NSBH EXN 1 2004

Question 1

Marks

(a) Evaluate, leaving answers in exact form:

(i) $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin 2x \, dx$ 2

(ii) $\int_{1}^{2} \frac{e^{2x}}{e^{2x}+1} \, dx$ 2

(b) Differentiate $y = e^x \ln x$ with respect to x. 2

(c) Evaluate $\cos^{-1}\left(\frac{1}{2}\right) - \sin^{-1}\left(-\frac{1}{2}\right)$, leaving your answer in terms of π. 1

(d) Use mathematical induction to show that $3^{2n+4} - 2^{2n}$ is divisible by 5 for all positive integers n. 5

Question 2 (Start a new page)

(a) Find the second derivative of $\sin^{-1} x$. 3

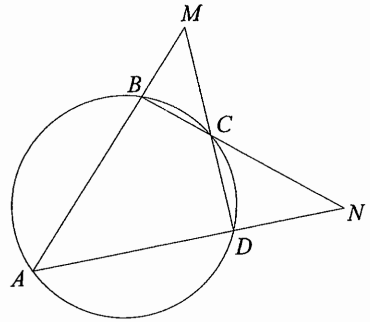
(b) Sketch the graph of $y = \frac{x-2}{x^2}$, showing any maximum and minimum turning points, points of inflexion, asymptotes, and intercepts with the coordinate axes. 6

(c) Use the substitution $u = 1 - x^2$ to find $\int \frac{x}{\sqrt{1-x^2}} \, dx$ 3

Question 3 (Start a new page)

(a) If $x^2 + y^2 = 7xy$, show that $\ln(x+y) = \ln 3 + \frac{1}{2}\ln x + \frac{1}{2}\ln y$. 3

(b)



In the figure, ABM, DCM and AND are straight lines.

Copy the diagram. Given that $\widehat{AMD} = \widehat{BNA}$, prove that

(i) $\widehat{ABC} = \widehat{ADC}$ 2

(ii) AC is the diameter of the circle. 3

(c) Prove that $\tan^{-1} 4 - \tan^{-1} \frac{3}{5} = \frac{\pi}{4}$. 4

Question 4 (Start a new page)

(a) The equation $\tan x - 2x = 0$ has a solution near $1{\cdot}1$. Use one application of Newton's method to find a better approximation to this solution. 3

(b) For the function $y = \cos\left(\sin^{-1} x\right)$

(i) state the domain 1

(ii) state the range 1

(iii) draw a neat sketch of the function 2

(c) The tangent at $P(2ap, ap^2)$ on the parabola $x^2 = 4ay$ meets the x axis at Q. Find the cartesian equation of the locus of the midpoint of PQ. 5

Question 5 (Start a new page)

(a) Find the coefficient of x^2 in the expansion of $(1 - 2x)^{18}(1 + 3x)^{17}$. **4**

(b) Factorize the polynomial $P(x) = x^3 - x^2 - 8x + 12$ completely, given that the equation $P(x) = 0$ has a repeated root. **4**

(c) If α, β and γ are the roots of $x^3 - x^2 + 4x - 1 = 0$, find the value of

(i) $\alpha\beta + \alpha\gamma + \beta\gamma$ **1**

(ii) $\alpha\beta\gamma$ **1**

(ii) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$ **2**

Question 6 (Start a new page)

(a) Prove that $\sin 3\theta = 3 \sin\theta - 4\sin^3\theta$. **4**

(b) Find an approximation for the area between the curve $y = 3\sin x$ and the x axis, between $x = 0$ and $x = 1$, using the trapezoidal rule with three function values. **4**

(c) ${}^nC_k$ is the coefficient of x^k in the binomial expansion of $(1 + x)^n$, where n is a positive integer. By differentiating the identity $x(1 + x)^n = \sum_{k=0}^{n} {}^nC_k x^{k+1}$, show that $\sum_{k=0}^{n} (k + 1)\,{}^nC_k = (n + 2) \cdot 2^{n-1}$. **4**

Question 7 (Start a new page)

(a) An object is placed in surroundings which remain at a constant temperature of 20°C. The temperature of the object (T °C) after t minutes is given by

$$T = 20 + (A - 20)\, e^{-kt}, \text{ where } A \text{ and } k \text{ are positive constants.}$$

(i) Prove that $\dfrac{dT}{dt} = -K(T - 20)$. **1**

(ii) Initially, the temperature of the object is 50°C and is falling at a rate of 6°C per minute. Find

(α) the values of A and K **2**

(β) the temperature (to the nearest degree) of the object after 10 minutes. **1**

(γ) the time required (to the nearest minute) for the temperature of the object to reach 21°C. **2**

(b) A stone is thrown at 20 ms^{-1} at an angle of 30° above the horizontal from the edge of the top of a building 40 metres high.

(i) Derive equations for the vertical and horizontal displacement of the stone in terms of time. Ignore air resistance, and assume that the acceleration due to gravity is 10 ms^{-2}. **2**

(ii) How long after projection will the stone strike the ground? **2**

(iii) Find the horizontal range of the flight. **2**

2004 N.S.B.H. 3U TRIAL M.S.C

i/ $\int_{\pi/3}^{\pi/2} \sin 2x\, dx$ 1

$= -\frac{1}{2}\cos 2x \Big]_{\pi/3}^{\pi/2}$ 1

$= \frac{1}{2} + \frac{1}{2}\cdot -\frac{1}{2}$

$= \frac{1}{4}$ 1

$\int_1^2 \frac{e^{2x}}{e^{2x}+1}\, dx$

$\frac{1}{2}\ln(e^{2x}+1)\Big]_1^2$ 1

$\frac{1}{2}\left[\ln(e^4+1) - \ln(e^2+1)\right]$

$\frac{1}{2}\ln\frac{(e^4+1)}{(e^2+1)}$ 1

$y = e^x \ln x$

$\frac{dy}{dx} = e^x \cdot \frac{1}{x} + e^x \ln x$ 2

$= e^x\left(\ln x + \frac{1}{x}\right)$

$\frac{\pi}{3} + \frac{\pi}{6} = \frac{\pi}{2}$

1

d) Let $\frac{3^{2n+4} - 2^{2n}}{5} = M \quad M \in J$

$3^{2n+4} - 2^{2n} = 5M$

Step 1 Prove true for $n=1$

$3^6 - 2^2 = \frac{725}{5}$

$\therefore$ True for $n=1$ 1

Step 2 Assume true for $n=k$

ie $3^{2k+4} - 2^{2k} = 5M$ 1

Prove true for $n = k+1$

$3^{2k+6} - 2^{2k+2} = 9\,3^{2k+4} - 4\,2^{2k}$ 1

$= 9(5M + 2^{2k}) - 4.2^{2k}$

$= 45M + 5.2^{2k}$

$= 5(9M + 2^{4})$ 1

which is $\div$ by 5 $\therefore$ true for $n=k+1$

Step 3. As the result is true for $n=1$ and we assumed the result for $n=k$ and proved it for the next value $n=k+1$, then it is true for $n=2$ and so on. 1

Q2)

a) $\frac{d}{dx}\sin^{-1}x$

$= \frac{1}{\sqrt{1-x^2}}$ 1

$\frac{d}{dx}\left((1-x^2)^{-1/2}\right) = -\frac{1}{2}(1-x^2)^{-3/2}\times -2x$ 1

$= \frac{x}{(1-x^2)^{3/2}}$ 1

c) $\int \frac{x}{\sqrt{1-x^2}}\, dx$

$u = 1-x^2$

$du = -2x\, dx$ 1

$-\frac{1}{2}du = x\, dx$

$-\frac{1}{2}\int \frac{du}{\sqrt{u}}$ 1

$= -u^{1/2} + c = -(1-x^2)^{1/2} + c.$ 1

$y = \frac{x-2}{x^2}$

$\frac{dy}{dx} = \frac{x^2.1 - (x-2).2x}{x^4}$

$= \frac{-x(x-4)}{x^{4}3}$ 1

Want $\frac{dy}{dx} = 0$

$x - 4 = 0$

$\therefore x = 4 \quad y = \frac{1}{8}$ 1

$\frac{d^2y}{dx^2} = \frac{x^3(-1) + (x-4)3x^2}{x^6}$

$= \frac{-x^3 + 3x^3 - 12x^2}{x^6}$

$= \frac{x^2(2x-12)}{x^{6}4}$ 3

at $x = 4$ 1

$\frac{d^2y}{dx^2} < 0 \therefore$ max

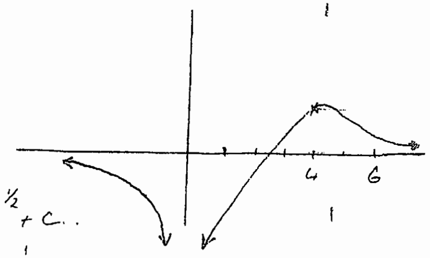
Want $\frac{d^2y}{dx^2} = 0$

$\therefore x = 6$ 1

x	5	6	7
$\frac{d^2y}{dx^2}$	<0	$=0$	>0

$\therefore$ infl at $(6, \frac{1}{9})$

1



1

)

a) $x^2+y^2 = 7xy$

$\therefore (x+y)^2 = 9xy$ 1

$\ln (x+y)^2 = \ln 9xy$

$2 \ln (x+y) = \ln 9 + \ln x + \ln y$ 1

$\therefore \ln (x+y) = \ln 3 + \frac{1}{2} \ln x + \frac{1}{2} \ln y$

b) i/ diagram

ii/ In $\triangle$ AMD and $A\hat{N}B$

α) $A\hat{M}D = A\hat{N}B$ given 1

$B\hat{A}D$ is common 1

$\therefore \triangle AMD \;|||\; \triangle ANB$

β) $A\hat{B}N = A\hat{D}M$ ($|||$ $\triangle$'s)

ABCD is cyclic 1

opp angles of cyclic quad $= 180°$ 1

$\therefore A\hat{B}N = A\hat{D}M = 90°$

$\therefore$ AC is a diameter 1

c) Let $\alpha = \tan^{-1} 4$

$\tan \alpha = 4$

$\beta = \tan^{-1} \frac{3}{5}$

$\tan \beta = \frac{3}{5}$ 1

$\therefore \tan(\alpha - \beta) = \dfrac{4 - \frac{3}{5}}{1 + 4 \cdot \frac{3}{5}}$ 1

$= \dfrac{\frac{17}{5}}{\frac{17}{5}}$

$= 1$ 1

$\therefore \alpha - \beta = \pi/4$

ie $\tan^{-1} 4 - \tan^{-1} \frac{3}{5} = \pi/4$ 1

Q4) a) Let $f(x) = \tan x - 2x$

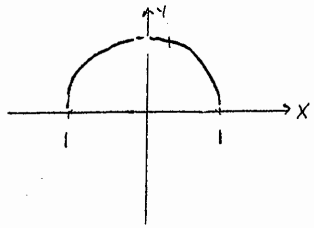
$f'(x) = \sec^2 x - 2$ 1

$x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$ 1

$= 1.1 - \dfrac{f(1.1)}{f'(1.1)}$ 1

$= 1.18$ (to 2D)

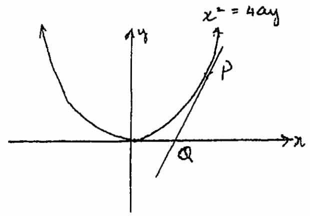
b) $y = \cos(\sin^{-1} x)$



i/ $1 \leq x < 1$ 1

ii/ $0 \leq y \leq 1$ 1

c)



$x^2 = 4ay$

$y = \dfrac{x^2}{4a}$

$\dfrac{dy}{dx} = \dfrac{x}{2a}$

at $x = 2ap$ $\dfrac{dy}{dx} = p$

$\therefore$ eqn of tangent

$y - ap^2 = p(x - 2ap)$

$y = px - ap^2$ 1

at $y = 0$

$x = ap$

$\therefore Q(ap, 0)$ 1

midpoint of PQ

$\left(\dfrac{3ap}{2}, \dfrac{ap^2}{2} \right)$ 1

$\therefore$ eqn of locus

$x = \dfrac{3ap}{2}$

$p = \dfrac{2x}{3a}$ 1

$\therefore y = \dfrac{a}{2} \left(\dfrac{2x}{3a} \right)^2$

$= \dfrac{2x^2}{9a}$ 1

4

a)

$\cdot 2x)^{18}(1+3x)^{17}$

${}^{18}C_1\, 2x + {}^{18}C_2 (2x)^2 \ldots\ldots)$ 1

${}^{17}C_1\, 3x + {}^{17}C_2 (3x)^2 + \ldots\ldots)$ 1

$9\ {}^{17}C_2 + 4\ {}^{18}C_2 - 6\ {}^{18}C_1 \cdot {}^{17}C$ 1

$1224 + 612 - 1836$

0

b) $P(x) = x^3 - x^2 - 8x + 12$

$P'(x) = 3x^2 - 2x - 8$ 1

Want $P'(x) = 0$

$(3x+4)(x-2) = 0$

$\therefore x = 2$ or $-\frac{4}{3}$ 1

$P(2) = 8 - 4 - 16 + 12 = 0$

$\therefore x = 2$ is the repeated root 1

$P(x) = (x-2)(x-2)(x+3)$ 1

c) $x^3 - x^2 + 4x - 1 = 0$

$\alpha+\beta+\gamma = +1$

$\alpha\beta + \alpha\gamma + \beta\gamma = 4$

$\alpha\beta\gamma = 1$

i/ $\alpha\beta + \alpha\gamma + \beta\gamma = \frac{4}{1}$ 1

ii/ $\alpha\beta\gamma = \frac{1}{1}$ 1

iii/ $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\alpha\beta + \alpha\gamma + \beta\gamma}{\alpha\beta\gamma}$ 1

$= 4.$ 1

Q6) a) $\sin 3\theta = \sin(2\theta + \theta)$ 1

$= \sin 2\theta \cos\theta + \sin\theta \cos 2\theta$

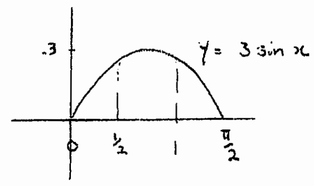
$= 2\sin\theta\cos^2\theta + \sin\theta(\cos^2\theta - \sin^2\theta)$

$= 2\sin\theta\cos^2\theta + \sin\theta\cos^2\theta - \sin^3\theta$

$= 3\sin\theta(1 - \sin^2\theta) - \sin^3\theta$

$= 3\sin\theta - 4\sin^3\theta$ 1

b)



$\int_0^1 3\sin x\, dx = \frac{\frac{1}{2}}{2}\left[0 + 3\sin 1 + 2\times 3\sin\frac{1}{2}\right]$ 1

$= 1.350\ u^2$ to 3 D. 1

c) $x(1+x)^n = {}^nC_0\, x + {}^nC_1\, x^2 + \ldots\ {}^nC_n\, x^{n+1}$

by diff^n

$(1+x)^n + nx(1+x)^{n-1} = {}^nC_0 + 2\,{}^nC_1\, x + \ldots\ldots\ (n+1)\,{}^nC_n\, x^n$ 1

$(1+x)^{n-1}(nx + 1 + x) =$ 1

$(1+x)^{n-1}(1 + (n+1)x) =$ 1

Let $x = 1$

$2^{n-1}\cdot(n+2) = {}^nC_0 + 2\,{}^nC_1 + \ldots\ldots + (n+1)\,{}^nC_n$

$= \sum_{k=0}^{n} (k+1)\,{}^nC_k.$ 1

a) i/ $T = 20 + (A-20)e^{-kt}$

$\frac{dT}{dt} = -k(A-20)e^{-kt}$

$= -k(T-20)$ 1

ii/ $T = 6 \quad t = 0 \quad \frac{dT}{dt} = -6$ when $t = 0$

α) $50 = 20 + (A-20)e^0$

$30 = A - 20$

$A = 50$ 1

$-6 = -k(A-20)e^0$

$6 = k(A-20)$

$6 = k\,30$

$k = \frac{1}{5}$ 1

β) $T(10) = 20 + 30e^{-\frac{1}{5}\times 10}$

$T(10) = 20 + 30e^{-2}$

$T = 24$ 1

γ) $21 = 20 + 30e^{-\frac{t}{5}}$

$1 = 30e^{-\frac{t}{5}}$

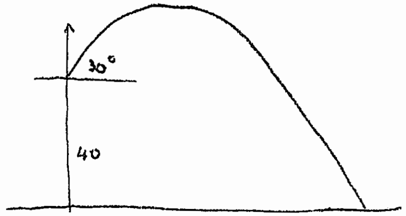
$\frac{1}{30} = e^{-2t}$

$-2t = \ln\frac{1}{30}$

$t = \frac{1}{5}\ln 30$

≈ 17 min 2

b)



i/ $\ddot{x} = 0$

$\dot{x} = t + c$

$t = 0 \quad \dot{x} = V\cos 30 = 20.\frac{\sqrt{3}}{2}$

$\therefore c = 10\sqrt{3}$

$x = 10\sqrt{3}\,t + d$

$x = 0,\ t = 0 \quad d = 0$

$x = 10\sqrt{3}\,t$

$\ddot{y} = -10$

$\dot{y} = -10t + c$

$t = 0 \quad \dot{y} = 20 \times \frac{1}{2} = 10$

$\therefore c = 10$

$\dot{y} = -10t + 10$

$y = -5t^2 + 10t + d$

$t = 0 \quad y = 0 \quad \therefore d = 0$

$y = -5t^2 + 10t$ 1

ii/ $y = -40$

$-40 = -5t^2 + 10t$

$t^2 - 2t - 8 = 0$

$(t-4)(t+2) = 0$

$\therefore t = 4$ or -2

but $t > 0$

$\therefore t = 4$ 2

iii/ $x = 10\sqrt{3}\,t$

$t = 4$

$\therefore x = 40\sqrt{3}$ 2

6