NORTH SYDNEY GIRLS HIGH SCHOOL

**2008**
**TRIAL HSC EXAMINATION**

# Mathematics Extension 1

## General Instructions

- Reading Time – 5 minutes
- Working Time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

## Total Marks – 84

Attempt Questions 1–7
All questions are of equal value

At the end of the examination, place your solution booklets in order and put this question paper on top. Submit one bundle. The bundle will be separated before marking commences so that anonymity will be maintained.

**Student Number:** ____________________ **Teacher:** ____________________________

**Student Name:** ___________________________________

| QUESTION | MARK |
|---|---|
| 1 | /12 |
| 2 | /12 |
| 3 | /12 |
| 4 | /12 |
| 5 | /12 |
| 6 | /12 |
| 7 | /12 |
| TOTAL | /84 |
| | % |

**Total Marks – 84**
**Attempt Questions 1–7**
**All questions are of equal value**

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

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**Marks**

**Question 1** (12 marks) Use a SEPARATE writing booklet.

(a) Find

(i) $\displaystyle\int \frac{1}{\sqrt{9-x^2}}\,dx$ **1**

(ii) $\displaystyle\int \frac{e^x}{e^x+2}\,dx$ **1**

(b) Differentiate $\cos^{-1}\left(\frac{2}{x}\right)$. **2**

(c) The interval $A(5, a)$ and $B(b, -1)$ is divided externally in the ratio 2:3 to give the point (7, 2). Find the values of $a$ and $b$. **2**

(d) Find the coefficient of $x^5$ in the expansion of $\left(2x - \dfrac{1}{x^2}\right)^{11}$. **3**

(e) Find the value of $\displaystyle\int_6^{11} x\sqrt{x-2}\,dx$ using the substitution $u^2 = x - 2$. **3**

**Marks**

**Question 2** (12 marks) Use a SEPARATE writing booklet.

(a) Solve $\dfrac{8}{x-3} \geq 1$. **2**

(b) Find the exact value of $\displaystyle\int_{\sqrt{3}}^{3} \frac{2}{9+x^2}\,dx$. **2**

(c) Consider the curve $y = 2\cos^{-1}(x-1)$.

(i) State the domain and range. **2**

(ii) Sketch the curve. **1**

(iii) Find the gradient of the tangent to the curve at the point where $x = 1$. **2**

(d) Find the acute angle between the curves $y = 6 - 2x$ and $y = 2x^2 + x - 8$ at the point where $x = 2$, giving your answer correct to the nearest degree. **3**

**Marks**

**Question 3** (12 marks) Use a SEPARATE writing booklet.

(a) (i) Show that $x^2 - 4x + \log_e x = 0$ has a root between $x = 3$ and $x = 4$. **1**

(ii) Using two applications of the method of halving the interval, find a smaller interval containing the root. **2**

(b) When the polynomial $P(x)$ is divided by $x^2 - x$, the quotient is $Q(x)$ and the remainder is $R(x) = ax + b$.

(i) Given that $P(1) = 3$, show that $R(1) = 3$. **1**

(ii) Further, when $P(x)$ is divided by $x$ the remainder is $-4$. Find $R(x)$. **2**

(c) Evaluate $\displaystyle\int_{\frac{\pi}{2}}^{\pi} \cos^2 2x \, dx$. **3**

(d) Prove that $5^n + 2(11)^n$ is divisible by 3 for all positive integer values of $n$. **3**

**Marks**

**Question 4** (12 marks) Use a SEPARATE writing booklet.

(a) Find the exact value of $\sin^{-1}\left(\cos\frac{2\pi}{3}\right)$. **1**

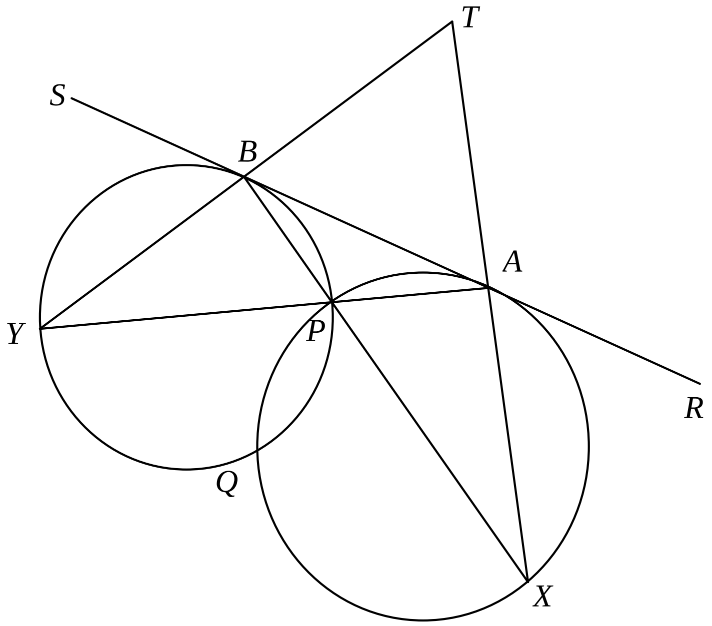
(b) Find $\dfrac{d}{dx}(x\tan x)$ and hence show that $\displaystyle\int_0^{\frac{\pi}{4}} x\sec^2 x\,dx = \frac{\pi}{4} - \frac{1}{2}\ln 2$. **3**

(c) Consider $(1+2x)^n$:

(i) Write an expression for the coefficient of the term in $x^4$. **1**

(ii) The ratio of the coefficient of $x^4$ to the coefficient of $x^6$ is $5:8$. **3**
Find the value of $n$.

(d) $AB$ is a common tangent to two circles which intersect at $P$ and $Q$ as illustrated in the diagram below.
$XPB$ and $YPA$ are straight lines. $XA$ and $YB$ intersect at $T$.

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(i) Copy or trace this diagram into your writing booklet.

(ii) Explain why $\angle SBY = \angle BPY$ **1**

(iii) Prove that $AT = TB$. **3**

**Marks**

**Question 5** (12 marks) Use a SEPARATE writing booklet.

Consider the function $f\left(x\right)=3e^{-x^2}$ .

(a) Show that the function is even. **1**

(b) Find the stationary point of $y=f\left(x\right)$. **2**

(c) Show that $\dfrac{d^2y}{dx^2}=6e^{-x^2}\left(2x^2-1\right)$ and hence find any points of inflexion. **3**

(d) Sketch the curve $y=f\left(x\right)$ using the same scale on both the *x* and *y* axis. **1**

(e) State the greatest positive domain of $y=f\left(x\right)$ for which an inverse function exists. **1**

(f) Sketch $y=f^{-1}\left(x\right)$ on the same diagram as (d) above. **1**

(g) Find the equation of the inverse function and state its domain. **2**

(h) Let $x = N$ where $N < 0$. Find the value of $f^{-1}\left(f\left(N\right)\right)$. **1**

**Marks**

**Question 6** (12 marks) Use a SEPARATE writing booklet.

(a) (i) Express $2\cos\theta + 3\sin\theta$ in the form $A\cos(\theta - \alpha)$, where $A > 0$ and $0 \le \alpha \le \dfrac{\pi}{2}$. **2**

(ii) Hence or otherwise find the maximum value of $2\cos\theta + 3\sin\theta - 3$. **1**

(b) In order to promote a new brand of bottled water, the word WIN is printed on the inside of some of the bottle caps.

The advertising slogan claims that 'one in every five bottles wins a prize'.

(i) Sandra buys 6 bottles of the new brand of water.

Find the probability that:

(α) The first bottle she opens does not win her a prize but the next one does. **1**

(β) She will win exactly twice after opening all six of the bottles. **1**

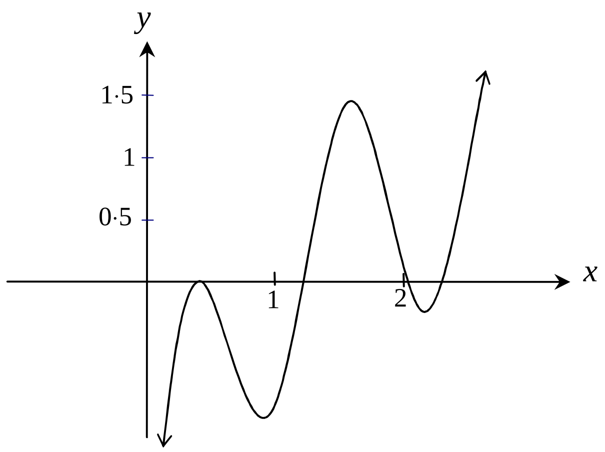
(ii) How many bottles would she have to open to ensure that her chance of winning a prize is at least 95%? **2**

(c) (i) Prove that $\sin 2X + \sin 2Y = 2\sin(X+Y)\cos(X-Y)$. **2**

(ii) Hence or otherwise, solve $\sin\theta + \sin 3\theta = \cos\theta$ if $0 \le \theta \le 2\pi$. **3**

**Question 7** (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) The function $f(x)$, where $f(x) = \ln x + \sin 5x$, has a zero between $x = 1$ and $x = 2$. This is illustrated in the diagram below.




DIAGRAM
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(i) Beginning with an approximation of $x = 1{\cdot}5$, attempt to find an improved value for this root using one application of Newton's method. **2**

(ii) Explain why this attempt fails. **1**

(b) An idle computer programmer decides to develop a 'signature tune' which will play each time she logs onto her computer. She plans to use six tones which she refers to as A, B, C, D, E and F. She plans to repeat two of the tones once only in the tune. For example: A B A C C F D E

(i) How many such 8 tone signature tunes will she be able to program if there is no restriction on when the repeated tones are played? **2**

(ii) For the sake of a more 'interesting' tune, the repeated tones are played together but the pairs are not to sound immediately after each other. That is, A A B C C… is allowed, but A A B B C… is not. If she applies this condition, how many 8 tone signature tunes will she then be able to program? **2**

(c) By considering the expansion of $(1+x)^{2n}$ prove that **3**

$$\sum_{k=0}^{2n} k(k-1)\binom{2n}{k} = n(2n-1)2^{2n-1}.$$

(d) Given that $2\log_y x + 2\log_x y = 5$, show that $\log_y x$ is equal to either 2 or $\frac{1}{2}$. **2**

**End of paper**

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## STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

# Trial HSC Mathematics Extension 1 Solutions 2008

## Question 1

(a) (i) $\displaystyle\int \frac{1}{\sqrt{9-x^2}}\,dx = \sin^{-1}\left(\frac{x}{3}\right)+C$

(ii) $\displaystyle\int \frac{e^x}{e^x+2}\,dx = \ln\left(e^x+2\right)+C$

(b) $$\frac{d}{dx}\left(\cos^{-1}\left(\tfrac{2}{x}\right)\right) = \frac{-1}{\sqrt{1-\left(\tfrac{2}{x}\right)^2}}\cdot\left(-2x^{-2}\right)$$
$$= \frac{2}{x^2\sqrt{1-\tfrac{4}{x^2}}}$$
$$= \frac{2}{x\sqrt{x^2-4}}$$

(c) $A(5,\ a)$ $\qquad B(b,\ -1)$

$-2:3$

$$(7,\ 2) = \left(\frac{5\times 3-2b}{-2+3},\ \frac{3a+(-2)(-1)}{-2+3}\right)$$
$$= \left(15-2b,\ 3a+2\right)$$

$\therefore 15-2b=7$ and $3a+2=2$

i.e. $b=4$ and $a=0$

(d) A general term of $\left(2x-\dfrac{1}{x^2}\right)^{11}$ has the form

$$\binom{11}{k}(2x)^{11-k}\left(-\frac{1}{x^2}\right)^k = \binom{11}{k}2^{11-k}(-1)^k x^{11-3k}$$

For the term in $x^5$: $\quad 11-3k=5$
$$3k=6$$
$$k=2$$

$\therefore$ the required coefficient is $\dbinom{11}{2}2^{11-2}(-1)^2 = 28160$

(e) $\displaystyle\int_6^{11} x\sqrt{x-2}\,dx$ $\qquad$ let $u^2 = x-2$

$$x = u^2+2$$
$$\frac{dx}{du} = 2u$$
$$dx = 2u\,du$$

If $x=11,\ u^2=9$

$u=3$ taking $u>0$

If $x=6,\ u^2=4$

$u=2$ taking $u>0$

Now $\displaystyle\int_6^{11} x\sqrt{x-2}\,dx = \int_2^3 \left(u^2+2\right)\sqrt{u^2}\,(2u)\,du$

$$= \int_2^3 \left(2u^4+4u^2\right)du$$

$$= \left[\frac{2u^5}{5}+\frac{4u^3}{3}\right]_2^3$$

$$= \frac{2(3)^5}{5}+\frac{4(3)^3}{3}-\left[\frac{2(2)^5}{5}+\frac{4(2)^3}{3}\right]$$

$$= 109\tfrac{11}{15} \qquad \left(\tfrac{1646}{15}=109.7\dot{3}\right)$$

## Question 2

(a)

$$\frac{8}{x-3}\ge 1 \qquad x\ne 3$$

$$8(x-3)\ge (x-3)^2$$

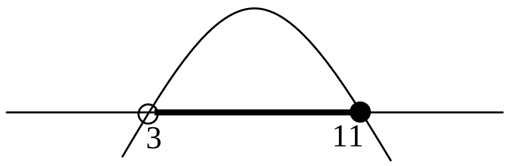
$$8(x-3)-(x-3)^2\ge 0$$

$$(x-3)\big(8-(x-3)\big)\ge 0$$

$$(x-3)(8-x+3)\ge 0$$

$$(x-3)(11-x)\ge 0$$

$$\therefore\ 3<x\le 11$$




(b) $\displaystyle\int_{\sqrt{3}}^{3}\frac{2}{9+x^2}\,dx = \left[\frac{2}{3}\tan^{-1}\left(\frac{x}{3}\right)\right]_{\sqrt{3}}^{3}$

$$= \frac{2}{3}\tan^{-1}\left(\frac{3}{3}\right)-\frac{2}{3}\tan^{-1}\left(\frac{\sqrt{3}}{3}\right)$$

$$= \frac{2}{3}\tan^{-1}1-\frac{2}{3}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$= \frac{2}{3}\left(\frac{\pi}{4}\right)-\frac{2}{3}\left(\frac{\pi}{6}\right)$$

$$= \frac{\pi}{18}$$

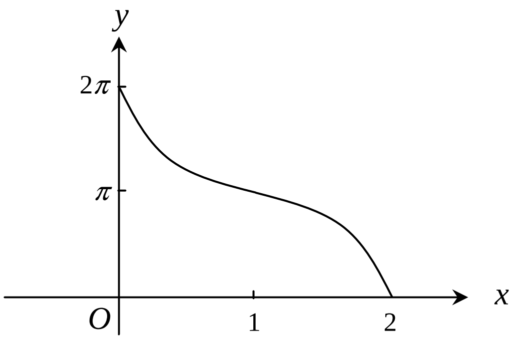
(c) $y = 2\cos^{-1}(x-1)$

(i) $\dfrac{y}{2} = \cos^{-1}(x-1)$

Domain: $-1\le x-1\le 1$ Range: $0\le \dfrac{y}{2}\le \pi$

$0\le x\le 2$ $0\le y\le 2\pi$

(ii)




(iii) $y = 2\cos^{-1}(x-1)$

$$\frac{dy}{dx} = \frac{-2}{\sqrt{1-(x-1)^2}}$$

At $x=1$: $\qquad \dfrac{dy}{dx} = \dfrac{-2}{\sqrt{1-(1-1)^2}} = -2$

$\therefore$ the gradient of the tangent at $x = 1$ is $-2$.

(d) $y = 6 - 2x$ has $m_1 = -2$

For $y = 2x^2 + x - 8$: $\qquad \dfrac{dy}{dx} = 4x+1$

At $x = 2$: $\qquad \dfrac{dy}{dx} = 4(2)+1 = 9 \qquad \therefore m_2 = 9$

Let $\theta$ be the acute angle between curves where $x = 2$, then

$$\tan\theta = \left|\frac{-2-9}{1+(-2)(9)}\right|$$

$$= \frac{11}{17}$$

$$\theta = 32^\circ 54'$$

$$= 33^\circ \quad \text{correct to the nearest degree}$$

**<u>Question 3</u>**

(a) (i) Let $f(x) = x^2 - 4x + \log_e x$

Now $f(3) = 3^2 - 4(3) + \log_e 3 = -1\cdot 901...$

and $f(4) = 4^2 - 4(4) + \log_e 4 = 1\cdot 386...$

$\therefore$ as the sign of the function changes over the interval $3 \le x \le 4$, and the function is continuous over this domain, there is a root between $x = 3$ and $x = 4$.

(ii) Now $f\left(\dfrac{3+4}{2}\right) = f(3\cdot 5) = 3\cdot 5^2 - 4(3\cdot 5) + \log_e 3\cdot 5 = -0\cdot 497...$

$\therefore$ the root lies in the interval $3\cdot 5 < x < 4$

$f\left(\dfrac{3\cdot 5+4}{2}\right) = f(3\cdot 75) = 3\cdot 75^2 - 4(3\cdot 75) + \log_e 3\cdot 75 = 0\cdot 384...$

$\therefore$ the root lies in the interval $3\cdot 5 < x < 3\cdot 75$

(b) (i) Let $P(x) = (x^2 - x)Q(x) + R(x)$

Now $P(1) = (1^2 - 1)Q(1) + R(1)$

i.e. $P(1) = R(1)$ but $P(1) = 3$ $\therefore R(1) = 3$

(ii) Now $P(x) = (x^2 - x)Q(x) + ax + b$ as $R(x) = ax + b$

When $P(x)$ is divided by *x*, the remainder is $-4$

i.e. $P(0) = -4 = R(0)$

Now $R(0) = -4$: $a(0) + b = -4$ $\therefore b = -4$

But $R(1) = 3$: $a(1) + b = 3$

Substituting $b = -4$: $a - 4 = 3$ $\therefore a = 7$

$\therefore R(x) = 7x - 4$

(c) $\displaystyle\int_{\frac{\pi}{2}}^{\pi} \cos^2 2x\,dx = \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} (\cos 4x + 1)\,dx$ as $\cos 4x = 2\cos^2 2x - 1$

$$= \frac{1}{2}\left[\frac{1}{4}\sin 4x + x\right]_{\frac{\pi}{2}}^{\pi}$$

$$= \frac{1}{2}\left[\frac{1}{4}\sin 4\pi + \pi\right] - \frac{1}{2}\left[\frac{1}{4}\sin 4\left(\frac{\pi}{2}\right) + \frac{\pi}{2}\right]$$

$$= \frac{\pi}{2} - \frac{\pi}{4}$$

$$= \frac{\pi}{4}$$

(d) Aim: Prove that $5^n + 2(11)^n$ is divisible by 3 for all positive integer values of *n*.

Test the result for $n = 1$: $5^1 + 2(11)^1 = 5 + 22$

$= 27$

$= 3(9)$ which is divisible by 3

$\therefore$ the result is true for *n* = 1

Let *n* = *k* be a value of *n* for which the result is true:

i.e. $5^k + 2(11)^k = 3M$ where *M* is an integer (1)

then $5^k = 3M - 2(11)^k$

Test the result for *n* = *k* + 1:

$$5^{k+1} + 2(11)^{k+1} = 5(5^k) + 2(11)(11)^k$$

$$= 5(5^k) + 22(11)^k$$

$$= 5\left[3M - 2(11)^k\right] + 22(11)^k$$

$$= 5(3M) + 12(11)^k \qquad \text{by (1)}$$
$$= 3\left[5M + 4(11)^k\right]$$

Now as $M$ and $k$ are both integral, $\left[5M + 4(11)^k\right]$ is an integer, say $N$

$\therefore 3\left[5M + 4(11)^k\right] = 3N$ where $N$ is an integer

and hence $5^{k+1} + 2(11)^{k+1} = 3N$ which is divisible by 3.

$\therefore$ by Mathematical induction, the result is true for all positive integral values of $n$.

## Question 4

(a) $\sin^{-1}\left(\cos\frac{2\pi}{3}\right) = \sin^{-1}\left(-\frac{1}{2}\right)$

$$= -\frac{\pi}{6}$$

(b) $\frac{d}{dx}(x\tan x) = x\sec^2 x + \tan x$

Now $x\sec^2 x = \frac{d}{dx}(x\tan x) - \tan x$

$$\therefore \int_0^{\frac{\pi}{4}} x\sec^2 x\,dx = \int_0^{\frac{\pi}{4}} \left\{\frac{d}{dx}(x\tan x) - \tan x\right\}dx$$

$$= \left[x\tan x\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan x\,dx$$

$$= \frac{\pi}{4}\tan\frac{\pi}{4} - 0\tan 0 - \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x}dx$$

$$= \frac{\pi}{4} + \left[\log(\cos x)\right]_0^{\frac{\pi}{4}}$$

$$= \frac{\pi}{4} + \log\left(\cos\frac{\pi}{4}\right) - \log(\cos 0)$$

$$= \frac{\pi}{4} + \log\left(\frac{1}{\sqrt{2}}\right) - \log 1$$

$$= \frac{\pi}{4} + \log(2)^{-\frac{1}{2}}$$

$$= \frac{\pi}{4} - \frac{1}{2}\ln 2$$

(c) (i) A general term of $(1+2x)^n = \binom{n}{k}(2x)^k$

$\therefore$ the coefficient of the term in $x^4 = \binom{n}{4}(2)^4$

(ii) The coefficient of the term in $x^6 = \binom{n}{6}(2)^6$

$$\therefore \quad \frac{\text{coefficient of } x^4}{\text{coefficient of } x^6} = \frac{5}{8}$$

$$\frac{\binom{n}{4}(2^4)}{\binom{n}{6}(2^6)} = \frac{5}{8}$$

$$8\binom{n}{4}(2^4) = 5\binom{n}{6}(2^6)$$

$$\frac{n!(2^7)}{4!(n-4)!} = \frac{5(n!)(2^6)}{6!(n-6)!}$$

$$\frac{2}{(n-4)(n-5)} = \frac{5}{6\times 5}$$

$$12 = (n-4)(n-5)$$

$$n^2 - 9n + 8 = 0$$

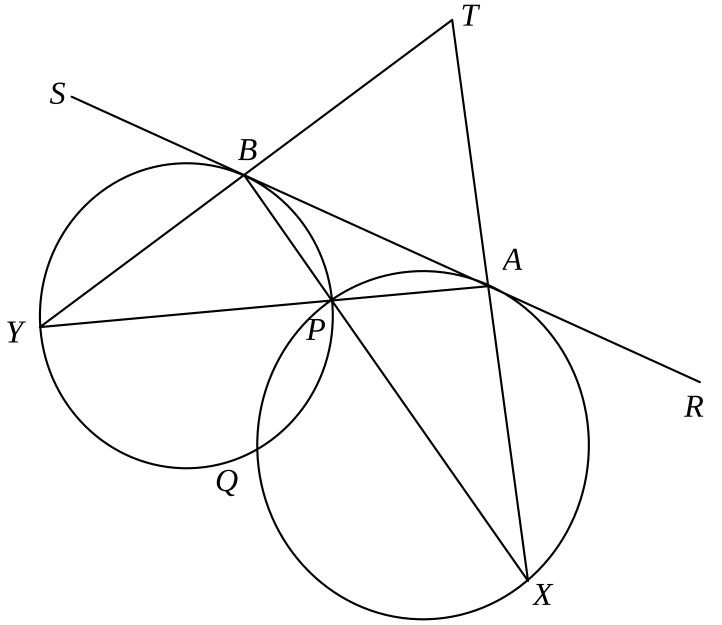
$$(n-8)(n-1) = 0$$

$$n = 1,\ 8$$

But $n \geq 6$ for the $x^6$ term to exist

$$\therefore\ n = 8$$

(d) (i)

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(ii) $\angle SBY = \angle BPY$ as $\angle SBY$ is the angle between the tangent $SR$ and the chord $BY$ and $\angle BPY$ is the angle in the alternate segment standing on $BY$.

(iii) $\angle TBA = \angle SBY$ (vertically opposite)

$= \angle BPY$ (angle in the alternate segment)

$= \angle APX$ (vertically opposite)

$= \angle RAX$ (angle between the tangent and the chord $AX$)

$= \angle TAB$ (vertically opposite)

$\therefore \angle TBA = \angle TAB$

$\therefore AT = TB$ (opposite equal sides in $\Delta TAB$)

## Question 5

(a)

$$f(x) = 3e^{-x^2}$$
$$f(-x) = 3e^{-(-x)^2}$$
$$= 3e^{-x^2}$$
$$= f(x)$$

$\therefore$ the function is even

(b)

$$f(x) = 3e^{-x^2}$$
$$f'(x) = -6xe^{-x^2}$$

Stationary points occur when $f'(x) = 0$

$\therefore \quad -6xe^{-x^2} = 0$

$x = 0$ or $e^{-x^2} = 0$

but $e^{-x^2} > 0$ for all values of $x$

$\therefore$ the only stationary point occurs at $x = 0$

$f(0) = 3e^{-0^2} = 3$

$\therefore (0,3)$ is the only stationary point.

(c)

$$f'(x) = -6xe^{-x^2}$$
$$f''(x) = -6x\left(-2xe^{-x^2}\right) + e^{-x^2}(-6)$$
$$= 6e^{-x^2}\left(2x^2 - 1\right) \text{ as required}$$

Points of inflexion occur when $f''(x) = 0$ and concavity changes sign

$\therefore \quad 6e^{-x^2}\left(2x^2 - 1\right) = 0$

$x^2 = \frac{1}{2}$ or $e^{-x^2} = 0$ but $e^{-x^2} > 0$ for all values of $x$

$\therefore \quad x = \pm\frac{1}{\sqrt{2}}$

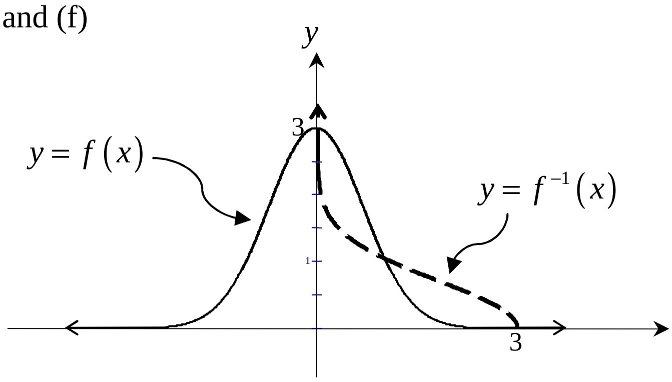
| $x$ | $\left(-\frac{1}{\sqrt{2}}\right)^-$ | $-\frac{1}{\sqrt{2}}$ | $\left(-\frac{1}{\sqrt{2}}\right)^+$ | $\left(\frac{1}{\sqrt{2}}\right)^-$ | $\frac{1}{\sqrt{2}}$ | $\left(\frac{1}{\sqrt{2}}\right)^+$ |
|---|---|---|---|---|---|---|
| $f''(x)$ | + | 0 | – | – | 0 | + |

$\therefore$ the concavity changes sign at both values of $x$.

$f\left(\frac{1}{\sqrt{2}}\right)=3e^{-\frac{1}{2}}$ and $f\left(-\frac{1}{\sqrt{2}}\right)=3e^{-\frac{1}{2}}$

i.e. inflexions occur at $\left(\frac{1}{\sqrt{2}},\, 3e^{-\frac{1}{\sqrt{2}}}\right)$ and $\left(-\frac{1}{\sqrt{2}},\, 3e^{-\frac{1}{\sqrt{2}}}\right)$

(d) and (f)




(e) **D**: $x \geq 0$

(f) See above

(g) Let $y=3e^{-x^2}$

Then the inverse is $x=3e^{-y^2}$

Now
$$\frac{x}{3}=e^{-y^2}$$
$$\ln\left(\frac{x}{3}\right)=-y^2$$
$$\ln\left(\frac{3}{x}\right)=y^2$$
$$y=\pm\sqrt{\ln\left(\frac{3}{x}\right)}$$

but the domain of the function is $x \geq 0$ so the range of the inverse function is $y \geq 0$

$\therefore$ the inverse function is $f'(x)=\sqrt{\ln\left(\frac{3}{x}\right)}$

The domain of the inverse function is **D**: $0 < x \leq 3$

(h) Let $x = N$ where $N < 0$.

Then $f^{-1}\left(f\left(N\right)\right)=f^{-1}\left(f\left(-N\right)\right)$ as $f\left(x\right)$ is an even function

$$=-N$$

**<u>Question 6</u>**

(a) (i) $2\cos\theta+3\sin\theta = A\cos\left(\theta-\alpha\right)$ where $A>0$ and $0\leq\alpha\leq\frac{\pi}{2}$

$$= A\cos\theta\sin\alpha+A\sin\theta\cos\alpha$$

Equating coefficients gives:

$$2=A\cos\alpha$$
$$3=A\sin\alpha$$

$$\therefore \quad \frac{A\sin\alpha}{A\cos\alpha}=\frac{3}{2}$$

$$\tan\alpha=\frac{3}{2}$$

$$\alpha=\tan^{-1}\left(\tfrac{3}{2}\right)$$

Also $A^2\sin^2\alpha+A^2\cos^2\alpha=A^2$

$$\therefore \quad 2^2+3^2=A^2$$

$$A=\sqrt{13} \text{ as } A>0$$

$$\therefore \quad 2\cos\theta+3\sin\theta=\sqrt{13}\cos\left(\theta-\tan^{-1}\left(\tfrac{3}{2}\right)\right)$$

(ii) Now $2\cos\theta+3\sin\theta-3=\sqrt{13}\cos\left(\theta-\tan^{-1}\left(\tfrac{3}{2}\right)\right)-3$ .

But the maximum value of $\sqrt{13}\cos(x)=\sqrt{13}$

$\therefore$ the maximum value of $\sqrt{13}\cos\left(\theta-\tan^{-1}\left(\tfrac{3}{2}\right)\right)-3=\sqrt{13}-3$

(b) (i) (α) $P(\text{win})=\frac{1}{5}$

$\therefore P(\text{lose then win})=\frac{4}{5}\times\frac{1}{5}=\frac{4}{25}$

(β) Probabilities are given by the terms of $\left(\frac{1}{5}+\frac{4}{5}\right)^6$

$$P(\text{win exactly twice})=P(X=2)$$

$$=\binom{6}{2}\left(\tfrac{1}{5}\right)^2\left(\tfrac{4}{5}\right)^4$$

$$=0\cdot 24576$$

(ii) Now probabilities are given by $\left(\frac{1}{5}+\frac{4}{5}\right)^n$ and we need $P(X\geq 1)=0\cdot 95$

$$\therefore \quad 1-P(X=0)=0\cdot 95$$

$$P(X=0)=0\cdot 05$$

$$\binom{n}{0}\left(\tfrac{1}{5}\right)^0\left(\tfrac{4}{5}\right)^n=0\cdot 05$$

$$\left(\tfrac{4}{5}\right)^n=0\cdot 05$$

$$n\ln\left(\tfrac{4}{5}\right)=\ln 0\cdot 05$$

$$n=\frac{\ln 0\cdot 05}{\ln\left(\frac{4}{5}\right)}$$

$$=13\cdot 425\ldots$$

$\therefore$ she would have to open 14 bottles

(c) (i) $2\sin(X+Y)\cos(X-Y)$

$$=2\left[\sin X\cos Y+\cos X\sin Y\right]\left[\cos X\cos Y+\sin X\sin Y\right]$$

$$=2\left(\sin X\cos^2 Y\cos X+\sin^2 X\cos Y\sin Y+\cos^2 X\sin Y\cos Y+\sin X\cos X\sin^2 Y\right)$$

$$= 2\left[\sin X\cos X\left(\cos^2 Y+\sin^2 Y\right)+\sin Y\cos Y\left(\cos^2 X+\sin^2 X\right)\right]$$
$$= 2\left[\sin X\cos X+\sin Y\cos Y\right]$$
$$= 2\left[\tfrac{1}{2}\sin 2X+\tfrac{1}{2}\sin 2Y\right]$$
$$= \sin 2X+\sin 2Y$$

(ii) Let $\sin\theta+\sin 3\theta=\sin 2X+\sin 2Y$

then $2X=\theta$ and $2Y=3\theta$ and hence $X+Y=2\theta$ and $X-Y=-\theta$

$\therefore$ as $\sin 2X+\sin 2Y=2\sin(X+Y)\cos(X-Y)$ from (i) above

i.e. $\sin\theta+\sin 3\theta=2\sin 2\theta\cos(-\theta)$ but $\cos(-\theta)=\cos\theta$

$\therefore$ $\sin\theta+\sin 3\theta=2\sin 2\theta\cos\theta$

Now $\sin\theta+\sin 3\theta=\cos\theta$

becomes $2\sin 2\theta\cos\theta=\cos\theta$

$\therefore$ $2\sin 2\theta\cos\theta-\cos\theta=0$

$\cos\theta(2\sin 2\theta-1)=0$

$\cos\theta=0$ or $\sin 2\theta=\tfrac{1}{2}$ but $0\le\theta\le 2\pi$

$\therefore$ $\theta=\tfrac{\pi}{2},\tfrac{3\pi}{2}$ or $2\theta=\tfrac{\pi}{6},\tfrac{5\pi}{6},\tfrac{13\pi}{6},\tfrac{17\pi}{6}$ as $0\le 2\theta\le 4\pi$

$\therefore$ $\theta=\tfrac{\pi}{2},\tfrac{3\pi}{2},\tfrac{\pi}{12},\tfrac{5\pi}{12},\tfrac{13\pi}{12},\tfrac{17\pi}{12}$

## Question 7

(a) (i) $f(x)=\ln x+\sin 5x$ then $f'(x)=\dfrac{1}{x}+5\cos 5x$

Let $x_0=1\cdot 5$

Then $x_1=1\cdot 5-\dfrac{f(1\cdot 5)}{f'(1\cdot 5)}$

$$=1\cdot 5-\frac{\ln 1\cdot 5+\sin 5(1\cdot 5)}{\frac{1}{1\cdot 5}+5\cos 5(1\cdot 5)}$$

$=0\cdot 940\ldots$ which is obviously not between 1 and 2

(ii) This attempt fails because a stationary point is very close to $x$ = 1·5 and consequently the tangent to the curve at $x$ = 1·5 has a small gradient. This causes the tangent to intersect the $x$-axis closer to the root between 0 and 1 than the root between 1 and 2. This argument is illustrated in the diagram below.

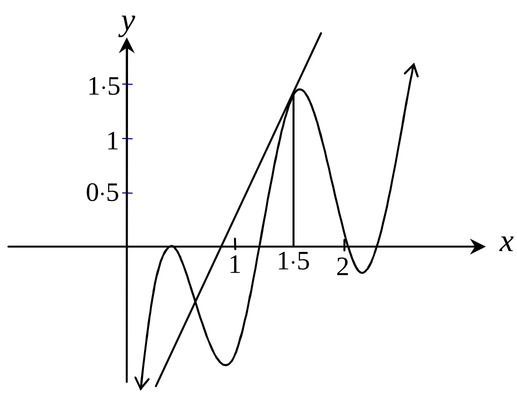



DIAGRAM
TO
SCALE

(b) (i) A B C D E F and 2 doubles

$$\text{Number of tunes } = {}^6C_2 \times \frac{8!}{2! \times 2!}$$
$$= 151\ 200$$

(ii) Number of choices for the double tones = ${}^6C_2$

The 4 remaining tones can be arranged in 4! ways.

__ tone __ tone__ tone__ tone__

Between these tones, there are 5 ‘gaps’, so the first double tone can be placed in any of these gaps. The second double tone then has only 4 gaps into which it can be placed.

Number of ways of placing the double tones = ${}^6C_2 \times 4! \times 5 \times 4$

$= 7200$

(c) Consider $(1+x)^{2n} = \binom{2n}{0} + \binom{2n}{1}x + \binom{2n}{2}x^2 + \binom{2n}{3}x^3 + \binom{2n}{4}x^4 + \ldots + \binom{2n}{2n}x^{2n}$

Differentiating both sides with respect to *x*:

$$2n(1+x)^{2n-1} = \binom{2n}{1} + 2\binom{2n}{2}x + 3\binom{2n}{3}x^2 + 4\binom{2n}{4}x^3 + \ldots + 2n\binom{2n}{2n}x^{2n-1}$$

Differentiating both sides again:

$$2n(2n-1)(1+x)^{2n-2} = 2\binom{2n}{2} + 3(2)\binom{2n}{3}x + 4(3)\binom{2n}{4}x^2 + \ldots + 2n(2n-1)\binom{2n}{2n}x^{2n-2}$$

Substituting $x = 1$:

$$2n(2n-1)(1+1)^{2n-2} = 2\binom{2n}{2} + 3(2)\binom{2n}{3} + 4(3)\binom{2n}{4} + \ldots + 2n(2n-1)\binom{2n}{2n}$$

$$2n(2n-1)(2)^{2n-2} = 2\binom{2n}{2} + 3(2)\binom{2n}{3} + 4(3)\binom{2n}{4} + \ldots + 2n(2n-1)\binom{2n}{2n}$$

Observing the pattern:

$$2n(2n-1)(2)^{2n-2} = 0(-1)\binom{2n}{0} + 1(0)\binom{2n}{1} + 2(1)\binom{2n}{2} + 3(2)\binom{2n}{3} + \ldots + 2n(2n-1)\binom{2n}{2n}$$

$$n(2n-1)(2)^{2n-1} = 0(-1)\binom{2n}{0} + 1(0)\binom{2n}{1} + 2(1)\binom{2n}{2} + 3(2)\binom{2n}{3} + \ldots + 2n(2n-1)\binom{2n}{2n}$$

$$n(2n-1)(2)^{2n-1} = \sum_{k=0}^{2n} k(k-1)\binom{2n}{k}$$

i.e. $\displaystyle\sum_{k=0}^{2n} k(k-1)\binom{2n}{k} = n(2n-1)2^{2n-1}$ as required

(e)

$$2\log_y x + 2\log_x y = 5$$

$$\frac{2\log x}{\log y} + \frac{2\log y}{\log x} = 5$$

$$2(\log x)^2 + 2(\log y)^2 = 5\log x \log y$$

$$2(\log x)^2 - 5\log x\log y + 2(\log y)^2 = 0$$

$$(2\log x - \log y)(\log x - 2\log y) = 0$$

$$\therefore \quad 2\log x = \log y \quad \text{or} \quad \log x = 2\log y$$

$$\frac{\log x}{\log y} = \frac{1}{2} \quad \text{or} \quad \frac{\log x}{\log y} = 2$$

$$\therefore \quad \log_y x = \tfrac{1}{2} \ \text{ or } \ 2$$

**End of Solutions**