100 Years

NORTH SYDNEY BOYS HIGH SCHOOL

1912 ~ 2012

# MATHEMATICS (EXTENSION 1)

2013 HSC Course Assessment Task 3 (Trial Examination)

June 26, 2013

**General instructions**

- Working time – 2 hours. (plus 5 minutes reading time)
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- Board approved calculators may be used.
- Attempt **all** questions.
- At the conclusion of the examination, bundle the booklets + answer sheet used in the correct order within this paper and hand to examination supervisors.

**SECTION I**

- Mark your answers on the answer sheet provided (numbered as page 9)

**SECTION II**

- Commence each new question on a new page. Write on both sides of the paper.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.

**STUDENT NUMBER:** .......................... **# BOOKLETS USED:** .....

**Class** (please ✔)

- ◯ 12M3A – Mr Lam
- ◯ 12M3B – Mr Berry
- ◯ 12M3C – Mr Lin
- ◯ 12M4A – Mr Fletcher
- ◯ 12M4B – Ms Beevers
- ◯ 12M4C – Ms Ziaziaris

Marker's use only.

| QUESTION | 1-10 | 11 | 12 | 13 | 14 | Total | % |
|---|---|---|---|---|---|---|---|
| MARKS | /10 | /15 | /15 | /15 | /15 | /70 | |

# Section I

**10 marks**
**Attempt Question 1 to 10**
**Allow approximately 15 minutes for this section**

Mark your answers on the answer sheet provided.

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**Questions** **Marks**

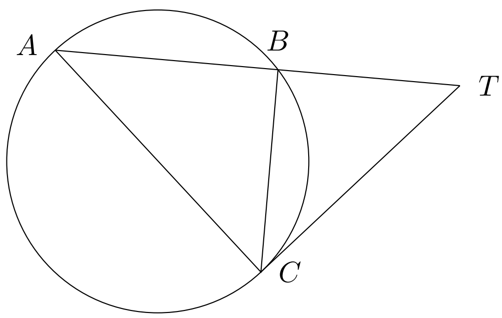
**1.** What is the acute angle between the lines $y = 2x - 3$ and $3x + 5y - 1 = 0$, correct to the nearest degree? **1**

(A) $32°$ (B) $50°$ (C) $82°$ (D) $86°$

**2.** The coordinates of the points that divide the interval joining $(-7, 5)$ and $(-1, -7)$ externally in the ratio $1 : 3$ are **1**

(A) $(-10, 8)$ (B) $(-10, 11)$ (C) $(2, 8)$ (D) $(2, 11)$

**3.** $CT$ is a tangent to the circle. $AB$ is a secant, intersecting the circle at $A$ and $B$. $AB$ intersects $CT$ at $T$. **1**




Which of the following statements is correct?

(A) $CT^2 = AC \times BC$

(B) $CT^2 = AB \times BC$

(C) $CT^2 = AC \times BT$

(D) $CT^2 = AT \times BT$

**4.** What is the domain and range of $y = \cos^{-1}\left(\dfrac{3x}{2}\right)$? **1**

(A) $D = \left\{x : -\dfrac{2}{3} \le x \le \dfrac{2}{3}\right\}$, $R = \{y : 0 \le y \le \pi\}$

(B) $D = \{x : -1 \le x \le 1\}$, $R = \{y : 0 \le y \le \pi\}$

(C) $D = \left\{x : -\dfrac{2}{3} \le x \le \dfrac{2}{3}\right\}$, $R = \{y : -\pi \le y \le \pi\}$

(D) $D = \{x : -1 \le x \le 1\}$, $R = \{y : -\pi \le y \le \pi\}$

**5.** Which of the following is equivalent to $\displaystyle\int \frac{dx}{4x^2+9}$, ignoring the constant of integration? **1**

(A) $\tan^{-1}\dfrac{2x}{3}$

(B) $\dfrac{1}{6}\tan^{-1}\dfrac{2x}{3}$

(C) $\dfrac{2}{3}\tan^{-1}\dfrac{2x}{3}$

(D) $\dfrac{3}{2}\tan^{-1}\dfrac{2x}{3}$

**6.** Which of the following is the general solution to $3\tan^2 x - 1 = 0$? **1**

(A) $n\pi \pm \dfrac{\pi}{6}$

(B) $2n\pi \pm \dfrac{\pi}{6}$

(C) $n\pi \pm \dfrac{\pi}{3}$

(D) $2n\pi \pm \dfrac{\pi}{3}$

**7.** What is the value of $\sin(\alpha+\beta)$ if $\sin\alpha = \dfrac{8}{17}$ and $\sin\beta = \dfrac{4}{5}$? **1**

(A) $\dfrac{36}{85}$

(B) $\dfrac{77}{85}$

(C) $\dfrac{84}{85}$

(D) $\dfrac{88}{85}$

**8.** Which of the following represents the exact value of $\displaystyle\int_0^{\frac{\pi}{8}} \cos^2 x\,dx$? **1**

(A) $\dfrac{\pi - 2\sqrt{2}}{16}$

(B) $\dfrac{\pi - 2\sqrt{2}}{8}$

(C) $\dfrac{\pi + 2\sqrt{2}}{16}$

(D) $\dfrac{\pi + 2\sqrt{2}}{8}$

**9.** Which of the following represents the derivative of $y = \cos^{-1}\left(\dfrac{1}{x}\right)$? **1**

(A) $-\dfrac{1}{\sqrt{x^2-1}}$

(B) $-\dfrac{1}{x\sqrt{x^2-1}}$

(C) $\dfrac{1}{\sqrt{x^2-1}}$

(D) $\dfrac{1}{x\sqrt{x^2-1}}$

**10.** Which of the following expressions is true? **1**

(A) $\tan^{-1} x = \sin^{-1}\dfrac{1}{\sqrt{1-x^2}}$

(B) $\tan^{-1} x = \sin^{-1}\dfrac{1}{\sqrt{1+x^2}}$

(C) $\tan^{-1} x = \sin^{-1}\dfrac{x}{\sqrt{1-x^2}}$

(D) $\tan^{-1} x = \sin^{-1}\dfrac{x}{\sqrt{1+x^2}}$

**Examination continues overleaf. . .**

# Section II

**60 marks**
**Attempt Questions 11 to 14**
**Allow approximately 1 hour and 45 minutes for this section.**

Write your answers in the writing booklets supplied. Additional writing booklets are available. Your responses should include relevant mathematical reasoning and/or calculations.

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**Question 11** (15 Marks) Commence a NEW page. **Marks**

(a) Solve for $x$: $\dfrac{5}{x-1} > 2$ **2**

(b) Find the value of $\theta$, such that **4**

$$\sqrt{3}\cos\theta - \sin\theta = 1$$

where $0 \leq \theta \leq 2\pi$.

(c) A freshly baked cake is cooling in a room of constant temperature of 20°C. At time $t$ minutes, its temperature $T$ decreases according to the equation

$$\frac{dT}{dt} = -k(T - 20)$$

where $k$ is a positive constant.

The initial temperature of the cake is 150°C and it cools to 100°C after 15 minutes.

i. Show that $T = 20 + Ae^{-kt}$ (where $A$ is a constant) is a solution to this equation. **1**

ii. Find the values $A$ and $k$, giving $k$ correct to 3 decimal places. **2**

iii. How long will it take for the cake to cool to 25°C? **2**
(Use the value of $k$ obtained in the previous part)

(d) A particle is moving in a straight line. After time $t$ seconds, it has displacement $x$ metres from a fixed point $O$ on the line. Its velocity $v$ $\text{ms}^{-1}$ is given by

$$v = \sqrt{x}$$

Initially, the particle is 1 m to the right of $O$.

i. Show that its acceleration $a$ is independent of $x$. **1**

ii. Express $x$ in terms of $t$. **2**

iii. Find the distance travelled by the particle during the third second of its motion. **1**

**Question 12** (15 Marks) Commence a NEW page. **Marks**

(a) Show that $\dfrac{\sin x}{1-\cos x} = \cot \dfrac{x}{2}$. **2**

(b) Use the substitution $u = \sin^2 x$ to evaluate **4**

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sin 2x}{1+\sin^2 x}\,dx$$

Give your answer in simplest exact form.

(c) Let $f(x) = x^2 - 2x$ for $x \geq 1$.

i. On the same set of axes, sketch the graphs of $y = f(x)$, $y = x$ and the inverse $y = f^{-1}(x)$. **3**

ii. Find an expression for $f^{-1}(x)$. **2**

iii. Evaluate $f^{-1}(2)$. **1**

(d) Use mathematical induction to show that for all positive integers $n \geq 2$, **3**

$$2 \times 1 + 3 \times 2 + 4 \times 3 + \cdots + n(n-1) = \frac{n(n^2-1)}{3}$$

**Question 13** (15 Marks) Commence a NEW page. **Marks**

(a) i. Show that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$. **2**

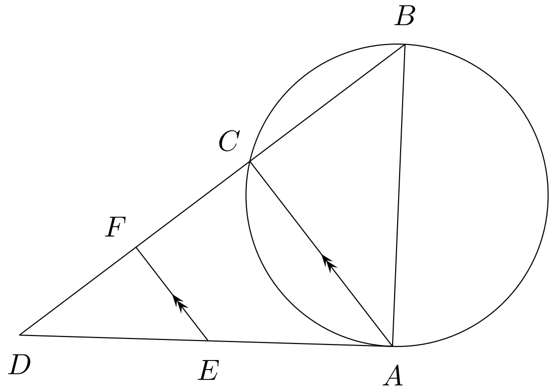
ii. Hence evaluate $\int 2\sin^3\theta \, d\theta$ **1**

(b) $P(x) = ax^3 - 7x^2 + kx + 4$ has $x - 4$ as a factor. When $P(x)$ is divided by $(x - 1)$, the remainder is $-6$.

i. Determine the values of $a$ and $k$. **2**

ii. Evaluate $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$. **2**

(c) $AB$ is a diameter of the circle and $C$ is a point on the circle. The tangent to the circle at $A$ meets $BC$ produced at $D$. $E$ is a point on $AD$ and $F$ is a point on $CD$ such that $EF \parallel AC$.

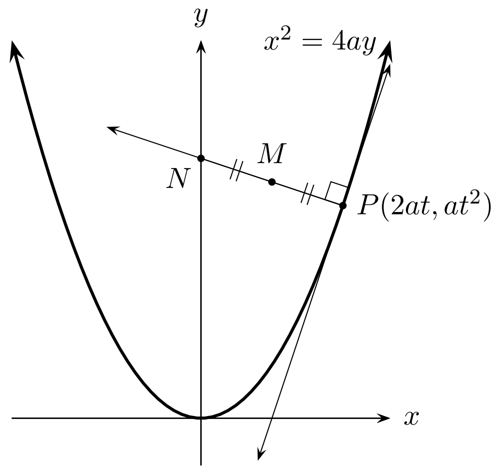



i. State why $\angle EAC = \angle ABC$. **1**

ii. Hence show that $EABF$ is a cyclic quadrilateral. **2**

iii. Show that $BE$ is a diameter of the circle through $E$, $A$, $B$ and $F$. **1**

(d) $P(2at, at^2)$ is a point on the parabola $x^2 = 4ay$. The normal to the parabola at $P$ cuts the $y$ axis at $N$. $M$ is the midpoint of $PN$.




i. Show that the normal has equation $x + ty = 2at + at^3$. **2**

ii. Find the equation of the locus of $M$ as $P$ moves on the parabola. **2**

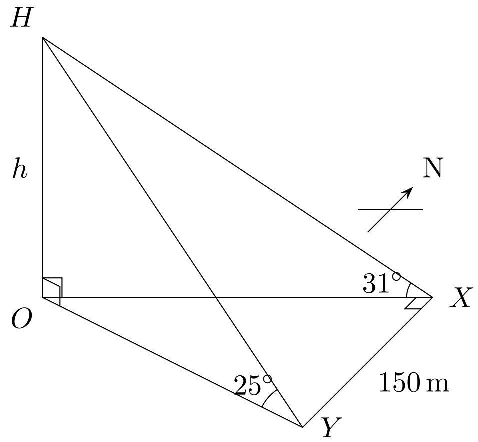
**Question 14** (15 Marks) Commence a NEW page. **Marks**

(a) The radius of a spherical balloon is expanding at a constant rate of $6\,\text{cms}^{-1}$.

i. At what rate is the volume of the balloon expanding when its radius is $4\,\text{cm}$? **2**

ii. At what rate is the surface area of the balloon expanding if the rate of change of volume is $15\,\text{cm}^3\text{s}^{-1}$? **3**

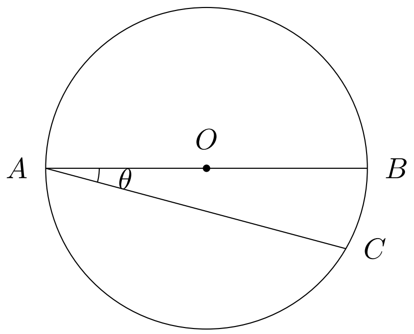
(b) From a point $X$ due east of a tower, the angle of elevation of the top of the tower $H$ is $31°$. From another point $Y$ due south of $X$, the angle of elevation is $25°$. $XY = 150\,\text{m}$.




i. Show that $h = \dfrac{150 \tan 25° \tan 31°}{\sqrt{\tan^2 31° - \tan^2 25°}}$. **3**

ii. Hence, find the height of the tower. **1**

(c) $AOB$ is the diameter of the circle with centre $O$ and radius 1 metre. $AC$ is a chord of the circle such that $\angle BAC = \theta$, where $0 < \theta < \frac{\pi}{2}$.




The area of the part of the circle contained between $AB$ and $AC$ is equal to one quarter of the area of the circle.

i. Show that $\theta + \dfrac{1}{2}\sin 2\theta - \dfrac{\pi}{4} = 0$. **2**

ii. Show that $0.4 < \theta < 0.5$. **2**

iii. Use one application of Newton's Method with an initial approximation of $\theta_0 = 0.4$ to find the next approximation to $\theta$, correct to 2 decimal places. **2**

**End of paper.**

**STANDARD INTEGRALS**

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C, \qquad n \neq -1; \quad x \neq 0 \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x + C, \qquad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C, \qquad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax + C, \qquad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C, \qquad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C, \qquad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax + C, \qquad a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C, \qquad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a} + C, \qquad a > 0, -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right) + C, \qquad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right) + C$$

NOTE: $\ln x = \log_e x,\ x > 0$

## Answer sheet for Section I

Mark answers to Section I by fully blackening the correct circle, e.g "●"

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- ◯ 12M4C – Ms Ziaziaris

**1 –** Ⓐ Ⓑ Ⓒ Ⓓ
**2 –** Ⓐ Ⓑ Ⓒ Ⓓ
**3 –** Ⓐ Ⓑ Ⓒ Ⓓ
**4 –** Ⓐ Ⓑ Ⓒ Ⓓ
**5 –** Ⓐ Ⓑ Ⓒ Ⓓ
**6 –** Ⓐ Ⓑ Ⓒ Ⓓ
**7 –** Ⓐ Ⓑ Ⓒ Ⓓ
**8 –** Ⓐ Ⓑ Ⓒ Ⓓ
**9 –** Ⓐ Ⓑ Ⓒ Ⓓ
**10 –** Ⓐ Ⓑ Ⓒ Ⓓ

# Suggested Solutions

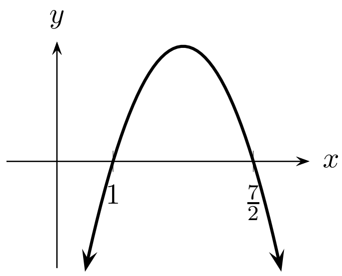
## Section I

**1.** (D) **2.** (B) **3.** (D) **4.** (A) **5.** (B) **6.** (A) **7.** (C)
**8.** (C) **9.** (D) **10.** (D)

## Question 11 (Ziaziaris)

(a) (2 marks)

$$\begin{aligned} \frac{5}{x-1} &> 2 \\ 5(x-1) &> 2(x-1)^2 \\ 5(x-1) - 2(x-1)^2 &> 0 \\ (x-1)\left(5-2(x-1)\right) &> 0 \\ (x-1)(7-2x) &> 0 \end{aligned}$$




$$\therefore 1 < x < \frac{7}{2}$$

(b) (4 marks)

$$\begin{aligned} \sqrt{3}\cos\theta - \sin\theta &= 1 \\ &\equiv R\cos(\theta+\alpha) \\ &= R\cos\theta\cos\alpha - R\sin\theta\sin\alpha \end{aligned}$$

$$\begin{cases} R\cos\alpha = \sqrt{3} & (1) \\ R\sin\alpha = 1 & (2) \end{cases}$$

$(1)^2 + (2)^2$:

$$\begin{aligned} R^2 &= 3+1 = 4 \\ \therefore R &= 2 \end{aligned}$$

(2) ÷ (1):

$$\begin{aligned} \tan\alpha &= \frac{1}{\sqrt{3}} \\ \therefore \alpha &= \frac{\pi}{6} \\ \therefore \sqrt{3}\cos\theta - \sin\theta = 2\cos\left(\theta + \frac{\pi}{3}\right) &= 1 \\ \therefore \cos\left(\theta+\frac{\pi}{6}\right) &= \frac{1}{2} \\ \theta + \frac{\pi}{6} &= \frac{\pi}{3}, \frac{5\pi}{3} \\ \therefore \theta &= \frac{\pi}{6}, \frac{3\pi}{2} \end{aligned}$$

(c) i. (1 mark)

$$\begin{aligned} T &= 20 + Ae^{-kt} \\ \therefore T - 20 &= Ae^{-kt} \\ \frac{dT}{dt} &= -k \cdot Ae^{-kt} \\ &= -k(T-20) \end{aligned}$$

ii. (2 marks)

$$T = 20 + Ae^{-kt}$$

When $t = 0$, $T = 150$:

$$\begin{aligned} 150 &= 20 + Ae^0 \\ \therefore A &= 130 \\ \therefore T &= 20 + 130e^{-kt} \end{aligned}$$

When $t = 15$, $T = 100$

$$\begin{aligned} 100 &= 20 + 130e^{-15k} \\ \frac{80}{130} &= e^{-15k} \\ -15k &= \log_e \frac{8}{13} \\ k = -\frac{1}{15}\log_e \frac{8}{13} &\approx 0.032 \text{ (3 dp)} \end{aligned}$$

iii. (2 marks)

$$\begin{aligned} 25 &= 20 + 130e^{-0.032t} \\ \frac{5}{13} &= e^{-0.032t} \\ -0.032t &= \log_e \frac{5}{13} \\ t &= \frac{1}{-0.032}\log_e \frac{5}{13} \\ &\approx 100.66\,\text{min} \end{aligned}$$

If exact values are used,

$$\begin{aligned} t &\approx 101.82\,\text{min} \\ &= 101\,\text{min}49\,\text{s} \\ &\approx 102\,\text{min} \end{aligned}$$

(d) i. (1 mark)

$$\begin{aligned} v &= \sqrt{x} \\ \therefore v^2 &= x \\ \ddot{x} &= \frac{d}{dx}\left(\frac{1}{2}v^2\right) \\ &= \frac{d}{dx}\left(\frac{1}{2}x\right) \\ &= \frac{1}{2} \end{aligned}$$

which is independent of $x$.

ii. (2 marks)

$$\frac{dx}{dt} = x^{\frac{1}{2}}$$

Separating variables,

$$\begin{aligned} \frac{dx}{x^{\frac{1}{2}}} &= dt \\ \int x^{-\frac{1}{2}}\,dx &= \int dt \\ \therefore t &= 2x^{\frac{1}{2}} + C \end{aligned}$$

When $t = 0$, $x = 1$.

$$\begin{aligned} 0 &= 2 + C \\ \therefore C &= -2 \\ \therefore t &= 2x^{\frac{1}{2}} - 2 \\ 2\sqrt{x} &= t + 2 \\ \therefore x &= \left(\frac{t+2}{2}\right)^2 \quad \left(= \frac{1}{4}t^2 + t + 1\right) \end{aligned}$$

iii. (1 mark)

$$\begin{aligned} x(2) &= \left(\frac{2+2}{2}\right)^2 = 4 \\ x(3) &= \left(\frac{3+2}{2}\right)^2 = \frac{25}{4} \\ \text{Dist travelled } &= \frac{25}{4} - 4 = 2.25\,\text{m} \end{aligned}$$

**Question 12** (Lin)

(a) (2 marks)
Let $t = \tan\left(\frac{\theta}{2}\right)$:

$$\begin{aligned} \frac{\sin x}{1 - \cos x} &= \frac{\frac{2t}{1+t^2}}{1 - \left(\frac{1-t^2}{1+t^2}\right)} \\ &= \frac{\frac{2t}{\cancel{1+t^2}}}{\frac{1+t^2-(1-t^2)}{\cancel{1+t^2}}} \\ &= \frac{2t}{2t^2} \\ &= \frac{1}{t} = \cot\frac{x}{2} \end{aligned}$$

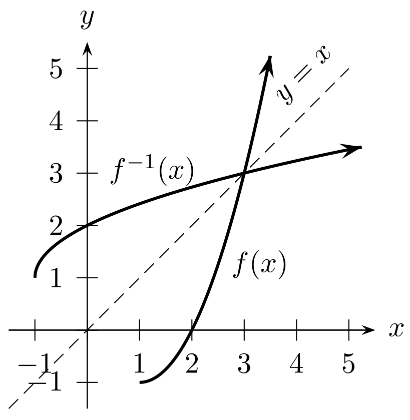
(b) (4 marks)

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sin 2x}{1 + \sin^2 x}\,dx$$

Let $u = \sin^2 x$:

$$\begin{aligned} \frac{du}{dx} &= 2\sin x\cos x = \sin 2x \\ \therefore du &= \sin 2x\,dx \\ x = \frac{\pi}{4} &\quad \to u = \sin^2\frac{\pi}{4} = \frac{1}{2} \\ x = \frac{\pi}{3} &\quad \to u = \sin^2\frac{\pi}{3} = \frac{3}{4} \\ \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sin 2x}{1+\sin^2 x}\,dx &= \int_{u=\frac{1}{2}}^{u=\frac{3}{4}} \frac{1}{1+u}\,du \\ &= \Big[\log_e(1+u)\Big]_{\frac{1}{2}}^{\frac{3}{4}} \\ &= \log_e\left(\frac{7}{4}\right) - \log_e\left(\frac{3}{2}\right) \\ &= \log_e\frac{\frac{7}{4}}{\frac{3}{2}} = \log_e\frac{7}{6} \end{aligned}$$

(c) i. (3 marks)



ii. (2 marks)

$$y = x^2 - 2x \quad x \geq 1$$

Interchanging variables,

$$\begin{aligned} x &= y^2 - 2y \quad y \geq 1 \\ x + 1 &= y^2 - 2y + 1 \\ \therefore (y-1)^2 &= x + 1 \end{aligned}$$

$y - 1 = \sqrt{x+1}$ (as $y \geq 1$, take positive root)

$$\therefore y = 1 + \sqrt{x+1}$$

iii. (1 mark)

$$f^{-1}(2) = 1 + \sqrt{2+1} = 1 + \sqrt{3}$$

(d) (3 marks)

Let $P(n)$ be the proposition

$$P(n): \quad 2\times 1 + 3\times 2 + 4\times 3 + \cdots + n(n-1) = \frac{n(n^2-1)}{3}$$

- Base case: $P(2)$ –

$$2(2-1) = 2$$
$$\frac{2\left(2^2-1\right)}{3} = \frac{2\times 3}{3} = 2$$

Hence $P(2)$ is true.

- Inductive hypothesis: assume that $P(k)$ is true for some $k \in \mathbb{Z}^+$, i.e.

$$P(k): \quad 2\times 1 + 3\times 2 + 4\times 3 + \cdots + k(k-1) = \frac{k(k^2-1)}{3}$$

Now examine $P(k+1)$:

$$\begin{aligned} &\overbrace{2\times 1 + 3\times 2 + 4\times 3 + \cdots + k(k-1)}^{=P(k)} \\ &\quad + (k+1)([k+1]-1) \\ &= \frac{k\left(k^2-1\right)}{3} + k(k+1) \\ &= \frac{k(k-1)(k+1) + 3k(k+1)}{3} \\ &= \frac{(k+1)\left(k^2 - k + 3k\right)}{3} \\ &= \frac{(k+1)\left(k^2+2k\right)}{3} \\ &= \frac{(k+1)\left(k^2+2k+1-1\right)}{3} \\ &= \frac{(k+1)\left((k+1)^2-1\right)}{3} \end{aligned}$$

**Question 13** (Berry)

(a) i. (2 marks)

$$\begin{aligned} \sin 3\theta &= \sin(2\theta + \theta) \\ &= \sin 2\theta \cos\theta + \cos 2\theta \sin\theta \\ &= 2\sin\theta\cos\theta\cos\theta + (\cos^2\theta - \sin^2\theta)\sin\theta \\ &= 2\sin\theta\cos^2\theta + \sin\theta\cos^2\theta - \sin^3\theta \\ &= 2\sin\theta(1-\sin^2\theta) + \sin\theta(1-\sin^2\theta) - \sin^3\theta \\ &= 2\sin\theta - 2\sin^3\theta + \sin\theta - \sin^3\theta - \sin^3\theta \\ &= 3\sin\theta - 4\sin^3\theta \end{aligned}$$

ii. (1 mark)

$$\begin{aligned} \int 2\sin^3\theta\,d\theta &= \frac{1}{2}\int 4\sin^3\theta\,d\theta \\ &= \frac{1}{2}\int (3\sin\theta - \sin 3\theta)\,d\theta \\ &= \frac{1}{2}\left[-3\cos\theta + \frac{1}{3}\cos 3\theta\right] + C \\ &= \frac{1}{6}\cos 3\theta - \frac{3}{2}\cos\theta + C \end{aligned}$$

(b) i. (2 marks)

$$P(x) = ax^3 - 7x^2 + k(4) + 4$$

As $(x-4)$ is a factor, by the factor theorem:

$$\begin{aligned} P(4) &= 0 \\ \therefore a\times(4^3) - 7(4^2) + 4k + 4 &= 0 \\ 64a - 112 + 4k + 4 &= 0 \\ 16a + k - 27 &= 0 \end{aligned}$$

Using the remainder theorem,

$$\begin{aligned} P(1) &= -6 \\ \therefore a(1^3) - 7(1^2) + k(1) + 4 &= -6 \\ a - 7 + k + 4 &= -6 \\ a + k - 3 &= -6 \\ k &= -3 - a \end{aligned}$$

$$\begin{cases} 16a + k = 27 & (1) \\ k = -3 - a & (2) \end{cases}$$

Substitute (2) to (1):

$$\begin{aligned} 16a + (-3 - a) &= 27 \\ 15a &= 30 \\ a &= 2 \\ \therefore k &= -5 \end{aligned}$$

ii. (2 marks)

$$\begin{aligned} \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} &= \frac{\alpha\beta + \alpha\gamma + \beta\gamma}{\alpha\beta\gamma} \\ &= \frac{\left(\frac{c}{a}\right)}{\left(-\frac{d}{a}\right)} = \frac{\left(\frac{-5}{2}\right)}{\left(-\frac{4}{2}\right)} \\ &= \frac{5}{4} \end{aligned}$$

(c) i. (1 mark)
The angle between a tangent and a chord through the point of contact is equal to the angle in the alternate segment.

ii. (2 marks)

- $\angle AEF = 180° - \angle EAC$ (Cointerior angles, $AC \parallel EF$)
- $\angle AEF = 180° - \angle ABC$ (Since $\angle EAC = \angle ABC$)

iii. (1 mark)

- $\angle EAB = 90°$ (The tangent to a circle is perpendicular to the diameter drawn to the point of contact)
- $\therefore BE$ is the diameter of the the circle through $E$, $A$, $B$ and $F$ as it subtends a right angle at the circumference of the circle at point $A$.

(d) i. (2 marks)

$$\begin{aligned} x^2 &= 4ay \\ y &= \frac{x^2}{4a} \\ \frac{dy}{dx} &= \frac{2x}{4a} = \frac{x}{2a} \end{aligned}$$

At the point $P(2at, at^2)$

$$\begin{aligned} m_{\tan} &= \frac{2at}{2a} = t \\ \therefore m_{\perp} &= -\frac{1}{t} \end{aligned}$$

Apply the point-gradient formula,

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - at^2 &= -\frac{1}{t}(x - 2at) \\ ty - at^3 &= -x + 2at \\ x + ty &= 2at + at^3 \end{aligned}$$

ii. (2 marks)
Find point $N$ by letting $x = 0$:

$$\begin{aligned} 0 + ty &= 2at + at^3 \\ \therefore y &= 2a + at^2 \qquad (t \neq 0) \end{aligned}$$

Finding the midpoint $M$ of $PN$

$$\begin{aligned} M &= \left(\frac{0 + 2at}{2}, \frac{2a + at^2 + at^2}{2}\right) \\ &= \left(\frac{2at}{2}, \frac{2at + 2at^2}{2}\right) \\ &= \left(at, a + at^2\right) \end{aligned}$$

$$\begin{cases} x = at & (1) \\ y = a + at^2 & (2) \end{cases}$$

Substitute (1) into (2) after rearranging,

$$\begin{aligned} y &= a + a\left(\frac{x}{a}\right)^2 \\ &= a + \frac{x^2}{a} \\ \therefore ay &= a^2 + x^2 \\ x^2 &= ay - a^2 = a(y - a) \end{aligned}$$

**Question 14** (Lam)

(a) i. (2 marks)

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$$

$$V = \frac{4}{3}\pi r^3 \quad \frac{dr}{dt} = 6\,\text{cms}^{-1}$$

$$\frac{dV}{dt} = 4\pi r^2$$

$$\begin{aligned}\frac{dV}{dt} &= 4\pi r^2 \times 6\Big|_{r=4}\\ &= 4 \times \pi \times 4^2 \times 6\\ &= 384\pi\end{aligned}$$

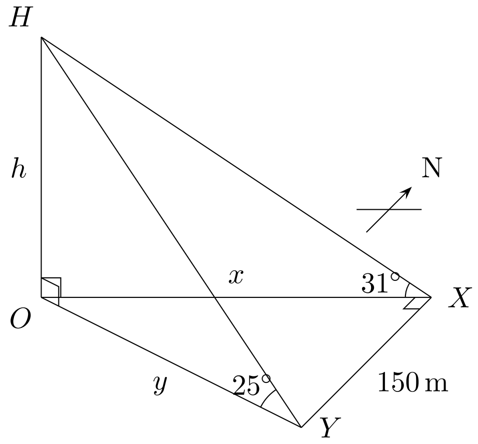
ii. (3 marks)

$$A = 4\pi r^2 \qquad \frac{dA}{dr} = 8\pi r$$

$$\begin{aligned}\frac{dV}{dt} &= \frac{dV}{dr} \times \frac{dr}{dt}\\ 15 &= 4\pi r^2 \times 6\\ \frac{15}{24\pi} &= r^2\\ \therefore r &= \sqrt{\frac{5}{8\pi}}\end{aligned}$$

$$\begin{aligned}\frac{dA}{dt} &= \frac{dA}{dr} \times \frac{dr}{dt}\\ &= 8\pi r \times 6\\ &= 8\pi \times \sqrt{\frac{5}{8\pi}} \times 6\\ &= \sqrt{1440\pi}\\ &= 67.3\,\text{cm}^2\text{s}^{-1} \text{ (1 dp)}\end{aligned}$$

(b) i. (3 marks)




- In $\triangle OHX$

$$\begin{aligned}\frac{h}{x} &= \tan 31^\circ\\ \therefore x &= \frac{h}{\tan 31^\circ}\end{aligned}$$

- Similarly in $\triangle OHY$,

$$\begin{aligned}\frac{h}{y} &= \tan 25^\circ\\ \therefore y &= \frac{h}{\tan 25^\circ}\end{aligned}$$

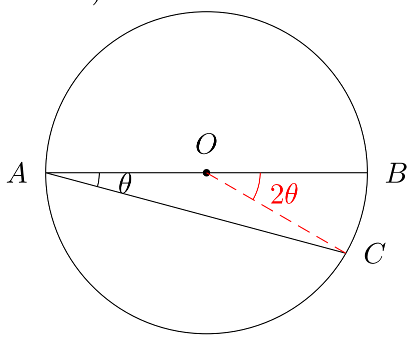
- In $\triangle OXY$,

$$\begin{aligned}x^2 + 150^2 &= y^2\\ \left(\frac{h}{\tan 31^\circ}\right)^2 + 150^2 &= \left(\frac{h}{\tan 25^\circ}\right)^2\\ \left(\frac{h}{\tan 25^\circ}\right)^2 - \left(\frac{h}{\tan 31^\circ}\right)^2 &= 150^2\\ h^2\left(\frac{1}{\tan^2 25^\circ} - \frac{1}{\tan^2 31^\circ}\right) &= 150^2\\ h^2\left(\frac{\tan^2 31^\circ - \tan^2 25^\circ}{\tan^2 25^\circ \tan^2 31^\circ}\right) &= 150^2\\ h^2 &= \frac{150^2 \tan^2 25^\circ \tan^2 31^\circ}{\tan^2 31^\circ - \tan^2 25^\circ}\\ \therefore h &= \frac{150 \tan 25^\circ \tan 31^\circ}{\sqrt{\tan^2 31^\circ - \tan^2 25^\circ}}\end{aligned}$$

ii. (1 mark)

$$h = 110.91\,\text{m (2 dp)}$$

(c) i. (2 marks)




$\angle COB = 2\theta$ as angle at the centre is twice the angle at the circumference, subtended by the same arc. (Can also use angle sum of $\triangle$ and exterior $\angle$)

$$A_{\text{sector } OBC} + A_{\triangle OAC} = \frac{1}{4}\pi r^2$$

$$\begin{aligned} A_{\text{sector } OBC} &= \frac{1}{2}r^2 \times (2\theta) \\ &= r^2\theta = \theta \quad (r = 1) \\ A_{\triangle OBC} &= \frac{1}{2}r^2 \times \sin(\pi - 2\theta) \\ &= \frac{1}{2}r^2 \sin 2\theta = \frac{1}{2}\sin 2\theta \end{aligned}$$

(As $\sin(\pi - x) = \sin x$)

Adding areas and equating,

$$\begin{aligned} \theta + \frac{1}{2}\sin 2\theta &= \frac{\pi}{4} \\ \therefore \theta + \frac{1}{2}\sin 2\theta - \frac{\pi}{4} &= 0 \end{aligned}$$

ii. (2 marks)

- When $\theta = 0.4$,

$$0.4 + \frac{1}{2}\sin 0.8 - \frac{\pi}{4} \approx -0.026\cdots$$

- When $\theta = 0.5$,

$$0.5 + \frac{1}{2}\sin 1 - \frac{\pi}{4} \approx 0.135$$

If $f(\theta) = \theta + \frac{1}{2}\sin 2\theta - \frac{\pi}{4}$, then $f(\theta)$ is continuous for all $x \in \mathbb{R}$ as $\theta$ and $\sin 2\theta$ are both continuous. From above, $f(0.4) < 0$ and $f(0.5) > 0$, therefore a zero crossing exists for $0.4 < \theta < 0.5$, satisfying the equation found in part (i).

iii. (2 marks)

$$\left| \begin{aligned} f(\theta) &= \theta + \frac{1}{2}\sin 2\theta - \frac{\pi}{4} \\ f'(\theta) &= 1 + \frac{1}{2} \times 2\cos 2\theta \\ &= 1 + \cos 2\theta \end{aligned} \right.$$

$$\begin{aligned} \theta_1 &= \theta_0 - \frac{f(\theta_0)}{f'(\theta_0)} \\ &= 0.4 - \frac{0.4 + \frac{1}{2}\sin 0.8 - \frac{\pi}{4}}{1 + \cos 0.8} \\ &\left(= 0.4 - \frac{-0.02672011\cdots}{1.696706\cdots}\right) \\ &= 0.41574\cdots \\ &= 0.42 \text{ (2 dp)} \end{aligned}$$