NORTH SYDNEY GIRLS' HIGH SCHOOL

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

1997

MATHEMATICS

3U/4U COMMON PAPER

Time allowed - Two hours
(Plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES:

* All questions may be attempted.
* All questions are of approximately equal value.
* Part marks for each question are shown in the right hand column.
* All necessary working must be shown.
* Marks may be deducted for careless or badly arranged work.
  * **Start each question on a NEW page**
* This examination is worth 50% of the H.S.C. Assessment Mark
* Standard integrals are printed on the back page which may be removed for your convenience. Approved calculators may be used.

This is a trial paper ONLY. The content and format of this paper do not necessarily reflect the content and format of the final Higher School Certificate examination paper.

Question 1. (Start a new page) **Marks**

(a) Evaluate $\int_4^{20} y \; dx \quad if \quad xy = 5$ 2

(b) Differentiate $y = \tan^{-1}\left(\frac{1}{x}\right)$ 3

(c) Sketch the curve $y = 2\sin(x + \pi)$ for $0 \le x \le 2\pi$ 2

(d) If $y = ae^{bx}$, show $\frac{d^2y}{dx^2} = b^2 y$ where a, b are constants 2

(e) Solve: $x^{\frac{2}{3}} + x^{\frac{1}{3}} - 6 = 0$ 3

Question 2. (Start a new page)

(a) State the domain and range of $y = 4\sin^{-1}2x$ and sketch the curve. 3

(b) Solve: $\cos^2 x - \cos 2x = 0$ for $0 \le x \le 2\pi$ 3

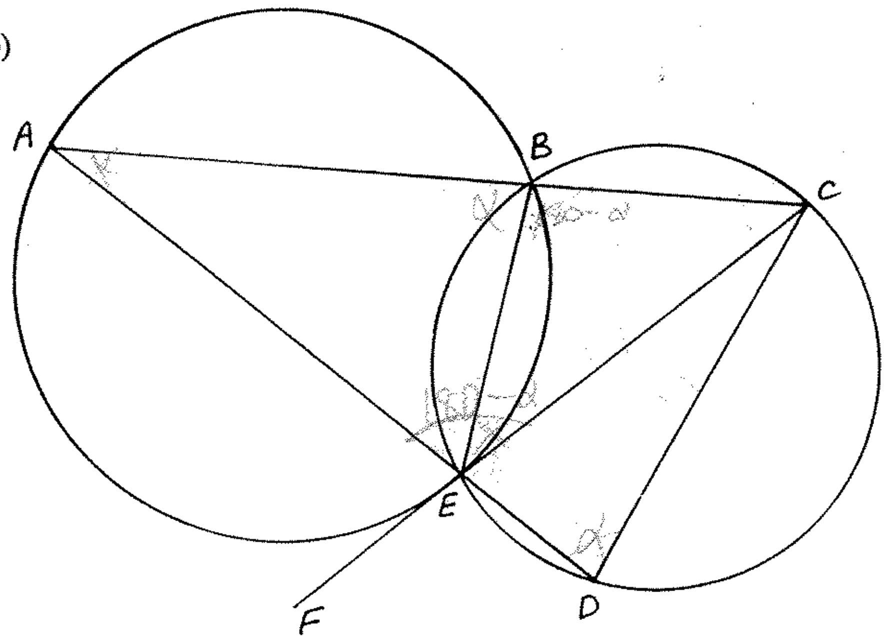
(c) Find the exact value of $\int_1^2 \frac{e^{\frac{1}{x}}}{x^2} dx$ using the substitution $u = \frac{1}{x}$ 3

(d) Use $x = 0.5$ to find an approximation to the root of $\cos x = x$ using one application of Newton's method. (Answer correct to two decimal places.) 3

Question 3. (Start a new page) **Marks**

(a) Solve: $\dfrac{x^2-3}{2x}>0$ 3

(b) 4



CEF is a tangent to circle AEB

ABC and AED are secants.

(i) Prove ΔACE is similar to ΔECB

(ii) Show that CE = CD

(c) $P(4p, 2p^2)$ and $Q(4q, 2q^2)$ are two variable points on the parabola $x^2 = 8y$. 5
R is the point of intersection of the tangents at P and Q.

(i) Show that the co-ordinates of R are $(2[p+q], 2pq)$.

(ii) Find the cartesian equation of the locus of R, if $p^2 + q^2 = 8$.

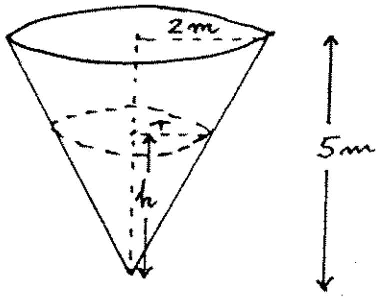
Question 4. (Start a new page) **Marks**

(a) Consider $P(x) = x^4 - x^3 - 3x^2 + 5x - 2$ 5

(i) Show that 1 and -2 are zeros of $P(x)$

(ii) Using sum and product of roots, or the division algorithm, factorise $P(x)$ into linear factors.

(b) An inverted right circular cone has height 5m and base radius 2m. Water is flowing from the apex (point) at a constant rate of $0.2m^3/\text{min}$. 5



(i) If h is the height when the radius is r, show that $r = \frac{2h}{5}$

(ii) At height h, show that V, the volume of water is given by $V = \frac{4\pi h^3}{75}$

(iii) Hence find the rate at which the water level is falling when the water is 4m deep.

(c) By letting $t = \tan\left(\frac{\theta}{2}\right)$, prove 2

$$\frac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta} = \tan\left(\frac{\theta}{2}\right)$$

Question 5. (Start a new page) **Marks**

(a) Find the acute angle between the tangents to the curves $y = x^2$ and $y = (x-2)^2$ at the point of intersection of these two curves. 3

(b) (i) Show that $T = P + Ae^{kt}$ is a solution of $\frac{dT}{dt} = k(T-P)$ where k, P and A are constants. 6

(ii) Meat, initially at 14°C is placed in a freezer whose temperature is a constant -10°C. After 25 seconds, the meat is 11°C.

(α) Show that A=24 and $k = -0.005$

(β) Find (to the nearest minute) when the temperature of the meat will reach -8°C.

(c) Find the co-ordinates of the point which divides the line joining the points (-1, 3) and (5, -7) externally in the ratio 4:3 3

Question 6. (Start a new page)

(a) (i) Show that $\sqrt{3}\cos x + \sin x$ can be expressed as $2\cos\left(x - \frac{\pi}{6}\right)$ 4

(ii) Hence state the greatest value of the expression $\sqrt{3}\cos x + \sin x$ and state the smallest positive value of x that gives this maximum value to the expression.

(b) Consider the graph $y = \frac{x^2}{1-x^2}$ 8

(i) Write down the domain of this function.

(ii) Find the turning point and determine its nature.

(iii) Prove that the function is even.

(iv) Find $\lim_{x\to\infty} \frac{x^2}{1-x^2}$

(v) Sketch the graph.

Question 7. (Start a new page) **Marks**

(a) Evaluate $\int_0^{\frac{2}{5}} \frac{dx}{\sqrt{4-25x^2}}$ 3

(b) (i) Using the fact $\cos 3x = 4\cos^3 x - 3\cos x$, find the general solutions of the equation $\cos 3x + 2\cos x = 0$ 6

(ii) What are the smallest and largest solutions for x in part (i) in the interval $0 \le x \le 2\pi$?

(c) Use mathematical induction to prove that $3^{2n+4} - 2^{2n}$ is divisible by 5, for $n \ge 1$. 3

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1)(a) $\int_4^{20} y\,dx$ if $xy = 5$

$= \int_4^{20} \frac{5}{x}\,dx$

$= 5\left[\log x\right]_4^{20}$

$= 5(\log 20 - \log 4)$

$= \underline{5\log 5}$ ②

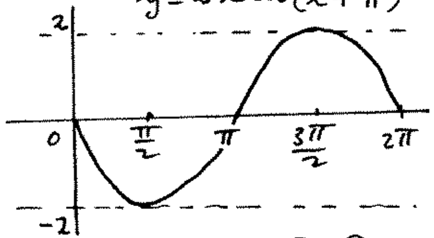
(b) $y = \tan^{-1}\left(\frac{1}{x}\right)$

$\frac{dy}{dx} = \frac{1}{1+\left(\frac{1}{x}\right)^2} \times -\frac{1}{x^2}$

$= \frac{x^2}{x^2+1} \times -\frac{1}{x^2}$

$= \underline{\frac{-1}{x^2+1}}$ ③

(c) $y = 2\sin(x+\pi)$



②

(d) $y = ae^{bx}$

$\frac{dy}{dx} = ab\,e^{bx}$

$\frac{d^2y}{dx^2} = ab^2e^{bx}$

$= \underline{b^2y}$. ②

(e) $x^{\frac{2}{3}} + x^{\frac{1}{3}} - 6 = 0$

let $m = x^{\frac{1}{3}}$

$\therefore m^2 + m - 6 = 0$

$(m+3)(m-2) = 0$

$\therefore m = -3 \quad , m = 2$

$\therefore x^{\frac{1}{3}} = -3 \quad \therefore x^{\frac{1}{3}} = 2$

$\therefore \underline{x = -27} \quad \therefore x = 8$ ③

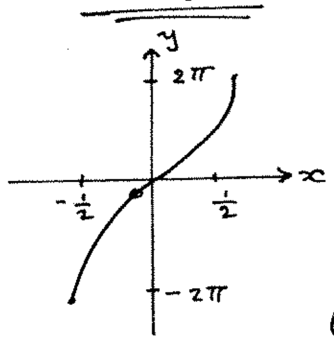
12

2)(a) D: $-1 \le 2x \le 1$

$\therefore \underline{-\frac{1}{2} \le x \le \frac{1}{2}}$

R: $4\times\left(-\frac{\pi}{2}\right) \le y \le 4\times\left(\frac{\pi}{2}\right)$

$\underline{-2\pi \le y \le 2\pi}$



③

(b) $\cos^2 x - \cos 2x = 0$

$\therefore \cos^2 x - (2\cos^2 x - 1) = 0$

$\therefore \cos^2 x - 2\cos^2 x + 1 = 0$

$\therefore \cos^2 x = 1$

$\therefore \cos x = \pm 1$

$\therefore x = \underline{0, \pi, 2\pi}$. ③

(c) $I = \int_1^2 \frac{e^{\frac{1}{x}}}{x^2}\,dx$

let $u = \frac{1}{x}$ $\begin{cases} x=1, u=1 \\ x=2, u=\frac{1}{2} \end{cases}$

$\therefore \frac{du}{dx} = -\frac{1}{x^2}$

$\therefore -du = \frac{dx}{x^2}$

$\therefore I = -\int_1^{\frac{1}{2}} e^u\,du$

$\therefore I = -\left[e^u\right]_1^{\frac{1}{2}}$

$= -\left\{e^{\frac{1}{2}} - e^1\right\}$

$= \underline{e - e^{\frac{1}{2}}}$ ③

(d) $a_1 = a - \frac{f(a)}{f'(a)}$

$a = 0.5$

$f(a) = \cos(0.5) - (0.5)$

$f'(a) = -\sin(0.5) - 1$

$a_1 = \underline{0.76}$ ③

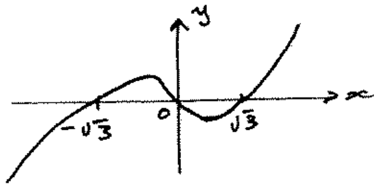
12

3)(a) $\frac{x^2-3}{2x} > 0$

$\therefore \frac{(x^2-3)\,x^2}{2x} > 0$

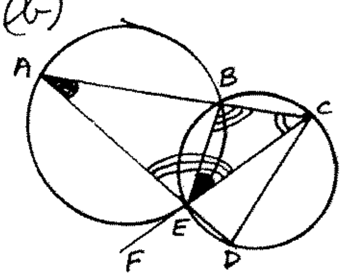
$\therefore x(x^2-3) > 0$

$x(x+\sqrt{3})(x-\sqrt{3}) > 0$



$\therefore \underline{-\sqrt{3} < x < 0}, \quad x > \sqrt{3}$ ③

(b)



(i) $\widehat{BAE} = \widehat{BEC}$ (∠ in alt. seg.)

$\widehat{BCE}$ is common

$\therefore \widehat{AEC} = \widehat{EBC}$ (∠ sum of Δ's)

$\therefore \underline{\Delta ACE\ |||\ \Delta ECB}$ (equiang...

(ii) $\widehat{EDC} = 180 - \widehat{EBC}$ (opp ∠, cyclic qu...

$\widehat{CED} = 180 - \widehat{AEC}$ (str. ∠).

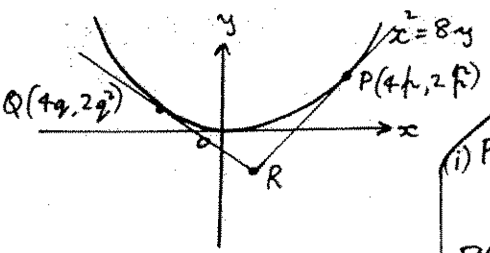
$\therefore \widehat{EDC} = \widehat{CED}$ ($\widehat{AEC} = \widehat{EBC}$ from (i)

$\therefore \Delta CED$ is isosceles

$\therefore \underline{CE = CD}$ (opp = ∠'s in isos ∠

④

c)



i) $x^2 = 8y$

$y = \frac{x^2}{8}$

$\frac{dy}{dx} = \frac{x}{4}$

$x = 4p$ $\frac{dy}{dx} = \frac{4p}{4}$
$= p$.

...ng at P is:

$y - 2p^2 = p(x - 4p)$

$y - 2p^2 = px - 4p^2$

$y = px - 2p^2 \cdots (1)$

...imilarly tang at Q is:

$y = qx - 2q^2 \cdots (2)$

...olving (1) and (2)

$px - 2p^2 = qx - 2q^2$

$x(p-q) = 2(p^2 - q^2)$

$\therefore x = 2(p+q)$

...b in (1)

$y = p.2(p+q) - 2p^2$

$y = 2p^2 + 2pq - 2p^2$

$y = 2pq$

R is $(2[p+q], 2pq)$.

) Let R be pt (x, y)

$\therefore x = 2(p+q)$ and $y = 2pq$

...ow $(p+q)^2 = p^2 + q^2 + 2pq$

$\therefore \left(\frac{x}{2}\right)^2 = 8 + y$

$\therefore \frac{x^2}{4} = 8 + y$

$\therefore x^2 = 4(y+8)$ (5)

12

4)(a)

$P(x) = x^4 - x^3 - 3x^2 + 5x - 2$

(i) $P(1) = 1 - 1 - 3 + 5 - 2$
$= 0$

$P(-2) = 16 + 8 - 12 - 10 - 2$
$= 0$

(ii) Let roots be $\alpha, \beta, 1, -2$

$\therefore \alpha + \beta + 1 - 2 = 1$

$\therefore \alpha + \beta = 2 \cdots (1)$

and $(\alpha\beta)(1)(-2) = -2$

$\therefore -2\alpha\beta = -2$

$\therefore \alpha\beta = 1 \cdots (2)$

Solving (1) and (2)

$\alpha(2-\alpha) = 1$

$2\alpha - \alpha^2 = 1$

$\alpha^2 - 2\alpha + 1 = 0$

$(\alpha - 1)^2 = 0$

$\therefore \alpha = 1$

$\beta = 1$

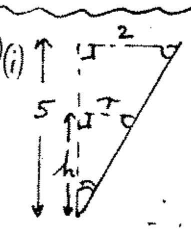
$\therefore P(x) = (x-1)^3(x+2)$

OR

$$\begin{array}{r} x^2 - 2x + 1 \\ x^2+x-2 \overline{)\, x^4 - x^3 - 3x^2 + 5x - 2} \\ x^4 + x^3 - 2x^2 \\ \hline -2x^3 - x^2 + 5x \\ -2x^3 - 2x^2 + 4x \\ \hline x^2 + x - 2 \\ x^2 + x - 2 \\ \hline 0 \end{array}$$

$\therefore P(x) = (x-1)^3(x+2)$ (5)

(b)(i)



By similar Δs

$\frac{r}{2} = \frac{h}{5}$

$\therefore r = \frac{2h}{5}$

(ii) $V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi\left(\frac{2h}{5}\right)^2 h$

$= \frac{4\pi h^3}{75}$

(iii) $\frac{dh}{dt} = ?$ $V = \frac{4\pi h^3}{75}$

$\frac{dV}{dt} = 0.2$ $\therefore \frac{dV}{dh} = \frac{4\pi h^2}{25}$

$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$

$= \frac{25}{4\pi h^2} \times 0.2$

$= \frac{5}{4\pi h^2}$ (5)

when $h = 4$: $\frac{dh}{dt} = \frac{5}{4\pi 16}$

$= \frac{5}{64\pi}$ m/min

(c) Prove (let $t = \tan\left(\frac{\theta}{2}\right)$).

$$\frac{1 + \sin\theta - \cos\theta}{1 + \sin\theta + \cos\theta} = \tan\left(\frac{\theta}{2}\right)$$

$$L.H.S. = \frac{1 + \frac{2t}{1+t^2} - \frac{1-t^2}{1+t^2}}{1 + \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}}$$

$$= \frac{1 + t^2 + 2t - 1 + t^2}{1 + t^2 + 2t + 1 - t^2}$$

$$= \frac{2t + 2t^2}{2 + 2t}$$

$$= \frac{2t(1+t)}{2(1+t)}$$

$= t$

$= RHS$. (2)

12

5)(a) $y = x^2$

$y = (x-2)^2$

$\therefore x^2 = x^2 - 4x + 4$

$\therefore 4x = 4$

$x = 1$, $y = 1$

Pt of int. is $(1, 1)$

For $y = x^2$

$\frac{dy}{dx} = 2x$

At $x = 1$: $\frac{dy}{dx} = 2$

$\therefore m_1 = 2$

For $y=(x-2)^2$

$\frac{dy}{dx} = 2(x-2)$

$t\ x=1 \quad \frac{dy}{dx} = -2$

$\therefore m_2 = -2$

ow $\tan\theta = \left|\frac{m_1 - m_2}{1+m_1 m_2}\right|$

$= \left|\frac{2+2}{1-4}\right|$

$= \left|\frac{4}{-3}\right|$

$= \frac{4}{3}$

$\therefore \theta = \underline{53°8'}$ (3)

)(i) $T = P + Ae^{kt}$

$\therefore \frac{dT}{dt} = kAe^{kt}$

but $Ae^{kt} = T - P$

$\therefore \frac{dT}{dt} = k(T-P)$

i)(α) $T = P + Ae^{kt}$

$t=0,\ P=-10,\ T=14$: $14 = -10 + A$, $A = 24$

$t=25,\ T=11$: $11 = -10 + 24e^{25k}$

$\frac{21}{24} = e^{25k}$

$\therefore k = \frac{1}{25}\log\left(\frac{21}{24}\right)$

$\doteqdot -0.005$

3) $T=-8,\ t=?$: $-8 = -10 + 24e^{-0.005t}$

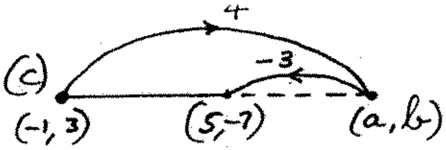
$\frac{2}{24} = e^{-0.005t}$

$\therefore \log\left(\frac{1}{12}\right) = -0.005t$

$\therefore t = \frac{\log\left(\frac{1}{12}\right)}{-0.005}$

$= 496.98$ sec

$\doteqdot \underline{8 \text{ min}}$ (6)

(c) 

$a = \frac{(-1)(-3)+(5)(4)}{4-3}$

$= \underline{23}$

$b = \frac{(3)(-3)+(-7)(4)}{4-3}$

$= \underline{-37}$

Pt is $\underline{(23, -37)}$ (3)

12

6)(a)(i) Let $\sqrt{3}\cos x + \sin x$

$\equiv A\cos\alpha\cos x + A\sin\alpha\sin x$

$\therefore A\cos\alpha = \sqrt{3}$

$A\sin\alpha = 1$

$\therefore \tan\alpha = \frac{1}{\sqrt{3}}$

$\therefore \alpha = \frac{\pi}{6}$

and $A^2(\cos^2\alpha + \sin^2\alpha) = 4$

$\therefore A = 2.$

$\therefore \sqrt{3}\cos x + \sin x = \underline{2\cos\left(x - \frac{\pi}{6}\right)}$

(ii) greatest value is $\underline{2}$

x value is $\underline{\frac{\pi}{6}}$ (4)

(b) $y = \frac{x^2}{1-x^2}$

(i) D : all real x, $x \neq \pm 1$.

(ii) $y = \frac{x^2}{1-x^2}$

$\frac{dy}{dx} = \frac{(1-x^2)2x - x^2(-2x)}{(1-x^2)^2}$

$= \frac{2x - 2x^3 + 2x^3}{(1-x^2)^2}$

$= \frac{2x}{(1-x^2)^2}$

$= 0$ when $x = 0$

$\therefore$ stat. pt at $(0,0)$

$\therefore f'(-1) < 0,\ f'(0) = 0,\ f'(1) > 0$

\ — /

$\therefore$ Min. t.p. at $\underline{(0,0)}$

(iii) $f(-x) = \frac{(-x)^2}{1-(-x)^2}$

$= \frac{x^2}{1-x^2}$

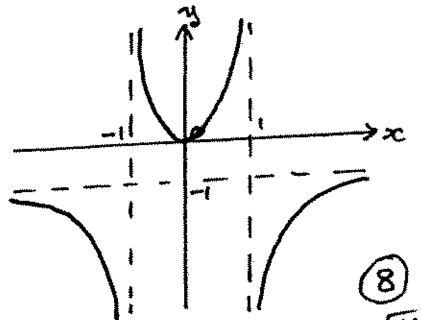
$= f(x)$

(iv) $\lim_{x\to\infty} \frac{x^2}{1-x^2}$

$= \lim_{x\to\infty} \frac{\frac{x^2}{x^2}}{\frac{1}{x^2} - \frac{x^2}{x^2}}$

$= \lim_{x\to\infty} \frac{1}{\frac{1}{x^2} - 1}$

$= \underline{-1}$ $\left(\lim_{x\to\infty}\frac{1}{x^2} = 0\right)$

(v) 

(8)

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7)(a) $\int_0^{\frac{2}{5}} \frac{dx}{\sqrt{4-25x^2}}$

$= \frac{1}{5}\int_0^{\frac{2}{5}} \frac{dx}{\sqrt{\left(\frac{2}{5}\right)^2 - x^2}}$

$= \frac{1}{5}\left[\sin^{-1}\left(\frac{x}{\frac{2}{5}}\right)\right]_0^{\frac{2}{5}}$

$= \frac{1}{5}\left[\sin^{-1}\frac{5x}{2}\right]_0^{\frac{2}{5}}$

$= \frac{1}{5}\left\{\sin^{-1}1 - \sin^{-1}0\right\}$

$= \frac{1}{5}\left(\frac{\pi}{2} - 0\right)$

$= \underline{\frac{\pi}{10}}$ (3)

(b) (i) If $\cos 3x = 4\cos^3 x - 3\cos x$

then $\cos 3x + 2\cos x = 0$ becomes

$4\cos^3 x - 3\cos x + 2\cos x = 0$

$\therefore \quad 4\cos^3 x - \cos x = 0$

$\cos x\,(4\cos^2 x - 1) = 0$

$\cos x\,(2\cos x - 1)(2\cos x + 1) = 0$

$\therefore \cos x = 0,\ \pm\frac{1}{2}$

General solutions:

$\cos x = 0$:

$x = 2n\pi \pm \cos^{-1} 0$

$x = 2n\pi \pm \frac{\pi}{2}$

$\cos x = \frac{1}{2}$:

$x = 2n\pi \pm \cos^{-1}(\frac{1}{2})$

$x = 2n\pi \pm \frac{\pi}{3}$

$\cos x = -\frac{1}{2}$:

$x = 2n\pi \pm \cos^{-1}(-\frac{1}{2})$

$= 2n\pi \pm (\pi - \cos^{-1}(\frac{1}{2}))$

$= 2n\pi \pm (\pi - \frac{\pi}{3})$

$x = 2n\pi \pm \frac{2\pi}{3}$

(ii) smallest:

($n=0$ in $x = 2n\pi + \frac{\pi}{3}$)

$x = \frac{\pi}{3}$

largest:

($n=1$ in $x = 2n\pi - \frac{\pi}{3}$)

$x = \frac{5\pi}{3}$

(6)

(c) $3^{2n+4} - 2^{2n}$

$n=1$: $3^6 - 2^2$

$= 725$

$\therefore$ divisible by 5

True for $n=1$

Assume true for $n=k$.

$\therefore 3^{2k+4} - 2^{2k} = 5M$

To prove true for $n=k+1$

Now $3^{2(k+1)+4} - 2^{2(k+1)}$

$= 3^{2k+6} - 2^{2k+2}$

$= 3^{(2k+4)+2} - 2^{2k+2}$

$= 9.3^{2k+4} - 4.2^{2k}$

$= 9(5M + 2^{2k}) - 4.2^{2k}$

$= 45M + 9.2^{2k} - 4.2^{2k}$

$= 45M + 5.2^{2k}$

$= 5[9M + 2^{2k}]$

Having assumed true for $n=k$, proven true for $n=k+1$

BUT true for $n=1$

$\therefore$ true for $n=2$

$\therefore$ True for $n = 3, 4, 5, ---, +$.

(3)

12

Total = 84