Ext 1

NORTH SYDNEY BOYS HIGH SCHOOL

2018 HSC ASSESSMENT TASK 3 (TRIAL HSC)

Mathematics

Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using blue or black pen
- Board approved calculators may be used
- All necessary working should be shown in every question
- Each new question is to be started on a **new page.**
- Attempt all questions

Class Teacher:
(Please tick or highlight)

- O Mr Berry
- O Mr Hwang
- O Mr Ireland
- O Dr Jomaa
- O Ms Lee
- O Ms Ziaziaris

Student Number:

(To be used by the exam markers only.)

Question No	1-10	11	12	13	14	Total	Total
Mark	/10	/15	/15	/15	/15	/70	/100

Section I

10 Marks

Attempt Questions 1 – 10

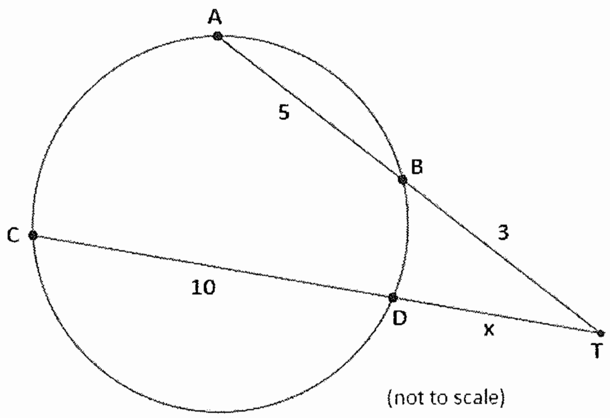
Use the multiple choice answer sheet for Questions 1 – 10

1 When the polynomial $P(x)$ is divided by $(x+1)(x-3)$ the remainder is $2x+7$.

The remainder when $P(x)$ is divided by $x-3$ is equal to 1

(A) 1 (B) 7 (C) 9 (D) 13

2 In the circle below, $CD=10$, $AB=5$, $BT=3$, and $DT=x$. 1



The value of x is

(A) $x = 1 \cdot 5$ (B) $x = 2$ (C) $x = 6$ (D) $x = 4$

3 Which of the following is equivalent to $\dfrac{\sin 2\theta + \sin\theta}{\cos 2\theta + \cos\theta + 1}$? 1

(A) $\cot\theta$ (B) $\sec\theta$ (C) $\sin\theta$ (D) $\tan\theta$

4 The acute angle between the lines $y = -x - 1$ and $4x + 2y = 1$ is 1

(A) $71° \, 34'$

(B) $45°$

(C) $30° \, 58'$

(D) $18° \, 26'$

5 Given n is an integer, the general solution of $\tan\left(2x+\frac{\pi}{4}\right)=\sqrt{3}$ is 1

(A) $x=\dfrac{(12n+1)\pi}{24}$

(B) $x=\dfrac{(3n+1)\pi}{6}$

(C) $x=\dfrac{(12n-1)\pi}{24}$

(D) $x=\dfrac{(6n+1)\pi}{6}$

6 The solution of $x-\cos 2x=0$ is known to be approximately $0\cdot 5$. Using one application of Newton's Method, a second approximation to this solution is 1

A) $x=0\cdot 485$ (B) $x=0\cdot 515$ (C) $x=0\cdot 983$ (D) $x=0\cdot 441$

7 A particle moves in a straight line, and its position at time t is given by $x=3\cos 2t+4\sin 2t$. Its greatest speed is: 1

(A) 6 (B) 10 (C) 12 (D) 5

8 The interval AB joining the points $A(1,3)$ and $B(a,b)$ is divided internally in the ratio 2:3 by the point $(3,13)$. The values of a and b are 1

(A) $a=6,\ b=28$

(B) $a=6,\ b=37$

(C) $a=9,\ b=28$

(D) $a=9,\ b=37$

9 If α,β,γ are the roots of the cubic equation $2x^3-5x^2-3x+1=0$, then the value of $\alpha^2+\beta^2+\gamma^2$ is 1

(A) $\dfrac{13}{4}$ (B) $\dfrac{37}{4}$ (C) $\dfrac{31}{4}$ (D) $\dfrac{19}{4}$

10 Which of the following is the domain of the function $y=log_e[x+\sqrt{x^2+1}\,]$? 1

(A) $\{x: x\le -1, x\ge 1\}$ (B) $\{x: -1\le x\le 1\}$

(C) $\{x: x \text{ is a real number}\}$ (D) $\{x: x\ge 1\}$

Section II

60 Marks
Attempt Questions 11-14

Answer each question on a NEW page. Extra writing booklets are available.

In Questions 11 – 14, your responses should include all relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Start a NEW page

(a) Find $\int sin^2 4x\, dx$ 2

(b) Evaluate the limit $\lim\limits_{x \to 0} \frac{1-\cos 2x}{6x^2}$ 2

(c) Use the substitution $u = x^2 + 1$ to evaluate $\int_0^2 \frac{x}{(x^2+1)^3}\, dx$ 3

(d) Find the exact value of $sin^{-1}\left(\frac{1}{\sqrt{5}}\right) + sin^{-1}\left(\frac{1}{\sqrt{10}}\right)$ 3

(e) Differentiate $y = tan^{-1}\frac{x}{4}$, giving your answer in fully simplified form. 2

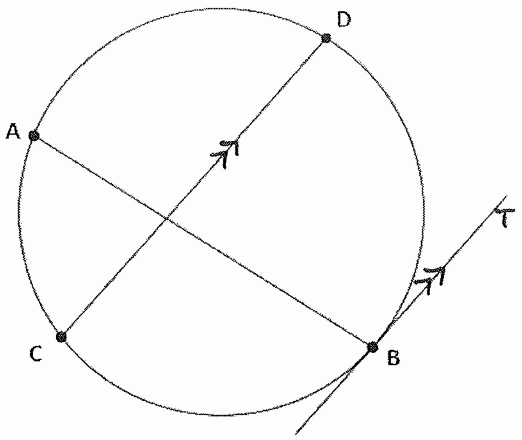
(f) Solve the equation $\sqrt{3}sin\theta = 1 + cos\theta$ in the domain $0 \leq \theta \leq 2\pi$. 3

Question 12 (15 marks) Start a NEW page

(a) The velocity v in cm/sec of a particle moving on the x-axis is given by $v^2 = -3x^2 + 20x + 7$. (x is the displacement from the origin, in centimetres).

Show that the particle is moving in simple harmonic motion, and find the centre, amplitude and period of that motion. 4

(b) AB and CD are two intersecting chords of a circle, and CD is parallel to the tangent to the circle at B.

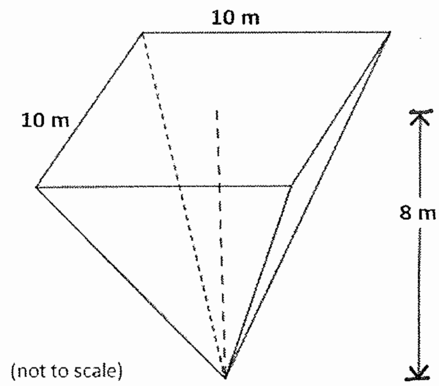


(i) Copy or trace the diagram to your writing booklet.

(ii) Prove that AB bisects $\angle CAD$. 3

(c) A loading chute is in the shape of an inverted square pyramid, with base length 10 metres and depth 8 metres.

Liquid is poured in the top at a constant rate of $4\ m^3/\text{min}$.



(i) Show that the volume V of the liquid when the level is h metres high is given by $V = \dfrac{25h^3}{48}$. [You may need to draw a diagram] 2

(ii) At what rate is the level of the liquid rising when its depth is 4m? 2

(d) For the function $y = \sin\ (\cos^{-1} x)$,

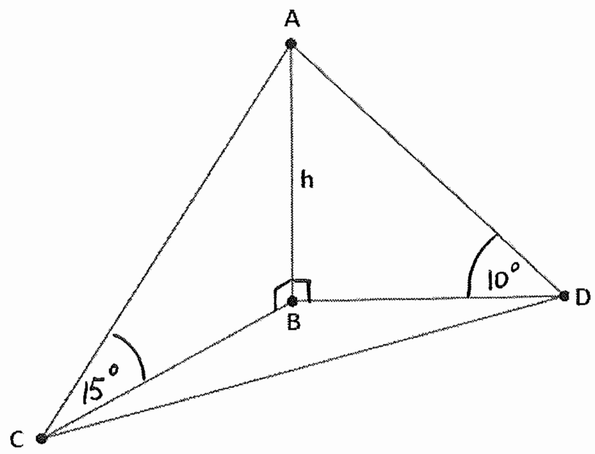
(i) State the domain. 1

(ii) State the range. 1

(iii) Sketch the function neatly, using at least 1/3 of a page. 2

Question 13 (15 marks) Start a NEW page

(a) AB represents a communications mast, height h metres. From points C and D in the same plane as the base of the mast, the angles of elevation of the top of the mast are $15°$ and $10°$ respectively. From the base of the mast, the bearings of points C and D are 230°T and 100°T respectively.



(i) Copy the diagram to your answer page.

(ii) Find the size of $\angle CBD$. 1

(iii) Show that $BD = h \cot 10°$. 1

(iv) If CD is 450 metres, find the height of the mast (to 1 decimal place). 3

(b) A particle P moves in a straight line with acceleration given by $\ddot{x} = 9(x-2)$, where x is the displacement in metres from the origin O after t seconds. Initially P is 4 metres to the right of O, with velocity $v = -6$ metres per second.

(i) Show that $v^2 = 9(x-2)^2$ 2

(ii) Find x as a function of t. 3

(c) A forensic scientist is called to a murder scene. At 10:23pm, she measures the temperature of the body as $26 \cdot 7°C$. The ambient temperature of the room is $20°C$. Exactly 60 minutes later, a second reading gives the body's temperature as $25 \cdot 8°C$.

Assume that the victim's body before death was a normal $37°C$, and that Newton's Law of cooling applies, that is, $\frac{dT}{dt} = -k(T - 20)$ where k is a constant, t is measured in minutes and temperature is in degrees Celsius,

(i)	Show that $T = 20 + Ae^{-kt}$ is a solution of the above equation.	1
(ii)	Evaluate A, then determine k to 6 decimal places.	2
(iii)	Calculate at what time of day the victim was murdered.	2

Examination continues on next page ->

Question 14 (15 marks) Start a NEW page

(a) Given that x is a positive integer, prove by the method of mathematical induction that $(1+x)^n - 1$ is divisible by x for all positive integers n ≥ 1. **3**

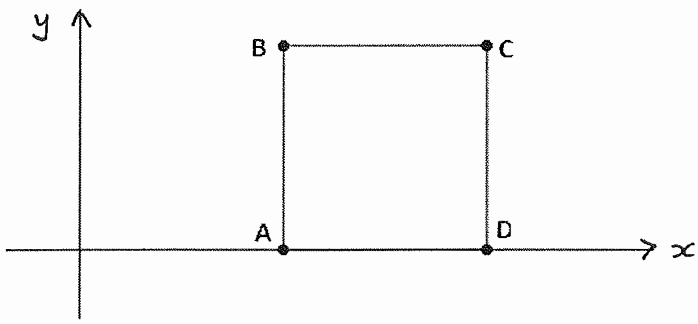
(b) A parabola has parametric equations $x = t,\ y = t^2$

(i) Sketch the parabola, marking the points P and Q that correspond to $t = -1$ and $t = 2$ respectively. **1**

(ii) Derive the tangents at P and Q, and show they intersect at $R(\frac{1}{2}, -2)$. **2**

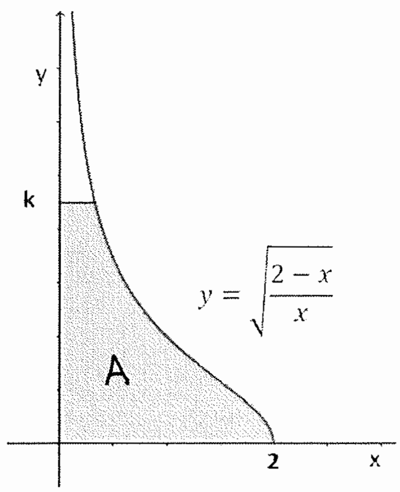
(iii) Let $T(t, t^2)$ be the point on the parabola between P and Q such that the tangent at T meets QR at the midpoint of QR. Show that the tangent at T is parallel to PQ. **3**

(c) The diagram shows a unit square with vertices $A(1,0), B(1,1), C(2,1), D(2,0)$.



A line passing through the origin with gradient m cuts the sides AB and CD at P and Q respectively. For what values of m does the line divide the area of the square in the ratio 2 : 1 ? **3**

(d) (i) Find the shaded area A. **2**



(ii) Find $\lim_{k \to \infty} A$ **1**

END OF EXAMINATION

M.C

2018 Y12 Ext.1 Task 3 Suggested Solutions

① $P(x) = (x+1)(x-3) \cdot Q(x) + 2x + 7$

$\therefore$ rem. $= P(3) = 2(3) + 7 = 13$ $\therefore$ (D)

② $8(3) = (10+x)(x)$

$24 = x^2 + 10x$

$x^2 + 10x - 24 = 0$ $\therefore$ (B)

$(x+12)(x-2) = 0$ $\therefore x = 2$

③ $\dfrac{\sin 2\theta + \sin\theta}{\cos 2\theta + \cos\theta + 1} = \dfrac{2\sin\theta\cos\theta + \sin\theta}{2\cos^2\theta - 1 + \cos\theta + 1}$

$= \dfrac{\sin\theta\,\cancel{(2\cos\theta + 1)}}{\cos\theta\,\cancel{(2\cos\theta + 1)}}$ $\therefore$ (D)

$= \tan\theta$

④ $\tan\theta = \left|\dfrac{-1-(-2)}{1+(-1)(-2)}\right| = \left|\dfrac{1}{3}\right|$

$\theta = 18^\circ 26'$ $\therefore$ (D)

⑤ $\tan\left(2x + \frac{\pi}{4}\right) = \sqrt{3}$

$\therefore 2x + \frac{\pi}{4} = \frac{\pi}{3} + n\pi$

$2x = \frac{\pi}{12} + n\pi$

$2x = \dfrac{\pi + 12n\pi}{12}$

$= \dfrac{\pi(1+12n)}{24}$ $\therefore$ (A)

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2018 Y12 Ext.1 Task 3 Suggested Solutions

(6) $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

$f(x) = x - \cos 2x$

$f'(x) = 1 + 2\sin 2x$

$\therefore \; x_2 = 0.5 - \dfrac{0.5 - \cos 1}{1 + 2\sin 1}$

$\doteqdot 0.515\,021\ldots$ $\quad \therefore$ (B)

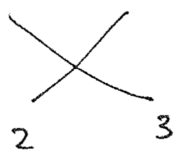
(7) $x = 3\cos 2t + 4\sin 2t$

$= 5\cos(2t - \alpha)$

$\dot{x} = -10\sin(2t - \alpha)$ $\quad \therefore$ (B)

$\therefore$ max. speed $= 10$

(8) $A(1,3)$ $\quad B(a,b)$



$\therefore \; 3 = \dfrac{3+2a}{5}$ & $13 = \dfrac{9+2b}{5}$

$\therefore \; 2a = 15 - 3$
$= 12$
$a = 6$

$65 = 9 + 2b$
$2b = 56$
$b = 28$

$\therefore$ (A)

(9) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$

$= \left(\frac{5}{2}\right)^2 - 2\left(-\frac{3}{2}\right)$

$= \frac{25}{4} + 3 = \frac{37}{4}$ $\quad \therefore$ (B)

(10) D: of $y = \ln\left[x + \sqrt{x^2+1}\right]$ is all real $\mathbb{R}$ $\quad \therefore$ (C)

11

2018	Y12 Ext.1	Task 3	Suggested Solutions

(a) $\int \sin^2 4x \, dx = \int \left(\frac{1}{2} - \frac{1}{2}\cos 8x\right) dx$ ✓

$= \frac{1}{2}x - \frac{1}{16}\sin 8x + c$ ✓

(b) $\lim_{x \to 0} \frac{1 - \cos 2x}{6x^2}$

$= \lim_{x \to 0} \frac{1 - (1 - 2\sin^2 x)}{6x^2}$ ✓

$= \lim_{x \to 0} \frac{2\sin^2 x}{6x^2}$

$= \frac{1}{3} \lim_{x \to 0} \left(\frac{\sin x}{x}\right)^2 = \frac{1}{3}$ ✓

(c) $I = \int_0^2 \frac{x}{(x^2+1)^3} \, dx$

$\left\{ \begin{array}{l} u = x^2 + 1 \\ du = 2x \, dx \\ x = 0 \to u = 1 \\ x = 2 \to u = 5 \end{array} \right]$ ✓

$= \frac{1}{2} \int_1^5 u^{-3} \, du$

$= \frac{1}{2} \left[\frac{u^{-2}}{-2}\right]_1^5$ ✓

$= -\frac{1}{4} \left[\frac{1}{5^2} - \frac{1}{1^2}\right] = \frac{1}{4}\left[1 - \frac{1}{25}\right]$

$= \frac{1}{4} \cdot \frac{24}{25} = \frac{6}{25}$ ✓

11 (d) Let $\alpha = \sin^{-1}\left(\frac{1}{\sqrt{5}}\right)$, $\beta = \sin^{-1}\left(\frac{1}{\sqrt{10}}\right)$

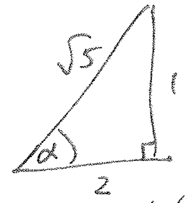
Then $\sin\left[\sin^{-1}\left(\frac{1}{\sqrt{5}}\right) + \sin^{-1}\left(\frac{1}{\sqrt{10}}\right)\right]$

$= \sin(\alpha + \beta)$

$= \sin\alpha\cos\beta + \cos\alpha\sin\beta$

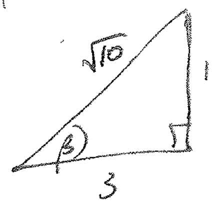
$= \frac{1}{\sqrt{5}} \cdot \frac{3}{\sqrt{10}} + \frac{2}{\sqrt{5}} \cdot \frac{1}{\sqrt{10}}$ ✓

$= \frac{5}{\sqrt{50}} = \frac{1}{\sqrt{2}}$



Let $\alpha = \sin^{-1}\left(\frac{1}{\sqrt{5}}\right)$

Let $\beta = \sin^{-1}\left(\frac{1}{\sqrt{10}}\right)$



But $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$\therefore \sin^{-1}\left(\frac{1}{\sqrt{5}}\right) + \sin^{-1}\left(\frac{1}{\sqrt{10}}\right) = \pi/4.$ ✓

(Needs working for marks!)

(e) $\frac{d}{dx}\left(\tan^{-1}\frac{x}{4}\right) = \frac{1}{4} \cdot \frac{1}{1 + \left(\frac{x}{4}\right)^2}$ ✓

$= \frac{1}{4} \cdot \frac{1}{1 + \frac{x^2}{16}}$

$= \frac{1}{4} \cdot \frac{1}{\frac{16 + x^2}{16}} = \frac{1}{4} \cdot \frac{16}{16 + x^2}$

$= \frac{4}{16 + x^2}$ ✓

(f) $\sqrt{3}\sin\theta - 1.\cos\theta = 1$

Let $\sqrt{3}\sin\theta - 1.\cos\theta = R\sin(\theta - \alpha)$

$= R\sin\theta\cos\alpha - R\cos\theta\sin\alpha$

$\therefore \begin{cases} R\cos\alpha = \sqrt{3} \\ R\sin\alpha = 1 \end{cases}$

$\therefore \tan\alpha = \frac{1}{\sqrt{3}}$, $\alpha = \pi/6$.

Also, $R = \sqrt{(\sqrt{3})^2 + 1^2} = 2$

$\therefore 2\sin\left(\theta - \frac{\pi}{6}\right) = 1$ ✓

$\sin\left(\theta - \pi/6\right) = \frac{1}{2}$, $-\frac{\pi}{6} \le \theta - \frac{\pi}{6} \le \frac{7\pi}{6}$.

$\therefore \theta - \frac{\pi}{6} = \frac{\pi}{6}$ or $\frac{5\pi}{6}$

$\therefore \theta = \frac{\pi}{3}, \pi$ ✓ ✓

alt. let $t = \tan\frac{\theta}{2}$ $\therefore \sin\theta = \frac{2t}{1+t^2}$, $\cos\theta = \frac{1-t^2}{1+t^2}$

Thus $\sqrt{3} \cdot \frac{2t}{1+t^2} - \frac{1-t^2}{1+t^2} = 1$ ✓

$2\sqrt{3}\,t - 1 + \cancel{t^2} = 1 + \cancel{t^2}$

$2\sqrt{3}\,t = 2$

$t = \frac{1}{\sqrt{3}}$ $\therefore \tan\frac{\theta}{2} = \frac{1}{\sqrt{3}}$, $0 \le \frac{\theta}{2} \le \pi$

$\frac{\theta}{2} = \frac{\pi}{6}$ $\therefore \theta = \pi/3$ ✓

But need to check $\theta = \pi$

$$\left.\begin{aligned} LHS &= \sqrt{3}\sin\pi - \cos\pi \\ &= \sqrt{3}(0) - (-1) \\ &= 1 = RHS \end{aligned}\right\}$$

$\therefore \theta = \pi$ is a soln.

$\therefore \theta = \pi/3, \pi$ ✓

(a) $v^2 = -3x^2 + 20x + 7$

$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d}{dx}\left(-\frac{3}{2}x^2 + 10x + \frac{7}{2}\right)$

$= -3x + 10$

So $\ddot{x} = -3\left(x - \frac{10}{3}\right)$

$\therefore \ddot{x} = -(\sqrt{3})^2\left(x - \frac{10}{3}\right)$

which is of form $\ddot{x} = -n^2(x - x_0)$ ✓

$\therefore$ motion is SHM.

Centre of motion is $x = \frac{10}{3}$. ✓

period $T = \frac{2\pi}{n} = \frac{2\pi}{\sqrt{3}}$ secs. $\left(= \frac{2\sqrt{3}\pi}{3}\text{ secs.}\right)$ ✓

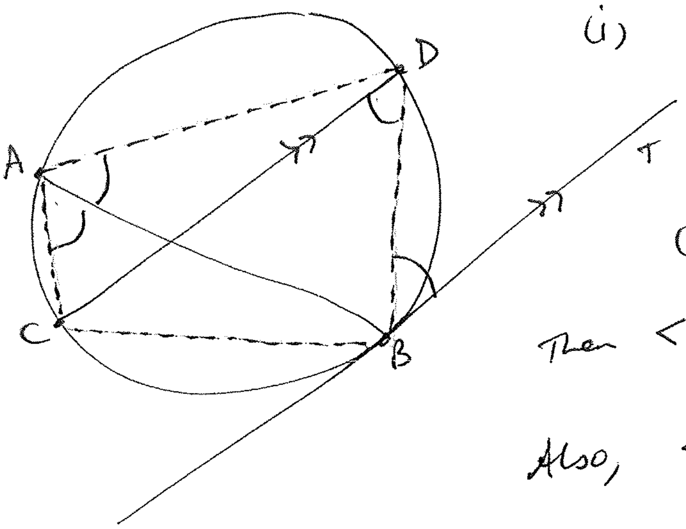
At endpts, $v = 0$ $\therefore$ $3x^2 - 20x - 7 = 0$

$(3x+1)(x-7) = 0$

$x = -\frac{1}{3}, 7$ are endpts.

$\therefore$ amplitude $= \frac{1}{2}\left(\frac{1}{3} + 7\right) = \frac{1}{2}\left(\frac{22}{3}\right) = \frac{11}{3}$ cm. ✓

(b)



(i)

(ii) Join the vertices A, B, C, D.

Then $\angle DBT = \angle BAD$ (alternate segment theorem) ✓

Also, $\angle DBT = \angle CDB$ (alternate angles, $CD \parallel BT$) ✓

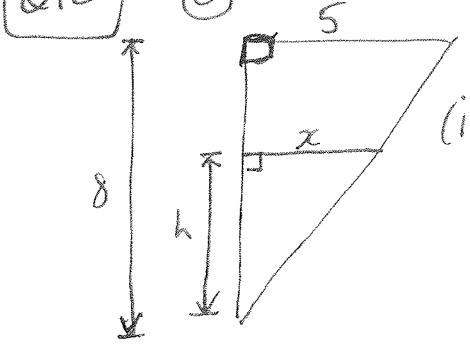
But $\angle CDB = \angle CAB$ (angles in same segment) ✓

$\therefore \angle BAD = \angle CAB$

ie AB bisects $\angle CAD$. #

Q12

(c)



(i) $\therefore \dfrac{x}{h} = \dfrac{5}{8}$ by similarity.

When height $= h$,

$$V = \frac{1}{3}(2x)^2 \cdot h$$

But $x = \dfrac{5h}{8}$ $\therefore$ $2x = \dfrac{5h}{4}$

$$\therefore V = \frac{1}{3}\cdot\left(\frac{5h}{4}\right)^2 \cdot h = \frac{25h^3}{48}$$

(ii) $$\frac{dV}{dt} = \frac{dV}{dh}\cdot\frac{dh}{dt}$$

$$\therefore \frac{dV}{dt} = \frac{25h^2}{16}\cdot\frac{dh}{dt}$$

$$\therefore 4 = \frac{25(4^2)}{16}\cdot\frac{dh}{dt} \qquad \therefore \frac{dh}{dt} = \frac{4}{25} \text{ m/min.}$$

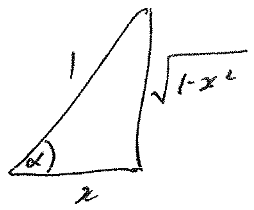
$(= 0.16$ m/min$)$

(d) $y = \sin(\cos^{-1}x)$

(i) $D: -1 \le x \le 1$

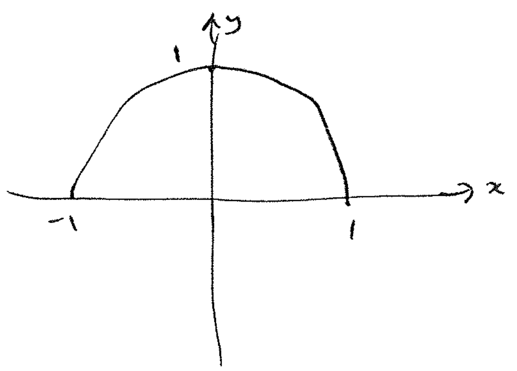
(ii) $R: 0 \le y \le 1$

(iii) Let $\alpha = \cos^{-1}x$ $\therefore$ $x = \cos\alpha$



$\therefore y = \sin\alpha = \sqrt{1-x^2}$

$\therefore$



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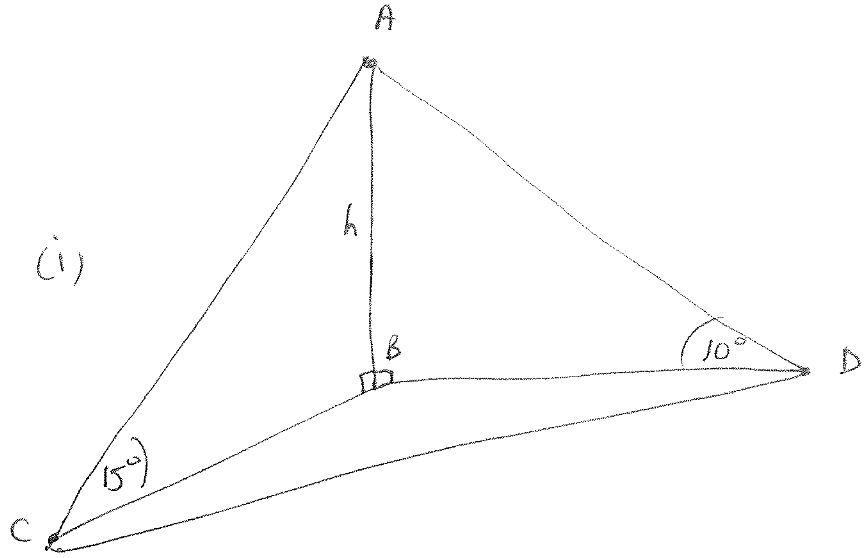
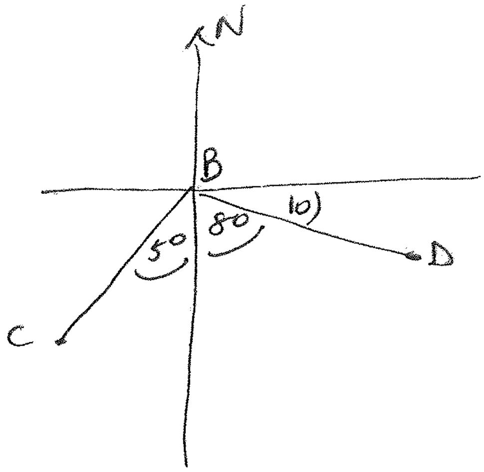
(a)

(i)



(ii) $\angle CBD = 130°$ ✓



(iii) $\frac{h}{BD} = \tan 10°$ ✓

$BD = \frac{h}{\tan 10°}$

$\therefore BD = h \cot 10°$

$(CD = h \cot 15)$

(iv) by cosine rule in ΔCBD,

$$450^2 = (h\cot 10°)^2 + (h\cot 15)^2 - 2 \cdot h\cot 10 \cdot h\cot 15 \cdot \cos 130$$

$$= h^2 \left[\cot^2 10 + \cot^2 15 - 2\cot 10 \cdot \cot 15 \cdot \cos 130 \right].$$

$$\therefore h^2 = \frac{450^2}{\cot^2 10 + \cot^2 15 - 2\cot 10 \cdot \cot 15 \cdot \cos 130}$$

$$\doteqdot 2762.5629 \ldots$$

$$\therefore h \doteqdot 52.5600$$

$$\therefore h = 52.6 \text{ m}$$ (1 d.p)

13 (b) $\ddot{x} = 9(x-2)$

at $t=0$, $x=+4$, $v=-6$.

(i) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 9x - 18$

$\frac{1}{2}v^2 = \frac{9x^2}{2} - 18x + c$

at $t=0$, $x=4$, $v=-6$

$\therefore \frac{1}{2}\cdot 36 = \frac{9}{2}\cdot 16 - 18\cdot 4 + c$

$\therefore c = 18$ ✓

$\therefore \frac{1}{2}v^2 = \frac{9x^2}{2} - 18x + 18$

$v^2 = 9x^2 - 36x + 36$

$= 9(x^2 - 4x + 4)$

$\therefore v^2 = 9(x-2)^2$ ✓

(ii) $v = \pm 3(x-2)$ But: when $x=+4$, $v=-6$

$\therefore v = -3(x-2)$

$\therefore \frac{dx}{dt} = -3(x-2)$ $\therefore \frac{dt}{dx} = -\frac{1}{3}\cdot\frac{1}{x-2}$ ✓

$\therefore t = -\frac{1}{3}\ln(x-2) + c$

at $t=0$, $x=4$

$\therefore 0 = -\frac{1}{3}\ln(2) + c$

$\therefore c = \frac{1}{3}\ln 2$. ✓

$\therefore t = -\frac{1}{3}\ln(x-2) + \frac{1}{3}\ln 2$

$= \frac{1}{3}(\ln 2 - \ln(x-2)) = \frac{1}{3}\ln\left(\frac{2}{x-2}\right)$

$\therefore e^{3t} = \frac{2}{x-2}$

$\therefore x - 2 = 2e^{-3t}$

$\therefore x = 2 + 2e^{-3t}$ #

13

(c) we're given $\frac{dT}{dt} = -k(T-20)$.

(i) If $T = 20 + Ae^{-kt}$, ie $Ae^{-kt} = T-20$,

then $\frac{dT}{dt} = -kAe^{-kt}$

$= -k(T-20)$ ✓

$\therefore$ a solution.

(ii) Let $t=0$ at 10:23 pm.

at $t=0$, $T = 26.7$

$\therefore \quad 26.7 = 20 + Ae^{0}$

$\boxed{A = 6.7}$ ✓

at $t=60$, $T = 25.8$

$\therefore \quad 25.8 = 20 + 6.7e^{-60k}$

$\therefore \quad e^{-60k} = \frac{5.8}{6.7}$

$-60k = \ln\left(\frac{5.8}{6.7}\right)$

$\therefore \quad k = -\frac{1}{60}\ln\left(\frac{5.8}{6.7}\right)$

$\boxed{k = 0.002404 \quad (6\text{dp.})}$ ✓

(iii) we want t when $T = 37$;

$\therefore \quad 37 = 20 + 6.7e^{-kt}$ ✓

$\therefore \quad e^{-kt} = \frac{17}{6.7} \qquad \therefore \quad -kt = \ln\left(\frac{17}{6.7}\right)$

$t = -\frac{1}{0.002404} \cdot \ln\left(\frac{17}{6.7}\right) = -387.315$ mins.

$= -6$ h, 27.3 mins

$\therefore$ $\boxed{3:56\text{pm}}$ was time of death. ✓

#

14

(a) When $n=1$

$(1+x)^1 - 1 = 1+x-1 = x$,

which is divisible by x.

$\therefore$ true for $n=1$ ✓

Assume true for $n=k$

ie assume $(1+x)^k - 1 = Mx$ for some integer M.

ie $(1+x)^k = Mx + 1$

Then for $n=k+1$,

$$(1+x)^{k+1} - 1 = (1+x)(1+x)^k - 1$$

$$= (1+x)(Mx+1) - 1 \quad \text{by assumption ✓}$$

$$= (Mx+1) + (Mx^2+x) - 1$$

$$= (M+1)x + Mx^2$$

$$= x(M+1+Mx) \quad ✓$$

$$= Nx \text{, } N \text{ an integer, as } M, x \text{ are integer.}$$

$\therefore$ true for $n=k+1$ if true for $n=k$.

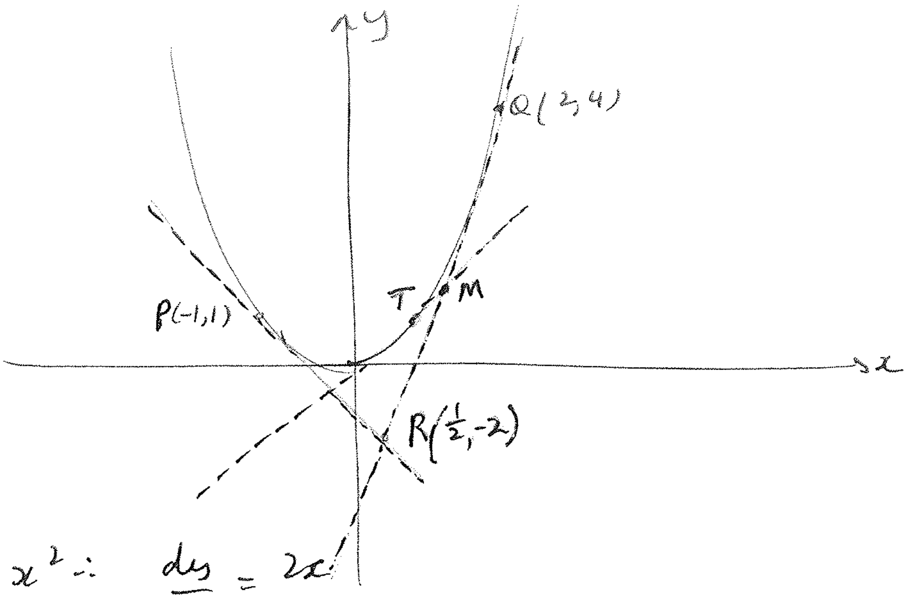
Hence true by induction. #

14 (b) $x = t$, $y = t^2$ $\therefore$ Cartesian eqn is $y = x^2$

$t = -1 \Rightarrow x = -1, y = 1$

$t = 2 \Rightarrow x = 2, y = 4.$

(i)



✓

(ii) $y = x^2 \therefore \frac{dy}{dx} = 2x$

$\therefore$ tangent at P is $y - 1 = -2(x+1)$

Likewise, tangent at Q is $y - 4 = 4(x-2)$ ✓

Solve simult. for R:

$1 - 2(x+1) = 4 + 4(x-2)$

$1 - 2x - 2 = 4 + 4x - 8$

$\therefore 6x = 3 \quad \therefore x = \frac{1}{2}$

Thus $y = 1 - 2(\frac{1}{2}+1) = 1 - 3 = -2$

$\therefore R = (\frac{1}{2}, -2)$ ✓

(iii) Tangent at T is

$y - t^2 = 2t(x - t)$

M = Midpt of QR is $\left(\frac{2+\frac{1}{2}}{2}, \frac{4-2}{2}\right) = \left(\frac{5}{4}, 1\right)$ ✓

M satisfies tangent at T

$\therefore 1 - t^2 = 2t(\frac{5}{4} - t)$

$4 - 4t^2 = 10t - 8t^2$

$4t^2 - 10t + 4 = 0$

$2t^2 - 5t + 2 = 0$, $(2t-1)(t-2) = 0$ ✓

$\therefore t = \frac{1}{2}$ or $t = 2$

→

(b) (ii) ctd.

But $t \neq 2$ (as t is 2 at Q)

$\therefore$ $t = \frac{1}{2}$ at T

tangent at is $y - t^2 = 2t(x-t)$

$\therefore$ M_T = gradient is $2t = 2(\frac{1}{2}) = 1$.

$M_{PQ} = \frac{4-1}{2+1} = 1$

$\therefore$ $M_T = M_{PQ}$ as required.

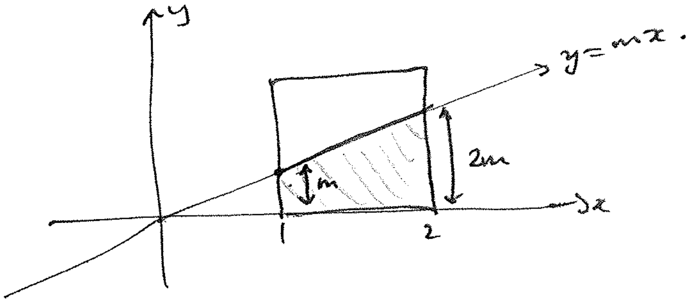
$\therefore$ parallel. #

(c)

Let the line be $y = mx$.

$\therefore$ when $x = 1$, $y = m$

when $x = 2$, $y = 2m$.



Thus shaded area is

$$A = \frac{m+2m}{2} \times 1 = \frac{3m}{2},$$

$\therefore$ remaining area is $1 - \frac{3m}{2} = \frac{2-3m}{2}$

Thus we have

$$\frac{\frac{3m}{2}}{\frac{2-3m}{2}} = \frac{2}{1} \quad \text{or} \quad \frac{\frac{3m}{2}}{\frac{2-3m}{2}} = \frac{1}{2}$$

$\therefore$ $3m = 4 - 6m$ or $6m = 2 - 3m$

$m = \frac{4}{9}$ or $m = \frac{2}{9}$

14

(d)

(i) We integrate along the y-axis:

Now, if $y = \sqrt{\frac{2-x}{x}}$ then

$$y^2 = \frac{2-x}{x}$$

$$\therefore \quad xy^2 = 2 - x$$

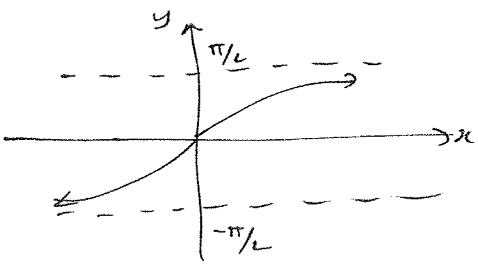
$$x(y^2+1) = 2$$

$$\therefore \quad x = \frac{2}{1+y^2}$$ ✓

$$\therefore \text{ Area } A = \int_{y=0}^{y=k} \frac{2}{1+y^2}\,dy = 2\left[\tan^{-1} y\right]_0^k$$

$$A = 2\tan^{-1} k \text{ units}^2.$$ ✓

(ii)



$\therefore$ as $k \to \infty$,

$\tan^{-1} k \to \frac{\pi}{2}$

$\therefore \quad A \to \pi$

So $\lim_{k \to \infty} A = \pi$ units^2. ✓

#