Ext 1

NORTH SYDNEY BOYS HIGH SCHOOL

2010 HSC ASSESSMENT TASK 3 (TRIAL HSC)

Mathematics
Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write on one side of the paper (with lines) in the booklet provided
- Write using blue or black pen
- Board approved calculators may be used
- All necessary working should be shown in every question
- Each new question is to be started on a **new page.**

- Attempt all questions

Class Teacher:
(Please tick or highlight)

- O Mr Ireland
- O Mr Lowe
- O Mr Rezcallah
- O Mr Barrett
- O Mr Trenwith
- O Mr Weiss

Student Number:

(To be used by the exam markers only.)

Question No	1	2	3	4	5	6	7	Total	Total
Mark	$\overline{12}$	$\overline{12}$	$\overline{12}$	$\overline{12}$	$\overline{12}$	$\overline{12}$	$\overline{12}$	$\overline{84}$	$\overline{100}$

Mathematics Extension 1 (HSC Course) Assessment Task 3 (Trial HSC)

Question 1

(a) Solve $\dfrac{x}{x-3} > 10$. **3**

(b) Find the acute angle between the lines $5x + 4y + 3 = 0$ and $3x + 8y - 1 = 0$. Give your answer correct to the nearest degree. **2**

(c) Show that $(x - 3)$ is a factor of the polynomial $f(x) = 2x^3 - 7x^2 - 7x + 30$, and find all linear factors of this polynomial. **3**

(d) Use the Table of Standard Integrals to evaluate $\displaystyle\int_0^{\frac{\pi}{6}} \sec 2x \tan 2x \, dx$. **2**

(e) Evaluate $\displaystyle\int_{-2}^{2\sqrt{3}} \frac{x}{x^2+1} \, dx$, leaving your answers in exact form. **2**

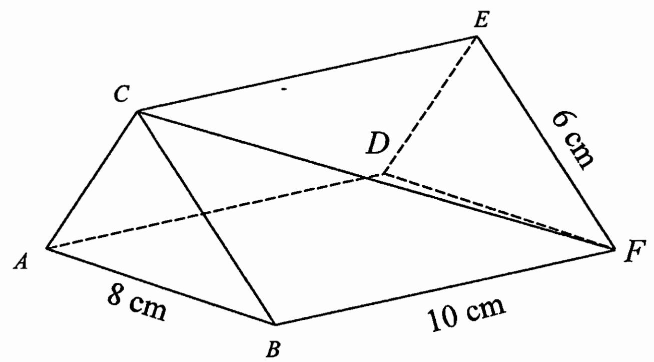
Question 2 (Start a new page)

(a) Use the substitution $u = 1 + x$ to evaluate $\int_{-1}^{3} x\sqrt{1+x}\,dx$. **3**

(b) (i) Show that the equation $\cos x = x$ has a root lying between $x = 0{\cdot}7$ and $x = 0{\cdot}8$. **1**

(ii) Using $x = 0{\cdot}75$ as a first approximation, use one application of Newton's Method to find a better approximation. Give your answer correct to 3 decimal places. **2**

(c) **3**



The roof above is a triangular prism, where the triangular faces are isosceles with $AC = BC$. $ABFD$ is horizontal. A snail walks along the roof in a straight line from F to C. What is the angle of elevation of its path?

(d) Air is pumped into a spherical balloon at a constant rate of $8\text{ cm}^3/\text{s}$. At what rate is the surface area of the balloon increasing when its volume is $24\pi\text{ cm}^3$? **3**

Question 3 (Start a new page)

(a) Find $\int \sin^2 3x \, dx$. **2**

(b) (i) Find the domain and range of the function $f(x) = 2\cos^{-1}(3 - 2x)$. **2**

(ii) Sketch a graph of the curve $y = f(x)$. **2**

(c) The polynomial $3x^3 - 2x^2 + 3x^2 - 4 = 0$ has zeros α, β and γ. **2**
Find the exact value of $\frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha}$.

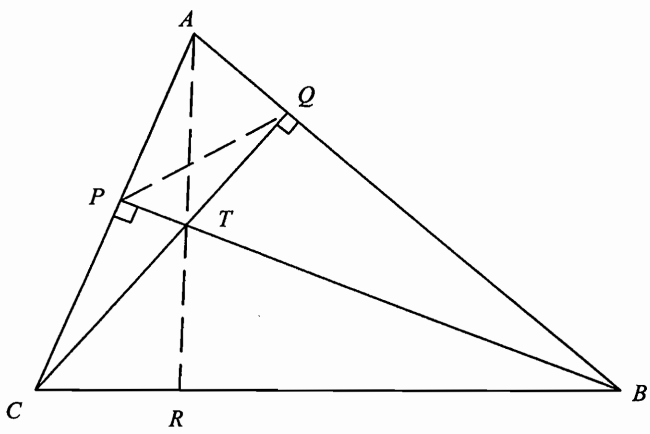
(d) (i) Differentiate $y = 2\cos^{-1}(3x)$ **2**

(ii) Find $\int \frac{dx}{\sqrt{9 - 4x^2}}$ **2**

Question 4 (Start a new page)

(a) $P(1, -3)$ divides AB externally in the ratio $2:3$, where B has coordinates $(4, 2)$. Find the coordinates of A. **2**

(b)



In the diagram, $BP \perp AC$, $CQ \perp AB$, and T is the point of concurrency of the lines CQ, BP and AR.

(i) State why $APTQ$ is cyclic. **1**

(ii) State why $CPQB$ is cyclic. **1**

(iii) Prove that $\angle TAQ = \angle QCB$. **2**

(iv) Prove that $AR \perp BC$. **2**

(c) (i) Differentiate $\dfrac{2x}{4+x^2} + \tan^{-1}\left(\dfrac{x}{2}\right)$. **2**

(ii) Hence evaluate $\displaystyle\int_0^2 \frac{dx}{(4+x^2)^2}$ **2**

Question 5 (Start a new page)

(a) (i) Write $\sin x - \cos x$ in the form $A\sin(x - \alpha)$. **2**

(ii) Hence solve the equation $\sin 2x - \cos 2x = 1$ for $0 \le x < 2\pi$. **2**

(b) A glass of water has a temperature of $4°C$, and is placed in a room with a temperature of $18°C$. The temperature of the water varies so that its rate of change is proportional to the difference between its temperature T and the temperature of the room at any time t minutes after the water is placed in the room.

(i) Show that the equation $T = A + Be^{-kt}$ satisfies the stated condition, where A, B and k are constants. **1**

(ii) After 5 minutes, the temperature of the water has risen to $10°C$. Find the values of A, B and k. **3**

(iii) Find the temperature of the water after a further 5 minutes have elapsed. **1**

(c) Find the exact value of $\sin^{-1}\dfrac{1}{\sqrt{5}} + \sin^{-1}\dfrac{1}{\sqrt{10}}$. Show all working. **3**

Question 6 (Start a new page)

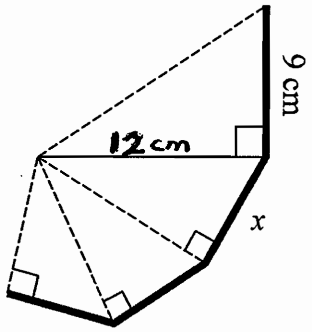
(a) A shell is fired at an angle α to the horizontal with an initial velocity of 75 m/s. It strikes a target on the same level as the gun, and 525 metres away. **6**

Let x be the horizontal displacement, and y the vertical displacement of the shell from the point of projection, and let g be the acceleration due to gravity.

(i) Using calculus, show that the position of the shell at any time t is given by $x = 75\,t\cos\alpha$, $y = 75\,t\sin\alpha - \frac{1}{2}gt^2$

(ii) Taking $g = 10$ m/s^2, show that there are two possible angles of elevation at which the gun can be fired. Find these angles correct to the nearest degree.

(b)



The diagram shows the first four segments of an **infinite** spiral, represented by the solid line. Each segment is one side of a right-angled triangle, and all such triangles are **similar**.

(i) Calculate the length of the side marked x. **2**

(ii) Find the length of the entire spiral. **1**

(c) The chance of a student catching a cold during the next school holiday is 0·2.

(i) What is the probability that three particular students all catch a cold during the next holiday? **1**

(ii) What is the probability that exactly two of three particular students catch a cold during the next school holidays? **2**

Question 7 (Start a new page)

(a) Use mathematical induction to show that for all positive integers, $n \geq 1$: **4**

$$\frac{3}{1 \times 2 \times 2^1} + \frac{4}{2 \times 3 \times 2^2} + \dots + \frac{n+2}{n(n+1)\cdot 2^n} = 1 - \frac{1}{(n+1)\cdot 2^n}$$

(b) $P\,(2ap, ap^2)$ and $Q\,(2aq, aq^2)$ are the points where the line $l:\ y = 3x + b$ meets the parabola $x^2 = 4ay$.

(i) Show that $p + q = 6$. **3**

(ii) The normal at P has equation $x + py = 2ap + ap^3$. **3**
(DO NOT prove this.)

Show that the normals at P and Q intersect at
$N\left[-apq(p+q),\ a(p^2+q^2+pq+2)\right]$

(iii) Show that the locus of N as the line l varies has equation: **2**
$x - 6y + 228a = 0$.

Mathematics Extension 1 (HSC Course) Assessment Task 3 (Trial HSC)

Question 1

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(a) $\frac{x}{x-3} > 10$

$x(x-3) > 10(x-3)^2$

$(x-3)[10(x-3)-x] < 0$

$3(x-3)(3x-10) < 0$

$3 < x < \frac{10}{3}$

3

(b) $m_1 = -\frac{5}{4}$

$m_2 = -\frac{3}{8}$

$\tan\theta = \left|\frac{-\frac{5}{4}-(-\frac{3}{8})}{1+(-\frac{5}{4})(-\frac{3}{8})}\right| \times \frac{32}{32}$

$= \left|\frac{-40+12}{32+15}\right|$

$= \frac{28}{47}$

$\theta = 31°$ (nearest degree)

2

(c) $f(x) = 2x^3 - 7x^2 - 7x + 30$

$f(3) = 2(27) - 7(9) - 7(3) + 30$

$= 0$

$\therefore x-3$ is a factor of $f(x)$

let roots be $3, \alpha, \beta$

$\therefore 3\alpha\beta = -15$ $\qquad 3+\alpha+\beta = \frac{7}{2}$

$\alpha\beta = -5$ $\qquad \alpha+\beta = \frac{1}{2}$

$x^2 - \frac{1}{2}x - 5 = 0$

$2x^2 - x - 10 = 0$

$(2x-5)\ (x+2) = 0$

$x = \frac{5}{2}, -2$

$\therefore f(x) = (x-3)(x+2)(2x-5)$

/3

(d) $\int_0^{\pi/6} \sec 2x \tan 2x \, dx = \frac{1}{2}\left[\sec 2x\right]_0^{\pi/6}$

$= \frac{1}{2}\left(\sec\frac{\pi}{3} - \sec 0\right)$

$= \frac{1}{2}(2-1)$

$= \frac{1}{2}$

/2

(e) $\int_{-2}^{2\sqrt{3}} \frac{x}{x^2+1} \, dx = \frac{1}{2}\left[\ln(x^2+1)\right]_{-2}^{2\sqrt{3}}$

$= \frac{1}{2}(\ln 13 - \ln 5)$

$= \frac{1}{2}\ln\frac{13}{5}$

/2

Question 2

(a) $\displaystyle\int_{-1}^{3} x\sqrt{1+x}\,dx$

$u = 1 + x$

$du = dx$

$x = -1,\ u = 0$

$x = 3,\ u = 4$

$= \displaystyle\int_0^4 (u-1)\sqrt{u}\,du$

$= \displaystyle\int_0^4 \left(u^{3/2} - u^{1/2}\right) du$

$= \left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_0^4$

$= \frac{64}{5} - \frac{16}{3}$

$= \frac{112}{15}$ 3

(b) (i) let $f(x) = \cos x - x$

$f(0.7) = 0.065$

$f(0.8) = -0.103$

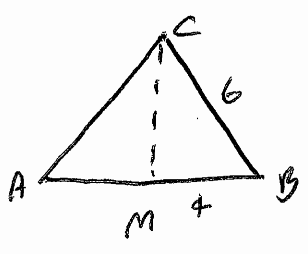
$\therefore f(x)$ changes sign $\Rightarrow$ root between $x = 0.7$ & $x = 0.8$ 1

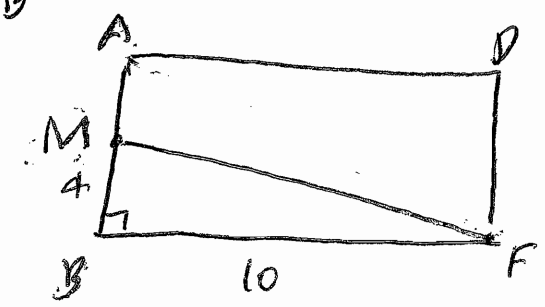
(ii) $f'(x) = -\sin x - 1$

$x_1 = 0.75 - \dfrac{\cos 0.75 - 0.75}{-\sin 0.75 - 1}$

$= 0.739$ 2

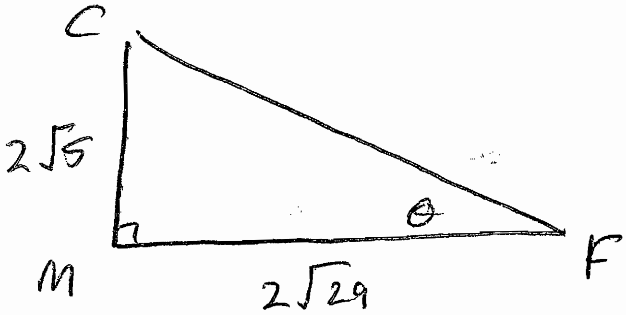
(c) let M be midpt of AB





$CM = 2\sqrt{5}$ (Pythagoras)

$AF = 2\sqrt{29}$ (Pythagoras)



$$\tan\theta = \frac{\sqrt{5}}{\sqrt{29}}$$

$$\theta = 22^\circ 33'$$

3

(d) $$\frac{dV}{dt} = 8$$

$$V = \frac{4}{3}\pi r^3 \qquad S = 4\pi r^2$$

$$\frac{dV}{dr} = 4\pi r^2 \qquad \frac{dS}{dr} = 8\pi r$$

$$\frac{dS}{dt} = \frac{dS}{dr} \cdot \frac{dr}{dV} \cdot \frac{dV}{dt}$$

$$= 8\pi r \cdot \frac{1}{4\pi r^2} \cdot 8$$

$$= \frac{16}{r}$$

$$24\cancel{\pi} = \frac{4}{3}\cancel{\pi} r^2$$

$$r^2 = 18$$

$$r = \sqrt{18}$$

$$\frac{dS}{dt} = \frac{16}{\sqrt{18}}$$

$$= 3.77 \text{ cm}^2/\text{s}.$$

3

Question 3

(a) $\int \sin^2 3x \, dx = \frac{1}{2}\int (1-\cos 6x)\, dx$

$\cos 6x = 1 - 2\sin^2 3x$

$\sin^2 3x = \frac{1}{2}(1-\cos 6x)$

$= \frac{1}{2}\left[x - \frac{1}{6}\sin 6x\right] + c$

$= \frac{x}{2} - \frac{1}{12}\sin 6x + c$

(b) (i) Domain: $-1 \le 3-2x \le 1$

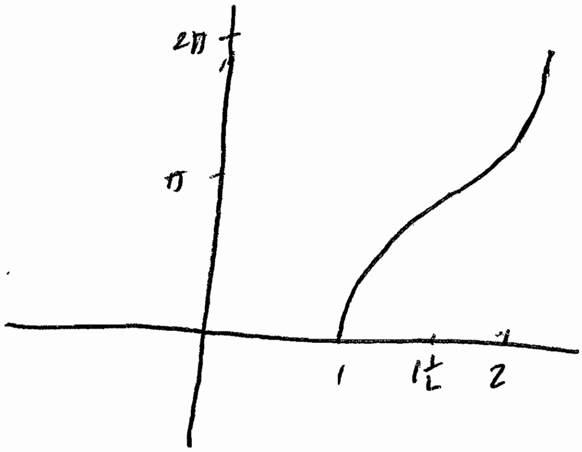
$-4 \le -2x \le -2$

$2 \ge x \ge 1$

$1 \le x \le 2$

Range: $0 \le y \le 2\pi$

(ii)



(c) $\frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha} = \frac{\alpha+\beta+\gamma}{\alpha\beta\gamma}$

$= \frac{2/3}{4/3}$

$= \frac{1}{2}$

(d) (i) $y = 2\cos^{-1}3x$

$$\frac{dy}{dx} = \frac{-2}{\sqrt{1-9x^2}} \times 3$$

$$= \frac{-6}{\sqrt{1-9x^2}}$$

(ii) $$\int \frac{dx}{\sqrt{9-4x^2}} = \frac{1}{2}\int \frac{dx}{\sqrt{\frac{9}{4}-x^2}}$$

$$= \frac{1}{2}\sin^{-1}\frac{2x}{3} + c$$

Question 4

(a) $A(x_1, y_1)$ $B(4, 2)$ $P(1, -3)$ $m:n = 2:-3$

$$1 = \frac{-3x_1 + 2(4)}{2-3} \qquad -3 = \frac{-3y_1 + 2(2)}{2-3}$$

$$-3x_1 + 8 = -1 \qquad -3y_1 + 4 = 3$$

$$3x_1 = 9 \qquad 3y_1 = 1$$

$$x_1 = 3 \qquad y_1 = \frac{1}{3}$$

$$\therefore A\left(3, \frac{1}{3}\right)$$

(b) (i) Opposite angles supplementary

(ii) CB subtends equal angles at P & Q

(iii) $T\hat{A}Q = T\hat{P}Q$ (angles subtended by chord QT in circle PAQT)

$T\hat{P}Q = Q\hat{C}B$ (" " " " QB " " CPQB)

$\therefore T\hat{A}Q = Q\hat{C}B$

(iv) In Δ's ARB & QQB

$A\hat{B}R = C\hat{B}Q$ (common)

$R\hat{A}B = Q\hat{C}B$ (from part (iii))

$\therefore A\hat{R}B = C\hat{Q}B$ (angle sum of Δ)

$= 90°$

ie. $AR \perp BC$

(c) (i) $\dfrac{d}{dx}\left[\dfrac{2x}{4+x^2} + \tan^{-1}\dfrac{x}{2}\right]$

$$= \frac{(4+x^2)\cdot 2 - 2x \cdot 2x}{(4+x^2)^2} + \frac{1}{1+\frac{x^2}{4}} \cdot \frac{1}{2}$$

$$= \frac{8+2x^2 - 4x^2}{(4+x^2)^2} + \frac{2}{4+x^2}$$

$$= \frac{8-2x^2 + 2(4+x^2)}{(4+x^2)^2}$$

$$= \frac{16}{(4+x^2)^2}$$

(ii) $\displaystyle\int_0^2 \frac{dx}{(4+x^2)^2} = \frac{1}{16}\int_0^2 \frac{16}{(4+x^2)^2}\,dx$

$$= \frac{1}{16}\left[\frac{2x}{4+x^2} + \tan^{-1}\frac{x}{2}\right]_0^2$$

$$= \frac{1}{16}\left[\left(\frac{1}{2}+\frac{\pi}{4}\right) - (0+0)\right]$$

$$= \frac{\pi+2}{64}$$

Question 5

(a) (i) let $\sin x - \cos x = A\sin(x - \alpha)$

$$= A(\sin x \cos\alpha - \cos x \sin\alpha)$$

$$= (A\cos\alpha)\sin x - (A\sin\alpha)\cos x$$

$\therefore A\cos\alpha = 1$ — ① $A\sin\alpha = 1$ — ②

① ÷ ② : $\tan\alpha = 1$

$\alpha = \frac{\pi}{4}$ (1st quad)

①^2 + ②^2 : $A^2 = 2$

$A = \sqrt{2}$

$\therefore \sin x - \cos x = \sqrt{2}\sin\left(x - \frac{\pi}{4}\right)$

(ii) $\sin 2x - \cos 2x = 1$

$\sqrt{2}\sin\left(2x - \frac{\pi}{4}\right) = 1$

$2x - \frac{\pi}{4} = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}$

$2x = \frac{\pi}{2}, \pi, \frac{5\pi}{2}, 3\pi$

$x = \frac{\pi}{4}, \frac{\pi}{2}, \frac{5\pi}{4}, \frac{3\pi}{2}$

(b) (i) $T = A + Be^{-kt}$

$\frac{dT}{dt} = -kBe^{-kt}$

$= -k(T - A)$

(ii) $\underline{A = 18}$

$4 = 18 + B$

$\underline{B = -14}$

$10 = 18 - 14e^{-5k}$

$e^{-5k} = \frac{4}{7}$

$\underline{k = -\frac{1}{5}\ln\frac{4}{7}} = 0.1119$

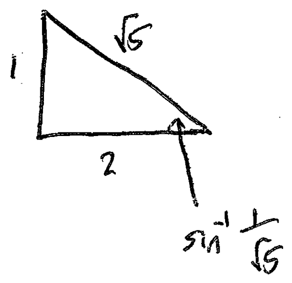
(iii) $t = 10$, $T = 18 - 14e^{2\ln\frac{4}{7}}$

$= 18 - 14\left(\frac{4}{7}\right)^2$

$= 18 - 14 \cdot \frac{16}{49}$

$= 13\frac{3}{7}°$

(c) $\sin\left[\sin^{-1}\frac{1}{\sqrt{5}} + \sin^{-1}\frac{1}{\sqrt{10}}\right] = \sin\left(\sin^{-1}\frac{1}{\sqrt{5}}\right)\cos\left(\sin^{-1}\frac{1}{\sqrt{10}}\right) + \cos\left(\sin^{-1}\left(\frac{1}{\sqrt{5}}\right)\right)\sin\left(\sin^{-1}\left(\frac{1}{\sqrt{10}}\right)\right)$

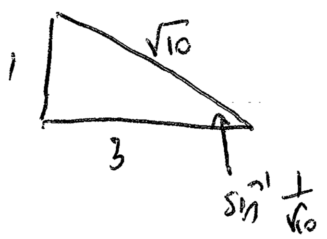


$= \frac{1}{\sqrt{5}} \cdot \frac{3}{\sqrt{10}} + \frac{2}{\sqrt{5}} \cdot \frac{1}{\sqrt{10}}$

No working

= No marks

$= \frac{3 + 2}{\sqrt{50}}$



$= \frac{5}{5\sqrt{2}}$

$= \frac{1}{\sqrt{2}}$

$\therefore \sin^{-1}\frac{1}{\sqrt{5}} + \sin^{-1}\frac{1}{\sqrt{10}} = \frac{\pi}{4}$ //

Question 6

(a) (i) $\ddot{x} = 0$

$\dot{x} = c$

$= 75\cos\alpha$

$x = 75t\cos\alpha + c$

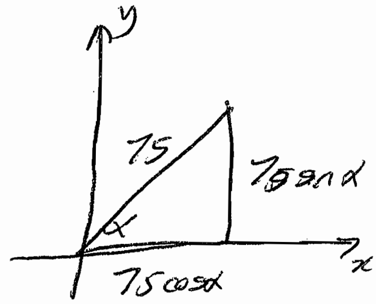
$= 75t\cos\alpha$

$\ddot{y} = -g$

$\dot{y} = -gt + c$

$= -gt + 75\sin\alpha$

$y = -\frac{1}{2}gt^2 + 75t\sin\alpha$



(ii) $x = 75t\cos\alpha$

$t = \dfrac{x}{75\cos\alpha}$

$y = -5t^2 + 75t\sin\alpha$

$= -5\left(\dfrac{x}{75\cos\alpha}\right)^2 + 75\left(\dfrac{x}{75\cos\alpha}\right)\sin\alpha$

$= \dfrac{-x^2}{1125}(1 + \tan^2\alpha) + x\tan\alpha$

$(525, 0)$: $\quad 0 = -\dfrac{525^2}{1125}(1 + \tan^2\alpha) + 525\tan\alpha$

$0 = -\dfrac{525}{1125} - \dfrac{525}{1125}\tan^2\alpha + \tan\alpha$

$525\tan^2\alpha - 1125\tan\alpha + 525 = 0$

$7\tan^2\alpha - 15\tan\alpha + 7 = 0$

$\tan\alpha = \dfrac{15 \pm \sqrt{29}}{14}$

$\alpha = 56°, 34°$

(b) (i) 3rd side of largest Δ = 15 cm (Pythagoras)

$\therefore \dfrac{15}{12} = \dfrac{9}{x}$

$x = 7.2$

(ii) $r = \dfrac{7.2}{9} = \dfrac{4}{5} \implies S_\infty = \dfrac{9}{1 - \frac{4}{5}} = \underline{45 \text{ cm}}$

(c) (i) $0.2^3 = 0.008$

(ii) $0.2 \times 0.2 \times 0.8 \times 3 = 0.096$

Question 7

(a) Test $n=1$: $LHS = \frac{3}{1\times2\times2} = \frac{3}{4}$

$RHS = 1 - \frac{1}{2\cdot2^1} = \frac{3}{4}$

$\therefore$ true for $n=1$

Assume true for n

Prove true for $n+1$

$$LHS = \frac{3}{1\times2\times2^1} + \frac{4}{2\times3\times3^2} + \dots + \frac{n+2}{n(n+1)\cdot2^n} + \frac{n+3}{(n+1)(n+2)\cdot2^{n+1}}$$

$$= 1 - \frac{1}{(n+1)\cdot2^n} + \frac{n+3}{(n+1)(n+2)\cdot2^{n+1}} \quad \text{(by assumption)}$$

$$= 1 - \frac{2(n+2)}{(n+1)(n+2)\cdot2^{n+1}} + \frac{n+3}{(n+1)(n+2)\cdot2^{n+1}}$$

$$= 1 - \frac{2n+4-n-3}{(n+1)(n+2)\cdot2^{n+1}}$$

$$= 1 - \frac{n+1}{(n+1)(n+2)\cdot2^{n+1}}$$

$$= 1 - \frac{1}{(n+2)\cdot2^{n+1}}$$

$\therefore$ ... conclusion.

(b) (i) $m_{SQ} = 3$

$$\frac{ap^2 - aq^2}{2ap - 2aq} = 3$$

$$\frac{\cancel{a}\cancel{(p-q)}(p+q)}{2\cancel{a}\cancel{(p-q)}} = 3$$

$$p+q = 6$$

(ii) $x + py = 2ap + ap^3$

$x + qy = 2aq + aq^3$

$-$ $\cancel{(p-q)}y = 2a\cancel{(p-q)} + a\cancel{(p-q)}(p^2+pq+q^2)$

$y = a(p^2+q^2+pq+2)$

$x + ap(p^2+q^2+pq+2) = 2ap + ap^3$

$x + \cancel{ap^3} + apq^2 + ap^2q + \cancel{2ap} = \cancel{2ap} + \cancel{ap^3}$

$x = -apq(p+q)$

$\therefore N[-apq(p+q),\ a(p^2+q^2+pq+2)]$

(iii) $x = -6apq$ $\quad y = a[(p+q)^2 - pq + 2]$

$pq = -\dfrac{x}{6a}$ $\quad y = a\left[36 + \dfrac{x}{6a} + 2\right]$

$= 36a + \dfrac{x}{6} + 2a$

$= \dfrac{x}{6} + 38a$

$6y = x + 228a$

$x - 6y + 228a = 0$