+C$

c) i) Area $\Delta AOB = 2\times$ area sector OCB

Let $OB = r$, $\frac{AB}{r} = \tan\theta$

$\frac{1}{2}OB\cdot AB = 2\times\frac{1}{2}r^2\theta$ 1

$\frac{1}{2}r\cdot r\tan\theta = r^2\theta$ 1 (2)

$\therefore \tan\theta = 2\theta$

ii) $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$f(x) = \tan\theta - 2\theta$ 1

$f'(x) = \sec^2\theta - 2$

$f(1.2) \doteqdot 0.17215$

$f'(1.2) \doteqdot 5.61596$ 1

$x_2 = 1.2 - \frac{0.17215}{5.61596}$ (3)

$\doteqdot 1.17$ correct to 2 dec. places 1

QUESTION 6

a) Prove $\cos(x+n\pi) = (-1)^n\cos x$

① If $n=1$

L.H.S $= \cos(\pi+x)$

$= -\cos x$

R.H.S $= (-1)^1\cos x$

$=$ L.H.S

$\therefore$ True for $n=1$. 1

② assume true for $n=k$.

$\cos(x+k\pi) = (-1)^k\cos x$

③ If $n=k+1$, R.T.P.

$\cos(x+k\pi+\pi) = (-1)^{k+1}\cos x$.

L.H.S $= \cos(\pi+x+k\pi)$ 1

$= -\cos(x+k\pi)$

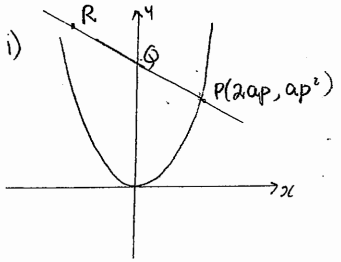
$= -(-1)^k\cos x$ from ② 1

$= (-1)^{k+1}\cos x$

④ Since it is true for $n=1$, it is true for $n=1+1$, and then true for $n=2+1$ and so on. 1 (4)

$\therefore$ True for all n

b) i)



$y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{x}{2a}$ at $P(2ap, ap^2)$, $m_T = p$ 1

$m_N = -\frac{1}{p}$

Eqn of normal: $y - ap^2 = -\frac{1}{p}(x-2ap)$ 1

$-py + ap^3 = x - 2ap$

$x + py = 2ap + ap^3$

ii) Q. $x=0$

$\therefore py = 2ap + ap^3$

$y = 2a + ap^2$

$Q(0, 2a+ap^2)$ 2

iii) $P(2ap, ap^2)$ $Q(0, 2a+ap^2)$

$2 : -1$

$R\left(\frac{2\times 0 + -1\times 2ap}{1}, \frac{2(2a+ap^2)+-ap^2}{1}\right)$

$R(-2ap, 4a+ap^2)$ 2

iv) $x = -2ap$

$p = \frac{x}{-2a}$

$y = 4a + a\left(\frac{x}{-2a}\right)^2$

$= 4a + \frac{x^2}{4a}$ 1

$\therefore x^2 = 4a(y-4a)$ 1

Parabola, focal length a, Vertex$(0, 4a)$

QUESTION 7

a) Let $t = \tan\frac{\theta}{2}$ 1

$$L.H.S = \frac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta}$$

$$= \frac{1+\frac{2t}{1+t^2}-\frac{1-t^2}{1+t^2}}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}$$ 2

$$= \frac{1+t^2+2t-1+t^2}{1+t^2+2t+1-t^2}$$

$$= \frac{2t^2+2t}{2+2t}$$

$$= \frac{2t(t+1)}{2(1+t)}$$ 1 (4)

$$= t$$

b) i) $\ddot{x} = 0$

$\dot{x} = C_1$

When $t=0$, $\dot{x} = 12\cos\alpha$

$\dot{x} = 12\cos\alpha$

$x = 12t\cos\alpha + C_2$

When $t=0$, $x=0$

$\therefore x = 12t\cos\alpha$ (1) from above

$\ddot{y} = -10$

$\dot{y} = -10t + C_3$

when $t=0$, $\dot{y} = 12\sin\alpha$

$\therefore \dot{y} = 12\sin\alpha - 10t$

$y = 12t\sin\alpha - 5t^2 + C_4$

when $t=0$, $y=-1$

$\therefore y = 12t\sin\alpha - 5t^2 - 1$ (1) from above (2)

ii) $\alpha = 30°$

Ball lands when $y=0$.

$\therefore 12t\,\frac{1}{2} - 5t^2 - 1 = 0$

$\therefore 6t - 5t^2 - 1 = 0$

$5t^2 - 6t + 1 = 0$

$(5t-1)(t-1) = 0$

$t = \frac{1}{5}$, $t = 1$ 1

ball lands on top

ii) When $t=1$

$x = 12\cos 30°$

$= 6\sqrt{3}$.

$\therefore$ Distance from A $= 6\sqrt{3} - 4$

$\doteqdot 6.4$m 1 (2)

iii) Max. Height. $\dot{y} = 0$.

$\therefore t = \frac{6\sin\alpha}{5}$, $\alpha = 30°$ 1 (1)

$= \frac{3}{5}$

iv) $x = 12t\cos\alpha$

$t = \frac{x}{12\cos\alpha}$

$\therefore y = -5\left(\frac{x}{12\cos\alpha}\right)^2 + 12\left(\frac{x}{12\cos\alpha}\right)\sin\alpha - 1$ 1

$= \frac{-5}{144}x^2\sec^2\alpha + x\tan\alpha - 1$

$x=4$, $y=0$

$\therefore \frac{-5}{9}\sec^2\alpha + 4\tan\alpha - 1 = 0$

$5(\tan^2\alpha + 1) - 36\tan\alpha + 9 = 0$ 1

$5\tan^2\alpha - 36\tan\alpha + 14 = 0$

$\tan\alpha = \frac{36 \pm \sqrt{36^2 - 4\times5\times14}}{10}$

$= 0.4125$, 6.7875

$\alpha = 22.4°$, $81.6°$

$\therefore$ The required range is

$23° < \alpha < 81°$ 1 (3)