Marks

Question 1 Begin a new page

(a) Find the value of $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ in terms of π. 1

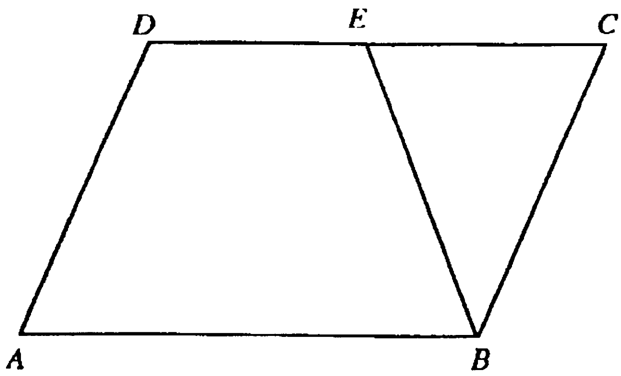
(b) The acute angle between the line $x-2y+3=0$ and the line $y=mx$ is $45°$. 3

(i) Show that $\left|\frac{2m-1}{m+2}\right|=1$

(ii) Find the possible values of m.

(c) Solve the equation $\ln\left(x^2+19\right)=2\ln(x+1)$. 3

(d) 5



$ABCD$ is a parallelogram. E is the point on CD such that $BE=BC$.

(i) Copy the diagram showing the above information.

(ii) Show that $ABED$ is a cyclic quadrilateral.

Marks

Question 2 Begin a new page

(a) Find $\lim_{x \to 0} \dfrac{\sin 2x}{x}$ 1

(b) Solve the inequality $\dfrac{x^2+9}{x} \le 6$ 3

(c) (i) Factorise $3x^3+3x^2-x-1$ 3

(ii) Solve the equation $3\tan^3\theta+3\tan^2\theta-\tan\theta-1=0$ for $0 \le \theta \le \pi$

(d) $P(2t, t^2)$ is a point on the parabola $x^2=4y$ with focus F. The point M divides the interval FP externally in the ratio 3 : 1. 5

(i) Show that as P moves on the parabola $x^2=4y$, then M moves on the parabola $x^2=6y+3$.

(ii) Find the coordinates of the focus and the equation of the directrix of the locus of M.

Question 3 Begin a new page

(a) Find the gradient of the tangent to the curve $y=\tan^{-1}\frac{1}{x}$ at the point on the curve where $x=1$. 2

(b) A function is given by the rule $f(x)=\dfrac{x+1}{x+2}$. Find the rule for the inverse function $f^{-1}(x)$. 2

(c) At any point on the curve $y=f(x)$ the gradient function is given by $\dfrac{dy}{dx}=2\cos^2 x+1$. If $y=\pi$ when $x=\pi$, find the value of y when $x=2\pi$. 4

(d) Use the substitution $x=u^2$, $u>0$, to express the value of $\displaystyle\int_1^{100} \frac{1}{x+2\sqrt{x}}\,dx$ in the form $\ln a$ for some constant $a>0$. 4

Marks

Question 4 **Begin a new page**

(a) Find the exact value of $\int_{\sqrt{2}}^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\, dx$. 2

(b) A particle is moving in a straight line. At time t seconds its displacement x metres from a fixed point O on the line is such that $t = x^2 - 3x + 2$. 2

(i) Find an expression for its velocity v in terms of x.

(ii) Find an expression for its acceleration a in terms of x.

(c) Consider the function $y = 2\cos^{-1}(1-x)$. 4

(i) Find the domain and range of the function.

(ii) Sketch the graph of the function.

(d) The radius r kilometres of a circular oil spill at time t hours after it was first observed is given by $r = \frac{1+3t}{1+t}$. Find the exact rate of increase of the area of the oil spill when the radius is 2 kilometres. 4

Question 5 **Begin a new page**

(a) Consider the function $f(x) = \frac{\ln x}{x}$. 6

(i) Find the coordinates and the nature of the stationary point on the curve $y = f(x)$.

(ii) Explain why $f(\pi) < f(e)$ and hence show that $\pi^e < e^\pi$.

(iii) $P(X, -2)$ is a point on the curve $y = f(x)$. Starting with an initial approximation of $X = 0\cdot5$, use one application of Newton's method to find an improved approximation to the value of X, giving the answer correct to 2 decimal places.

Marks

Question 5 (Cont)

(b) A machine which initially costs \$49 000 loses value at a rate proportional to the difference between its current value $\$M$ and its final scrap value \$1000. After 2 years the value of the machine is \$25 000. **6**

(i) Explain why $\dfrac{dM}{dt} = -k\,(M - 1000)$ for some constant $k > 0$, and verify that $M = 1000 + Ae^{-kt}$, A constant, is a solution of this equation.

(ii) Find the exact values of A and k.

(iii) Find the value of the machine, and the time that has elapsed, when the machine is losing value at a rate equal to one quarter of the initial rate at which it loses value.

Question 6 Begin a new page

(a) If α, β and χ are the roots of $3x^3 + 5x^2 - 7x + 4 = 0$, find the values of **2**

(i) $\alpha + \beta + \chi$

(ii) $\alpha\beta + \alpha\chi + \beta\chi$

(b) Two circles touch internally at a point P. A line through P cuts the smaller circle at A and the larger circle at B. A second line through P cuts the smaller and larger circles at C and D respectively. **4**

(i) Draw a diagram showing this information.

(ii) Prove that AC is parallel to BD.

(c) A particle moving in a straight line is performing Simple Harmonic Motion. At time t seconds its displacement x metres from a fixed point O on the line is given by $x = 2\sin 3t - 2\sqrt{3}\cos 3t$. **6**

(i) Express x in the form $x = R\sin(3t - \alpha)$ for some constants $R > 0$ and $0 < \alpha < \frac{\pi}{2}$.

(ii) Describe the initial motion of the particle in terms of its initial position, velocity and acceleration

(iii) Find the exact value of the first time that the particle is 2 metres to the left of O and moving towards O.

Marks

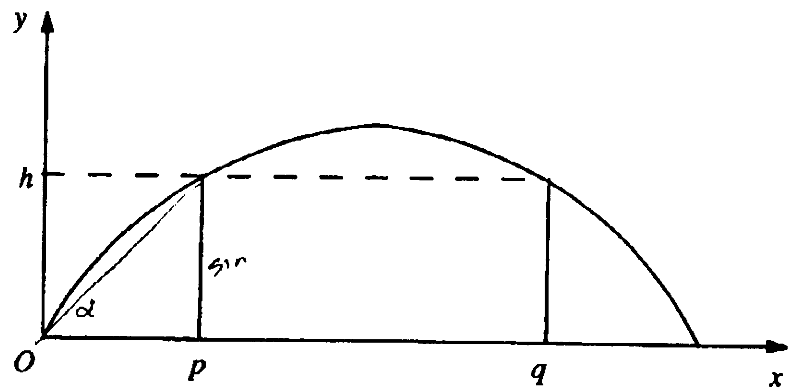
Question 7 Begin a new page

(a) Use the method of mathematical induction to prove that $7^n - 5^n$ is even, for all positive integers $n \geq 1$. 4

(b) Given that ABCD is a cyclic quadrilateral, show that 2

$$\tan A + \tan B + \tan C + \tan D = 0$$

(c) 6



A particle is projected with velocity V ms^{-1} from a point O at an angle of elevation α. Axes Ox and Oy are taken horizontally and vertically through O. The particle just clears two vertical chimneys of height h metres at horizontal distances of p metres and q metres from O. The acceleration due to gravity is taken as 10 ms^{-2} and air resistance is ignored

(i) Write down expressions for the horizontal displacement x and the vertical displacement y of the particle after time t seconds.

(ii) Show that $V^2 = \dfrac{5p^2\left(1+\tan^2\alpha\right)}{p\tan\alpha - h}$.

(iii) Show that $\tan\alpha = \dfrac{h(p+q)}{pq}$.

Solutions to NSGHS 1999 3U Trial

1 (a) $\frac{5\pi}{6}$

(b)(i) $x-2y+3=0$ has gradient $\frac{1}{2}$

$\therefore \tan 45^\circ = \left|\frac{m-\frac{1}{2}}{1+\frac{m}{2}}\right|$

$1 = \left|\frac{2m-1}{2+m}\right|$

(ii) $\frac{2m-1}{m+2}=1$ or $\frac{2m-1}{m+2}=-1$

$2m-1=m+2$ $\quad$ $2m-1=-m-2$

$m=3$ $\quad$ $3m=-1$

$\therefore m=3$ or $-\frac{1}{3}$

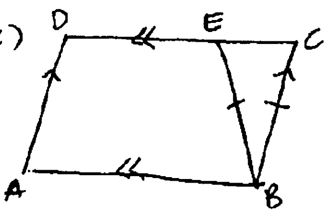
(c) $\ln(x^2+19)=\ln(x+1)^2$

$x^2+19=x^2+2x+1$

$18=2x$

$x=9$

(d)(i)



(ii) $\angle BCE=\angle BEC$ (equal angles opposite equal sides in $\triangle BCE$)

Also $\angle BCE=\angle BAD$ (opposite angles of parallelogram)

$\therefore \angle BEC=\angle BAD$

$\therefore$ ABED is a cyclic quadrilateral (exterior angle equal to opposite interior angle)

2 (a) $\lim_{x\to 0}\frac{\sin 2x}{x} = 2\lim_{x\to 0}\frac{\sin 2x}{2x}$

$=2\times 1$

$=2$

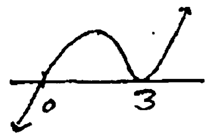
(b) $\frac{x^2+9}{x}\times x^2 \le 6x^2$

$x^3+9x \le 6x^2$

$x^3-6x^2+9x \le 0$

$x(x^2-6x+9)\le 0$

$x(x-3)^2 \le 0$



$x\le 0$ or $x=3$

But $x\ne 0$

$\therefore x<0$ or $x=3$

(c)(i) $3x^2(x+1)-1(x+1)=(x+1)(3x^2-1)$

(ii) Let $x=\tan\theta$

$\therefore (\tan\theta+1)(3\tan^2\theta-1)=0$

$\tan\theta=-1$ or $\tan^2\theta=\frac{1}{3}$, $0\le\theta\le\pi$

$\tan\theta=\pm\frac{1}{\sqrt{3}}$

$\theta=\frac{3\pi}{4}, \frac{\pi}{6}, \frac{5\pi}{6}$

(d)(i) $x=\frac{3\times 2t-1\times 0}{3-1}$, $y=\frac{3\times t^2-1\times 1}{3-1}$

$=\frac{6t}{2}$ $\quad$ $y=\frac{3t^2-1}{2}$ ②

$x=3t$ ①

From ①, $t=\frac{x}{3}$, so from ②, $y=\frac{3.\frac{x^2}{9}-1}{2}$

$2y=\frac{x^2}{3}-1$

$x^2=6y+3$

(ii) $x^2=6(y+\frac{1}{2})$ Focus $(0,1)$, dir $y=-2$

3 (a) $y = \tan^{-1}\frac{1}{x}$

$$\frac{dy}{dx} = \frac{1}{1+\frac{1}{x^2}} \cdot \frac{-1}{x^2}$$

$$= \frac{-1}{x^2+1}$$

When $x=1$, $\frac{dy}{dx} = \frac{-1}{2}$ = gradient of tangent

(b) Inverse is $x = \frac{y+1}{y+2}$

$$xy+2x = y+1$$

$$y(x-1) = 1-2x$$

$$\therefore y = \frac{1-2x}{x-1}$$

$$\therefore f^{-1}(x) = \frac{1-2x}{x-1}$$

(c) $\frac{dy}{dx} = 2\cos^2 x + 1$ $\qquad \cos 2x = 2\cos^2 x - 1$

$$= 1+\cos 2x + 1$$

$$= 2+\cos 2x$$

$$y = 2x + \tfrac{1}{2}\sin 2x + c$$

When $x=\pi$, $y=\pi$

$$\therefore \pi = 2\pi + \tfrac{1}{2}\sin 2\pi + c$$

$$\therefore c = -\pi$$

$$y = 2x + \tfrac{1}{2}\sin 2x - \pi$$

When $x = 2\pi$, $y = 4\pi + \frac{1}{2}\sin 4\pi - \pi$

$$= 3\pi$$

(d) $x = u^2$ $\qquad$ When $x=1$, $u=1$

$\frac{dx}{du} = 2u$ $\qquad$ " $x=100$, $u=10$

$dx = 2u.du$

$$\therefore \int_1^{10} \frac{2u.du}{u^2+2u} = \int_1^{10}\frac{2\,du}{u+2}$$

$$= 2\left[\ln(u+2)\right]_1^{10} = 2\ln\frac{12}{3}$$

$$= \ln 16$$

4 (a) $\int_{\sqrt2}^{\sqrt3}\frac{dx}{\sqrt{4-x^2}} = \left[\sin^{-1}\frac{x}{2}\right]_{\sqrt2}^{\sqrt3}$

$$= \sin^{-1}\frac{\sqrt3}{2} - \sin^{-1}\frac{\sqrt2}{2}$$

$$= \frac{\pi}{3}-\frac{\pi}{4}$$

$$= \frac{\pi}{12}$$

(b) (i) $t = x^2-3x+2$

$$\frac{dt}{dx} = 2x-3$$

$$\therefore v = \frac{dx}{dt} = \frac{1}{2x-3}$$

(ii) $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$$= \frac{d}{dx}\left[\tfrac{1}{2}(2x-3)^{-2}\right]$$

$$= -(2x-3)^{-3}.\,2$$

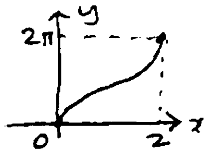
$$= -\frac{2}{(2x-3)^3}$$

(c) (i) Domain is $-1\le 1-x\le 1$

$$-2\le -x\le 0$$

$$0\le x\le 2$$

Range is $0\le y\le 2\pi$



(d) $\frac{dr}{dt} = \frac{(1+t).3-(1+3t).1}{(1+t)^2}$ $\qquad$ and $A=\pi r^2$

$\frac{dA}{dr} = 2\pi r$

$$= \frac{2}{(1+t)^2}$$

$$\frac{dA}{dt} = \frac{dA}{dr}\cdot\frac{dr}{dt}$$

$$= \frac{4\pi r}{(1+t)^2}$$

When $r=2$, $2 = \frac{1+3t}{1+t}$

$$2+2t = 1+3t$$

$$t=1$$

$$\therefore \frac{dA}{dt} = \frac{8\pi}{4} = 2\pi$$

$\therefore$ Rate of increase is 2π km/h

5) (a) (i) $\frac{dy}{dx} = \frac{x \cdot \frac{1}{x} - \ln x \cdot 1}{x^2}$

$= \frac{1 - \ln x}{x^2}$

$\frac{d^2y}{dx^2} = \frac{x^2 \cdot -\frac{1}{x} - (1-\ln x).2x}{x^4}$

$= \frac{2x\ln x - 3x}{x^4}$

$= \frac{2\ln x - 3}{x^3}$

$\frac{dy}{dx} = 0 \Rightarrow \frac{1-\ln x}{x^2} = 0$

$\ln x = 1$

$x = e$

When $x = e$, $\frac{d^2y}{dx^2} = -\frac{1}{e^3} < 0$

$\therefore$ Max. turning pt at $(e, \frac{1}{e})$

(ii) Since there is a maximum at $x = e$, and $\frac{dy}{dx} < 0$ for $x > e$, then, since $\pi > e$,

$f(\pi) < f(e)$

ie $\frac{\ln \pi}{\pi} < \frac{\ln e}{e}$

$e \ln \pi < \pi \ln e$

$\ln \pi^e < \ln e^\pi$

ie $\pi^e < e^\pi$

(iii) $\frac{\ln X}{X} = -2$

$\ln X = -2X$

$\ln X + 2X = 0$

Let $P(X) = \ln X + 2X$

$P'(X) = \frac{1}{X} + 2$

If $X_1 = 0.5$,

$X_2 = 0.5 - \frac{P(0.5)}{P'(0.5)}$

$= 0.42$ (2d p)

(b) (i) $\frac{dM}{dt} \propto M - 1000$ and $\frac{dM}{dt} < 0$

$\therefore \frac{dM}{dt} = -k(M-1000)$ ($k > 0$)

$M = 1000 + Ae^{-kt}$

$\frac{dM}{dt} = -Ake^{-kt}$

$= -k(M-1000)$

(ii) When $t = 0$, $M = 49000$

$\therefore 49000 = 1000 + A$

$\therefore A = 48000$

Then $M = 1000 + 48000e^{-kt}$

When $t = 2$, $M = 25000$

$25000 = 1000 + 48000e^{-2k}$

$24000 = 48000e^{-2k}$

$0.5 = e^{-2k}$

$-2k = \ln 0.5$

$k = \frac{-\ln 0.5}{2}$

$= \frac{\ln 2}{2}$

(iii) When $M = 49000$,

$\frac{dM}{dt} = -\frac{\ln 2}{2}(49000 - 1000)$

$= -24000 \ln 2$

Let $\frac{dM}{dt} = \frac{-24000 \ln 2}{4} = -\frac{\ln 2}{2}(M - 1000)$

$12000 = M - 1000$

$\therefore M = 13000$

When $M = 13000$, $13000 = 1000 + 48000e^{-kt}$

$12000 = 48000e^{-kt}$

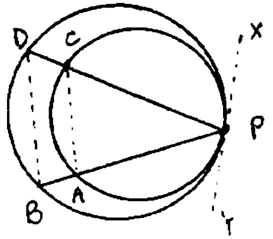
$0.25 = e^{-kt}$

$-kt = \ln 0.25$

$t = -\frac{\ln 0.25}{k} = 4$

) (a)(i) $\alpha+\beta+\gamma = -\frac{5}{3}$

(ii) $\alpha\beta+\alpha\gamma+\beta\gamma = -\frac{7}{3}$

(b) 

(ii) Draw common tangent through P. Call it XY.

Then $\angle XPC = \angle PAC$ (angle between tangent and chord equal to angle in alternate segment)

and $\angle XPC = \angle PBD$ for large circle

$\therefore \angle PAC = \angle PBD$

$\therefore AC \parallel BD$ (corresponding angles equal)

(c)(i) Let $2\sin 3t - 2\sqrt{3}\cos 3t = R\sin(3t-\alpha)$
$= R(\sin 3t\cos\alpha - \cos 3t\sin\alpha)$

$\therefore \begin{cases} R\cos\alpha = 2 \\ R\sin\alpha = 2\sqrt{3} \end{cases}$

$R^2 = 2^2+(2\sqrt{3})^2$ and $\tan\alpha = \sqrt{3}$
$= 4+12$

$\therefore R = 4$ and $\alpha = \frac{\pi}{3}$

$\therefore x = 4\sin(3t-\frac{\pi}{3})$

(ii) $\dot{x} = 12\cos(3t-\frac{\pi}{3})$

$\ddot{x} = -36\sin(3t-\frac{\pi}{3})$

When $t=0$, $x = -2\sqrt{3}$

$\dot{x} = 6$

$\ddot{x} = 18\sqrt{3}$

$\therefore$ Initially, the particle is $2\sqrt{3}$ m to the left of O, moving at 6 m/s to the right, speeding up at a rate of $18\sqrt{3}$ m/s².

(iii) When $x=-2$, $-2 = 4\sin(3t-\frac{\pi}{3})$

$\sin(3t-\frac{\pi}{3}) = -\frac{1}{2}$

$3t-\frac{\pi}{3} = -\frac{\pi}{6}, \frac{7\pi}{6}, \ldots$

$3t = \frac{\pi}{6}, \frac{3\pi}{2}, \ldots$

$t = \frac{\pi}{18}, \frac{\pi}{2}, \ldots$

When $t=\frac{\pi}{18}$, $\dot{x} = 12\cos(\frac{\pi}{6}-\frac{\pi}{3})$
$= 6\sqrt{3} > 0$

$\therefore$ First time is $\frac{\pi}{18}$ seconds.

(7) (a) When $n=1$, $7^1-5^1 = 2$, which is even

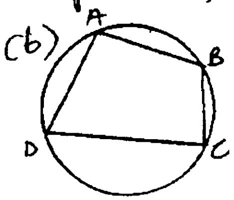
$\therefore$ True for $n=1$

Assume true for $n=k$, ie $7^k-5^k = 2p$, where p is a positive integer

When $n=k+1$, $7^{k+1}-5^{k+1} = 7.7^k-5.5^k$
$= 7(2p+5^k)-5.5^k$ using the assumption
$= 14p+2.5^k$
$= 2(7p+5^k)$

which is divisible by 2, as $7p+5^k$ is a pos. int.

$\therefore$ True for $n=k+1$ if true for $n=k$. Since true for $n=1$, then true for all integers $n \geq 1$.

(b) 

$C = 180° - A$ and $D = 180° - B$ (opposite angles of cyclic quadrilateral supplementary)

$\therefore \tan A+\tan B+\tan C+\tan D$
$= \tan A+\tan B-\tan A-\tan B$
$= 0$

(b)(c) (i) $\ddot{x} = 0$

$\dot{x} = c_1$

When $t = 0$, $\dot{x} = V\cos\alpha$

$\therefore c_1 = V\cos\alpha$

$\therefore \dot{x} = V\cos\alpha$

$x = Vt\cos\alpha + c_2$

When $t = 0$, $x = 0$

$\therefore c_2 = 0$

$\therefore x = Vt\cos\alpha$

$\ddot{y} = -10$

$\dot{y} = c_3 - 10t$

When $t = 0$, $\dot{y} = V\sin\alpha$

$\therefore c_3 = V\sin\alpha$

$\therefore \dot{y} = V\sin\alpha - 10t$

$y = Vt\sin\alpha - 5t^2 + c_4$

When $t = 0$, $y = 0$

$\therefore c_4 = 0$

$\therefore y = Vt\sin\alpha - 5t^2$

(ii) When $x = p$, $y = h$

$\therefore t = \dfrac{p}{V\cos\alpha}$

and $h = V\sin\alpha . \dfrac{p}{V\cos\alpha} - 5 . \dfrac{p^2}{V^2\cos^2\alpha}$

$\dfrac{5p^2}{V^2\cos^2\alpha} = p\tan\alpha - h$

$\dfrac{V^2\cos^2\alpha}{5p^2} = \dfrac{1}{p\tan\alpha - h}$

$V^2 = \dfrac{5p^2\sec^2\alpha}{p\tan\alpha - h}$

$= \dfrac{5p^2(1+\tan^2\alpha)}{p\tan\alpha - h}$

(iii) Similarly $V^2 = \dfrac{5q^2(1+\tan^2\alpha)}{q\tan\alpha - h}$

$\therefore \dfrac{5p^2(1+\tan^2\alpha)}{p\tan\alpha - h} = \dfrac{5q^2(1+\tan^2\alpha)}{q\tan\alpha - h}$

$p^2(q\tan\alpha - h) = q^2(p\tan\alpha - h)$

$(p^2q - q^2p)\tan\alpha = (p^2 - q^2)h$

$\tan\alpha = \dfrac{(p+q)(p-q)h}{pq(p-q)}$

$= \dfrac{h(p+q)}{pq}$