N.S.B.H. 2006 Ex I TRIAL

Question 1 **Marks**

a) Find $\frac{d}{dx}(e^{3x}\sin 2x)$ **2**

b) Evaluate $\int_{-\sqrt{3}}^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}dx$ **2**

c) Evaluate $\lim_{x\to 0} \frac{x}{\sin 5x}$ **1**

d) Use the substitution $u = 1 + x^4$ to evaluate $\int_0^1 \frac{x^3}{1+x^4}dx$ **3**

e) If the roots of the equation $x^3 + 2x^2 - x - p = 0$ are c, $c+1$ and $c+3$, find the value of p. **2**

f) The lines $y = mx + 3$ and $y = 4x - 5$ intersect at an acute angle α. Find possible values of m, if $\tan\alpha = 2$. **2**

Question 2 (START A NEW PAGE)

a) Solve $\frac{5}{4-x} \geq 1$ **3**

b) Consider the polynomial $P(x) = (x+2)(x-3)Q(x) + ax + b$

Given that $P(x)$ has remainders 1 and 6 when divided by $(x+2)$ and $(x-3)$ respectively, find a and b. **2**

c) A person wishes to receive a superannuation payout of $250000 at the end of 20 years. $d is to be paid at the beginning of each year. Interest is paid at the end of each year at a rate of 8% p.a. compounding annually. Find the value of d. **3**

Marks

d) The region bounded by the curve $y = \sin^{-1}x$, the y axis and $y = \frac{\pi}{6}$, is rotated about the y axis to form a solid.

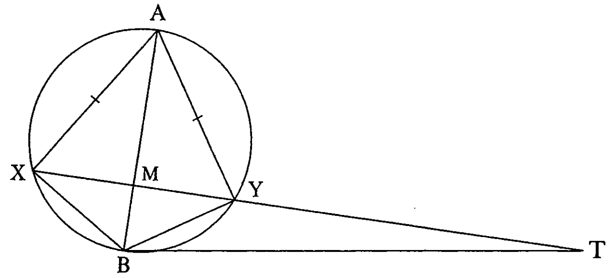
i) Show that the volume of the solid obtained is given by **1**

$$V = \pi\int_0^{\frac{\pi}{6}} \sin^2 y\, dy$$

ii) Find the volume of the solid. **3**

Question 3 (START A NEW PAGE)

a) A, X, B and Y are points on the circumference of a circle such that AX = AY.
XY meets AB at the point M.
The tangent at B meets the chord XY produced at T.



i) Explain why AB bisects $\angle XBY$. **1**

ii) Prove that BT = MT. **3**

b) Consider the functions $f(x) = 4 - x^2$ and $g(x) = \ln x$.

i) Sketch the graphs of $f(x) = 4 - x^2$ and $g(x) = \ln x$ on the same set of axes for $x > 0$. **2**

ii) Use your graph to show that the equation $x^2 - 4 + \ln x = 0$ has only one root which is near 1·5. **1**

iii) Use one application of Newton's method to find a better approximation of the root of the equation $x^2 - 4 + \ln x = 0$. **2**

Marks

c) Solve the equation

$\sqrt{5}\sin x + 2\cos x = -2, \quad 0° \le x \le 360°$ **3**

Question 4 (START A NEW PAGE)

a) Prove the following identity

$$\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta} + \frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta} = 2\sec\theta$$ **2**

b) When a patient takes a pill, the medicine is absorbed into the blood stream at a rate given by $\frac{dM}{dt} = -k(M - a)$, where M is the concentration of the medicine in the blood t minutes after taking the pill, and a and k are constants.

i) Show that $M = a(1 - e^{-kt})$ satisfies the given equation. **1**

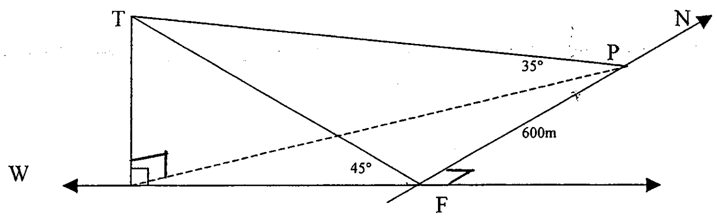
ii) What is the limiting value of the concentration? **1**

iii) Find k, if the concentration reaches 99% of the limiting value after 2 hours. **1**

iv) The patient starts to notice relief when the concentration reaches 10% of the limiting value. When will this occur, correct to the nearest second? **1**

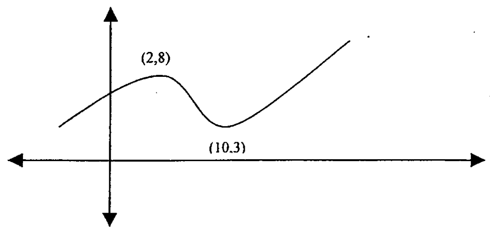
c) At the Mount Snow Gum ski fields there are two chairlifts that meet at Top Station (T). The first chairlift starts in the valley below from Wombat Flat (F) and runs due west as it rises at an angle of 45°. The second lift starts from Pot Hole Creek (P), 600m horizontally due north of Wombat Flat in the same valley. It rises at an angle of 35° to Top Station.

How high is Top Station above the valley below? **3**



Marks

d) A cubic function $y = f(x)$ is sketched below.



i) If $g(x) = f'(x)$, sketch $y = g(x)$. **1**

ii) Hence find the area of the region bounded $y = g(x)$ and the x axis. (Do not find the equation of the function $y = f(x)$) **2**

Question 5 (START A NEW PAGE)

a) Let $f(x) = \frac{x}{1 + x^2}$ for all real values of x.

i) Sketch the graph of f(x) showing the coordinates of the turning points and the x and y intercepts. Do not find the points of inflexion. **4**

ii) What is the largest domain containing the value $x = 2$ for which f(x) has an inverse function? **1**

iii) Sketch the graph of the inverse function $y = f^{-1}(x)$ on the same axes as your graph in part (i). **1**

iv) Find an expression for $y = f^{-1}(x)$ in terms of **2**

b) Use the process of Mathematical Induction to show that for all $n \ge 1$

$$\frac{1}{1\times 3} + \frac{1}{3\times 5} + \frac{1}{5\times 7} + \ldots\ldots\ldots + \frac{1}{(2n-1)(2n+\ )} = \frac{n}{2n+1}$$ **4**

Question 6 (START A NEW PAGE) **Marks**

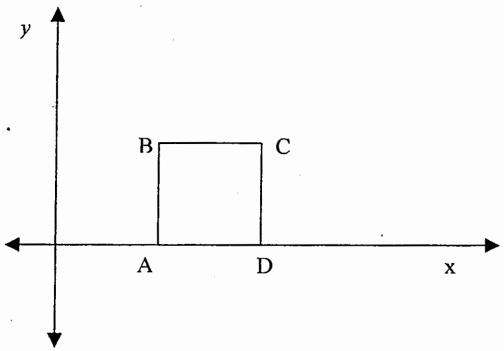
a) A boat is being pulled at a speed of 20m/min toward a wharf by a rope. The rope is attached to a point on the wharf 7 m vertically above the boat. Find the rate at which the rope is being drawn in when the boat is 24m from the wharf. 3

b) P is a variable point $(2ap, ap^2)$ on the parabola $x^2 = 4ay$ and S is the focus. The line joining P and S is produced to Q, so that PS = SQ.

i) Write down the coordinates of Q in terms of p. 2

ii) Find the cartesian equation of the locus of Q 2

c) The diagram shows a unit square with vertices A(1,0), B(1,1), C(2,1) and D(2,0).



A line passing through the origin with gradient m cuts the sides AB and CD at P and Q respectively.

i) Comment on possible values of m. 1

ii) For what value(s) of m does the line divide the area of the square in the ratio of 2:1? 4

Question 7 (START A NEW PAGE) **Marks**

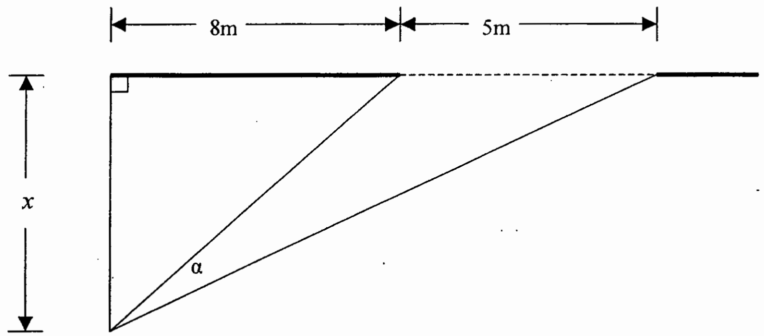
a) A large underground water main bursts, sending out fragments of road, rock, soil etc and water into the air with an initial speed of 12·5m/s and at an angle of θ. You may assume g = 10m/s.

i) Beginning with $\ddot{x} = 0$ and $\ddot{y} = -10$, derive the other four equations of projectile motion. 2

ii) How far away must a person stand from the hole to be certain not to be hit by any fragments or water? 2

iii) What is the greatest height the water can reach? 2

b) Hagar the Horrible wanted to attack a fortress with a 5 metre opening along a front wall. The strategy was to stand at the point P, x metres away from the wall and 8 metres to the left of the opening, thus giving an angle of vision α, through which to fire arrows from a crossbow.



Using the measurements on the diagram

i) Show that the angle of vision α is given by 1

$$\alpha = \tan^{-1}\left(\frac{13}{x}\right) - \tan^{-1}\left(\frac{8}{x}\right)$$

ii) Find the value of x in order to give the maximum angle of vision and hence find the maximum angle of vision in radians. 5

END OF EXAMINATION

i)

$e^{3x} \times 2\cos 2x + \sin 2x \cdot 3e^{3x}$

$= e^{3x}(2\cos 2x + 3\sin 2x)$ [2]

$\left[\sin^{-1}\frac{x}{2}\right]_{-\sqrt{3}}^{\sqrt{3}}$

$\sin^{-1}\frac{\sqrt{3}}{2} - \sin^{-1}\frac{-\sqrt{3}}{2}$

$= \frac{\pi}{3} - -\frac{\pi}{3} = \frac{2\pi}{3}$ [2]

$\frac{1}{5}$ [1]

$u = 1+x^4$

$du = 4x^3 dx$

$x=1 \Rightarrow u = 2$

$x=0 \Rightarrow u = 1$ [3]

$\int_1^2 \frac{1}{4}\frac{du}{u}$

$\frac{1}{4}\left[\ln u\right]_1^2$

$\frac{1}{4}\ln 2 - 0$

e)

$c + c+1 + c+3$

$= 3c+4$

Sum of roots $= +2 = -(3c+4)$

$2 = -3c - 4$

$6 = -3c$

$c = -2$

∴ Product of roots

$= c(c+1)(c+3)$

$= -2 \times -1 \times 1 = 2$ [2]

∴ $p = 2$

f)

$\left|\frac{m-4}{1+4m}\right| = 2$

∴ $\frac{m-4}{1+4m} = \pm 2$

$m-4 = 2+8m$

$-6 = 7m$

$\frac{-6}{7} = m$ [2]

or

$m-4 = -2-8m$

$9m = 2$

$m = \frac{2}{9}$

Q2

a) $\frac{5}{4-x} \geq 1$

$5(4-x) \geq (4-x)^2$

$20-5x \geq 16-8x+x^2$

$0 \geq -4-3x+x^2$

$x^2-3x-4 \leq 0$

$(x-4)(x+1) \leq 0$

∴ $-1 \leq x < 4$ [3]

Note inequality. (-1 it incorrect)

b) $P(-2) = -2a+b = 1$ (1)

$P(3) = 3a+b = 6$ (2)

(2) − (1)

$5a = 5$

$a = 1$

∴ $b = 3$ [2]

(c) Compound interest: 0 marks.

Answer of 5463.05 with working – 2 marks.

$d \times 1.08^{20} + d \times 1.08^{19} \ldots + d \times 1 \cdot 08 = 250{,}000$

$\frac{d \times 1.08[1.08^{20}-1]}{1.08-1} = 250000$

$d \times 49.4192\ldots = 250000$

$d = \$5058.38$ [3]

d)

i) $V = \pi\int_a^b x^2 dy$

$x = \sin y$

∴ $x^2 = \sin^2 y$

∴ $V = \pi\int_0^{\pi/6} \sin^2 y$ [1]

ii) $= \pi\int_0^{\pi/6} \frac{1}{2}(1-\cos 2y)\,dy$

$= \frac{\pi}{2}\left[y - \frac{1}{2}\sin 2y\right]_0^{\pi/6}$ [3]

$= \frac{\pi}{2}\left[\frac{\pi}{6} - \frac{1}{2}\times\frac{\sqrt{3}}{2} - (0-0)\right]$

$= \frac{\pi^2}{12} - \frac{\sqrt{3}\pi}{8}$ cubic units

4. a)

$LHS = \frac{c+s}{c-s} + \frac{c-s}{c+s}$

$= \frac{(c+s)^2+(c-s)^2}{(c-s)(c+s)}$

$= \frac{c^2+2cs+s^2+c^2-2cs+s^2}{c^2-s^2}$

$= \frac{2}{\cos 2\theta}$

$= 2\sec 2\theta$ [2]

b) i) $M = a(1-e^{-kt})$ (1)

$= a - ae^{-kt}$

$\frac{dM}{dt} = +ake^{-kt}$ (2)

Now $\frac{dM}{dt} = -k(M-a)$ (3)

Subst (1) into (3).

$\frac{dM}{dt} = -k(a(1-e^{-kt}) - a)$

$= ke^{-kt}$ ⇒

[1] (3) = (2). $t=0 \Rightarrow M=0$

ii) $\lim_{t\to\infty}$ [1]

$M = a$

iii) $t = 2$ $M = .99a$

∴ $.99 = (1-e^{-120k})$

$-.01 = -e^{-120k}$

$.01 = e^{-120k}$ [1]

∴ $\ln .01 = -120k$

$k = \frac{\ln .01}{-120} = .03837$

iv) $.1 = 1-e^{-kt}$

$-.9 = -e^{-kt}$

$\frac{\ln .9}{-k} = t$ [1]

$t = 2$ min 45 secs

c)

$\angle NFP = 90°$

$WF = h\cot 45°$

$FP = 600$

$WP = h\cot 35°$

$WP^2 = WF^2 + 600^2$

$h^2\cot^2 35 = h^2\cot^2 45 + 600^2$

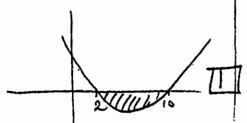
$h^2(\cot^2 35 - 1) = 600^2$

$h^2 = \frac{600^2}{\cot^2 35 - 1}$

$h^2 = 346684.8$

$= 588.5$ m [3]

d)



[1]

$\int_2^{10} g(x)\,dx = \left|\left[G(x)\right]_2^{10}\right|$ [2]

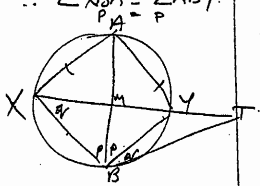
$= |3-8|$

$= |-5| = 5$ sq units.

Q3

As AX and AY are equal chords they subtend equal angles on the circumference

∴ $\angle XBA = \angle ABY$



$\angle YBT = \angle BXY$ (Tgt chord thm)

$= q$

$\angle BMT = \angle BXM + \angle XBM$ (ext $\angle$ th)

$= p+q$

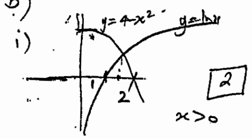
$= \angle MBY + \angle YBT$

∴ $\triangle BMT$ is isosceles (= base $\angle$s)

∴ $MT = BT$

b)

i)



[2]

$x > 0$

ii) Graphs would suggest pt of intersection about ½ way between $x=1$ and 2 [1]

iii) Consider

$y = \ln(x) + x^2 - 4$

$\frac{dy}{dx} = \frac{1}{x} + 2x$

$x_1 = x_0 - \frac{y_0}{y_0'}$

$= 1.5 - \frac{\ln 1.5 + 1.5^2 - 4}{\frac{1}{1.5} + 3}$

$\doteqdot 1.87$ [2]

c)

$\sqrt{5}\sin x + 2\cos x = -2$

$A\sin(x+\alpha)$

$= A\sin x\cos\alpha + A\cos x\sin\alpha$

∴ $A\cos\alpha = \sqrt{5}$

$A\sin\alpha = 2$

$\tan\alpha = \frac{2}{\sqrt{5}}$

$\alpha \doteqdot 41°49'$

$A = \sqrt{5+4}$

$= 3$

∴ $3\sin(x+41°49') = -2$

$\sin(x+41°49') = \frac{-2}{3}$

∴ $x + 41°49' = 221°49'$

$x = 180°$

Or

$x + 41°49' = 318°11'$

$x = 276°23'$ [3]

Q5

$y = \frac{x}{1+x^2}$

$y' = \frac{(1+x^2)\times 1 - x\times 2x}{(1+x^2)^2}$

$= \frac{1+x^2-2x^2}{(1+x^2)^2}$

$= \frac{1-x^2}{(1+x^2)^2}$

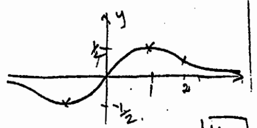
$y' = 0 \Rightarrow x = \pm 1$

Test

x	-2	-1	0	1	2
y'	$-$	0	1	0	$-$

∴ Min at $x=-1$

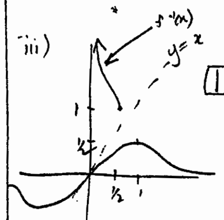
Max at $x=1$



[4]

ii) $x \geq 1$ [1]

iii)



[1]

iv) $x = \frac{y}{1+y^2}$

$x + xy^2 - y = 0$

$xy^2 - y + x = 0$

$y = \frac{1 \pm \sqrt{1-4x^2}}{2x}$

as $y \geq 1$ y is only possible.

$y = \frac{1+\sqrt{1-4x^2}}{2x}$ [2]

b)

1. $n=1$

$LHS = \frac{1}{1\times 3} = \frac{1}{3}$

$RHS = \frac{1}{2\times 1+1} = \frac{1}{3}$

∴ True for $n=1$

2. Assume true for $n=k$

i.e. $S_k = \frac{k}{2k+1} = \frac{1}{1\times 3} + \ldots \frac{1}{(2k-1)(2k+1)}$

Consider

$S_k + T_{k+1}$

$= \frac{k}{2k+1} + \frac{1}{(2(k+1)-1)(2(k+1)+1)}$

$= \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)}$

$= \frac{k(2k+3)+1}{(2k+1)(2k+3)}$

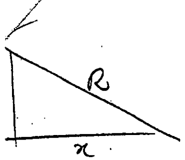
$= \frac{2k^2+3k+1}{(2k+1)(2k+3)}$

$= \frac{(2k+1)(k+1)}{(2k+1)(2k+3)} = \frac{k+1}{2k+3}$ by formula

[4]

∴ If it is true for $n=1$ & if true for $n=k$ it is true for $n=k+1$ by Math induction ∴ true for $n=1, 2, 3, \ldots$ all integral values of n

4



$R^2 = x^2 + 49$

w.r.t. t

$\therefore 2R\frac{dR}{dt} = 2x \times \frac{dx}{dt}$

$x = 24 \Rightarrow R = 25$

$\frac{dx}{dt}$ 20m/min

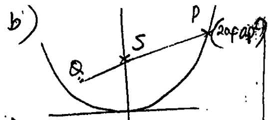
$2 \times 25 \times \dot{R} = 2 \times 24 \times 20$

$\dot{R} = \frac{24 \times 20}{25} = \frac{24 \times 4}{5}$

$= \frac{96}{5}$ m/min

[3]

b)



i). $QS = PS$

Let $Q(m, n)$

$\therefore S(0, a)$ is midpt PQ

$\frac{m + 2ap}{2} = 0 \Rightarrow m = -2ap$

$\frac{n + ap^2}{2} = a$

$n = 2a - ap^2$

$n = a(2 - p^2)$

$Q(-2ap, a(2-p^2))$ [2]

ii) $\Rightarrow x = -2ap$

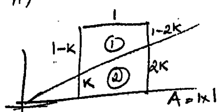
$p = \frac{-x}{2a}$

$\therefore y = a\left(2 - \frac{x^2}{4a^2}\right)$

$y = 2a - \frac{x^2}{4a}$ [2]

i) $0 \le m \le \frac{1}{2}$ ✓

ii)



$A_1 = 2 \times A_2 = \frac{2}{3}$

$A_1 \Rightarrow \frac{1}{2}[1 - 2k + 1 - k] = \frac{2}{3}$

$\frac{1}{2}[2 - 3k] = \frac{2}{3}$

$2 - 3k = \frac{4}{3}$

$-3k = -\frac{2}{3}$

$k = \frac{2}{9} \Rightarrow m = \frac{k}{1}$, $m = \frac{2}{9}$ [2]

OR

$A_2 = \frac{1}{2}A_2 = \frac{1}{3}$

$A_1 \Rightarrow \frac{1}{2}[2 - 3k] = \frac{1}{3}$

$A_1 = 2 - 3k = \frac{2}{3}$

$-3k = -\frac{4}{3}$

$k = \frac{4}{9}$

$m = \frac{k}{1}$, $m = \frac{4}{9}$ [2]

$V = \frac{25}{2}$, $t = 0$ $\dot{x} = V\cos\theta$, $\dot{y} = V\sin\theta$

$V\cos\theta$

$Vt\cos\theta$

-10

$-10t + C$

$C = V\sin\theta$

$\therefore -10t + V\sin\theta = -10t + \frac{25}{2}\sin\theta$

$-5t^2 + V\sin\theta + K$

$\therefore K = 0$ [2]

$= -5t^2 + Vt\sin\theta$

$= -10t^2 + \frac{25}{2}t\sin\theta$

Max Range @

$= 0$

$Vt\sin\theta - 5t^2 = 0$

$t(V\sin\theta - 5t) = 0$

$t = 0$ or $t = \frac{V\sin\theta}{5}$

$x_{max} = \frac{V^2\sin 2\theta}{10}$

$= \frac{25^2\sin 2\theta}{4 \cdot 40}$ [2]

max at $\theta = 45°$

$\therefore \frac{25^2}{40} = \frac{625}{40}$ 15.625m

iii) Max height when $\dot{y} = 0$ and $\theta = 90°$

$\dot{y} = 0 \Rightarrow V\sin\theta = 10t$

$t = \frac{V\sin\theta}{10}$

$= \frac{25}{2} \times \frac{1}{10} = \frac{25}{20} = \frac{5}{4}$

$\therefore y = -5 \times \frac{5^2}{16} + \frac{25}{2} \times \frac{5}{4} \times 1$

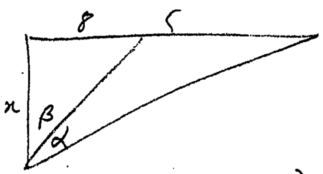
$= -\frac{125}{16} + \frac{125}{8}$

$= \frac{125}{8}$

$= 15.5/8$ m high [2]

b)

i)



$\tan\beta = \frac{8}{x}$

$\beta = \tan^{-1}\left(\frac{8}{x}\right)$

$\tan(\alpha + \beta) = \frac{13}{x}$

$\alpha + \beta = \tan^{-1}\left(\frac{13}{x}\right)$

$\therefore \alpha = \tan^{-1}\left(\frac{13}{x}\right) - \tan^{-1}\left(\frac{8}{x}\right)$ [1]

ii) $\frac{d\alpha}{dx} = \frac{\frac{-13}{x^2}}{1 + \frac{169}{x^2}} + \frac{\frac{8}{x^2}}{1 + \frac{64}{x^2}}$

$\frac{d\alpha}{dx} = \frac{-13}{x^2 + 169} + \frac{8}{x^2 + 64}$

$\alpha' = 0 \Rightarrow \frac{8}{x^2 + 64} = \frac{13}{x^2 + 169}$

$8(x^2 + 169) = 13(x^2 + 64)$

$8x^2 + 1352 = 13x^2 + 832$

$5x^2 = 520$

$x^2 = 104$

$x = \sqrt{104} = 2\sqrt{26}$ [5]

Test

x	$\sqrt{104}^-$	$\sqrt{104}$	$\sqrt{104}^+$
α'	+	0	−

$\therefore$ Max at $\sqrt{104}$

$\alpha(\sqrt{104}) = \tan^{-1}\left(\frac{13}{\sqrt{104}}\right) - \tan^{-1}\left(\frac{8}{\sqrt{104}}\right)$

$= .906 - .685$

$= .24$ radians

5