NORTH SYDNEY GIRLS HIGH SCHOOL

2011
TRIAL HSC EXAMINATION

Mathematics Extension 1

General Instructions

- Reading Time – 5 minutes
- Working Time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total Marks – 84

Attempt Questions 1–7
All questions are of equal value
At the end of the examination, place your writing booklets in order and put this question paper on top. Submit one bundle. The bundle will be separated before marking commences so that anonymity will be maintained.

Student Number: ____________________________ **Teacher:** ___________________________

Student Name: ______________________________

QUESTION	MARK
1	/12
2	/12
3	/12
4	/12
5	/12
6	/12
7	/12
TOTAL	/84

Total marks – 84
Attempt Questions 1 – 7

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (12 marks) Use a SEPARATE writing booklet.

(a) Find $\int \frac{1}{(3+2x)^3}\,dx$. **1**

(b) Differentiate $2\sin^{-1}(3x)$. **2**

(c) Evaluate $\lim_{x\to 0}\frac{\tan 5x}{2x}$. **1**

(d) Let $f(x)=x\log_e(x-1)$. What is the domain and range of $f(x)$? **2**

(e) Solve the inequality $\frac{x-3}{2x}>1$. **3**

(f) Use the substitution $u=1-x$ to evaluate $\int_0^1 2x\sqrt{1-x}\,dx$. **3**

Question 2 (12 marks) Use a SEPARATE writing booklet.

(a) Consider the polynomial $P(x) = x^3 - 7x^2 + 8x + 16$.

(i) Show that $P(x)$ has a zero at $x = 4$. **1**

(ii) Write $P(x)$ as a product of its factors. **2**

(iii) Sketch the polynomial over the domain $-2 \leq x \leq 5$. **2**

(iv) Write an expression for the exact area enclosed by the curve, $x = -2$, $x = 5$ and the *x*-axis. Do not evaluate the integral. **2**

(b) The point *X* divides the interval *AB* externally in the ratio 3:5. If *A* is $(3,4)$ and *B* is $(0,-5)$ find the coordinates of *X*. **2**

(c) Find the co-efficient of the x^2 term in the expansion of $\left(2x + \frac{1}{x^2}\right)^8$. **3**

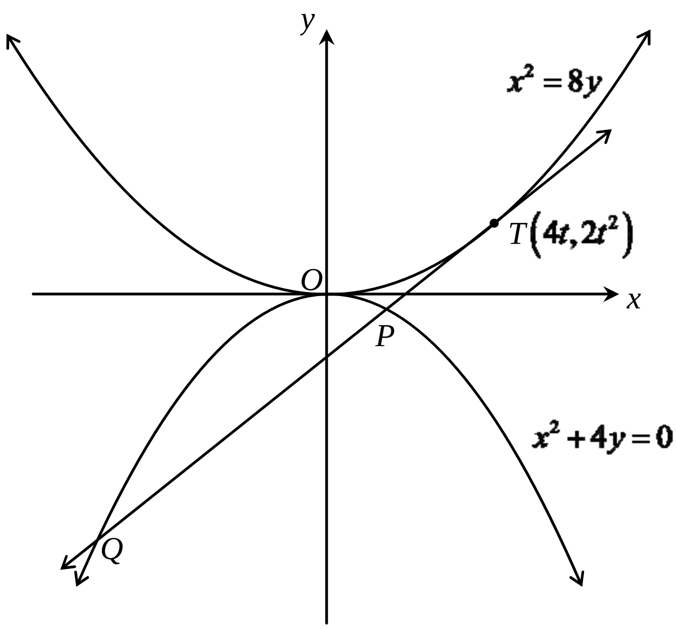
Question 3 (12 marks) Use a SEPARATE writing booklet.

(a) (i) Find the derivative of $f(x) = \tan\left(x^2\right)$ **1**

(ii) Hence or otherwise evaluate $\int_{-a}^{a} x\sec^2\left(x^2\right)dx$. **2**

(b) Find a simplified expression for $\sin\left[\cos^{-1}\left(x-1\right)\right]$. **2**

(c) Consider the point $T\left(4t, 2t^2\right)$ on the parabola $x^2 = 8y$.



(i) Show that the equation of the tangent at T has equation $y - tx + 2t^2 = 0$. **2**

(ii) This tangent meets the parabola $x^2 + 4y = 0$ at two points $P\left(x_1, y_1\right)$ and $Q\left(x_2, y_2\right)$ as shown. Show that x_1 and x_2 are the roots of the quadratic equation $x^2 + 4tx - 8t^2 = 0$. **1**

(iii) Write an expression for $\dfrac{x_1 + x_2}{2}$. **1**

(iv) If $M(x, y)$ is the midpoint of PQ, find the coordinates of M in terms of t. **2**

(v) Find the locus of M as T varies. **1**

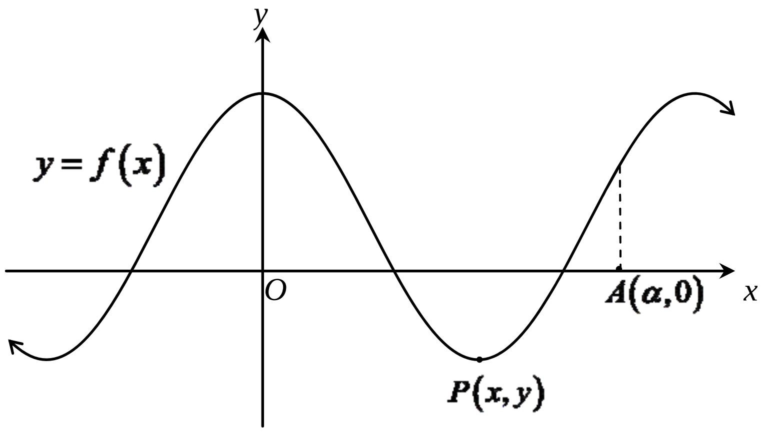
Question 4 (12 marks) Use a SEPARATE writing booklet.

(a) Five people enter a restaurant and sit at a rectangular table in which there are six seats, three on one side of the table and three on the opposite side.

(i) In how many ways can they take their seats if any person can occupy any seat? **1**

(ii) What is the probability that persons *X* and *Y* sit facing each other? **1**

(b) Let $f(x) = 1 + 3\cos\frac{x}{2}$. The diagram shows the graph $y = f(x)$.



(i) State the period and the amplitude of the curve, given that *x* is expressed in radians. **2**

(ii) The point $P(x, y)$ is a turning point on the curve. Find its coordinates. **2**

(iii) What is the largest positive domain, containing $x = 0$, for which $f(x)$ has an inverse function? **1**

(iv) Find the equation of $f^{-1}(x)$ for this restricted domain of $f(x)$. **1**

(v) Sketch the curve $y = f^{-1}(x)$. **2**

(vi) $A(\alpha, 0)$ lies to the right of *P* as indicated in the diagram above. Find a simplified expression for the exact value of $f^{-1}(f(\alpha))$. **2**

Question 5 (12 marks) Use a SEPARATE writing booklet.

(a) Find the volume of the solid generated when the part of the curve $y = \sin 2x$ between $x = 0$ and $x = \dfrac{\pi}{3}$ is rotated about the x-axis. **3**

(b) Five players are selected at random from three sporting teams *A*, *B* and *C*. Each team consists of seven players numbered 1 to 7.

(i) Two brothers play for different teams. What is the probability that of the five selected players, both brothers are selected? **2**

(ii) Jason wanted to find the number of ways to select five players so that no team misses out in the selection. His answer was **2**

$$ {}^3C_1\,{}^7C_1\,{}^7C_1\,{}^7C_3 + {}^3C_1\,{}^7C_1\,{}^7C_2\,{}^7C_2 \,. $$

Explain why this expression is correct.

(c) From three points *A*, *B* and *C* on level ground, the angles of elevation of the top *T* of a hill are $32°$, $30°$ and $25°$ respectively. Point *A* is due west and *C* is due south of the hill *H*. Point *B* lies on the straight line *AC*.

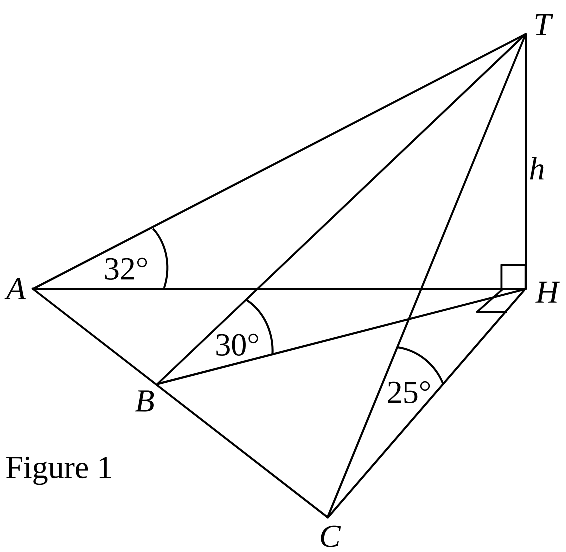


Figure 1

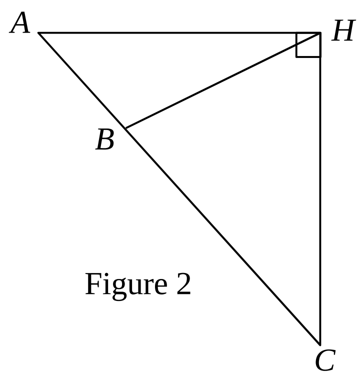


Figure 2

Copy Figure 2 into your writing booklet.

(i) Find an expression for *AH* in terms of *h*. **1**

(ii) Show that $\angle HCA = 36°44'$. **2**

(iii) Find $\angle HBC$ correct to the nearest minute. **2**

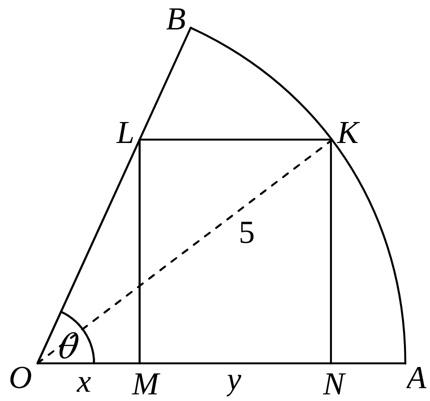
Question 6 (12 marks) Use a SEPARATE writing booklet.

(a) (i) Determine the gradient of the tangent to the curve $y = \tan^{-1}\left(e^x\right)$ at $x = 0$. **2**

(ii) Hence find the equation of the tangent at that point. **2**

(iii) Discuss the behaviour of the curve as $x \to \infty$. **1**

(b) The diagram shows a rectangle *KLMN* drawn in a sector *OAB* of a circle, radius 5 units. $\angle AOB = \theta$, $OM = x$ and $MN = y$.



(i) Given $\tan\theta = 3$ show that $KN = 3x$. **1**

(ii) Hence show that $(x+y)^2 = 25 - 9x^2$. **1**

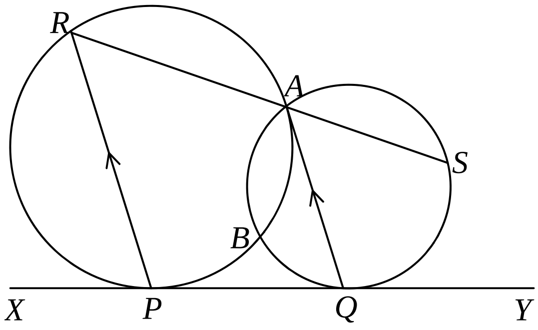
(iii) Show that the perimeter, *P*, of the rectangle is given by **2**

$$P = 4x + 2\sqrt{25 - 9x^2}\ .$$

(iv) Find the value of *x*, correct to 2 decimal places, which maximises the perimeter of the rectangle. **3**

Question 7 (12 Marks) Use a SEPARATE booklet.

(a) Two circles meet in A and B. PQ is a common tangent and $PR \parallel QA$.
RA produced meets the other circle in S.



Copy the diagram into your writing booklet.

(i) Prove that $PRSQ$ is cyclic. **2**

(ii) Prove that $PA \parallel QS$. **2**

(b) Prove by induction that **3**

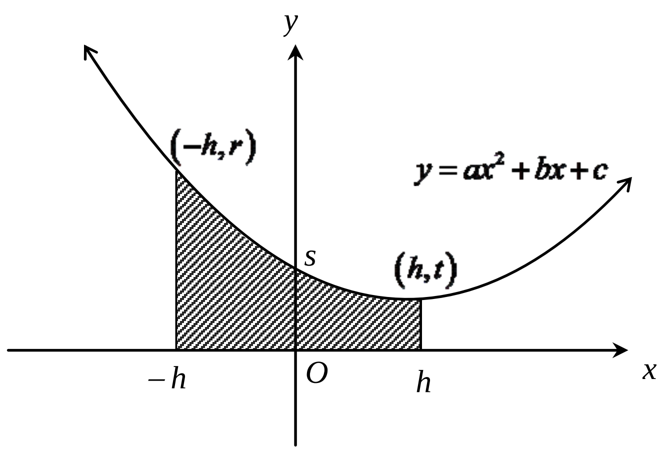
$$\frac{2\times1}{2\times3}+\frac{2^2\times2}{3\times4}+\frac{2^3\times3}{4\times5}+\ldots+\frac{2^n\times n}{(n+1)(n+2)}=\frac{2^{n+1}}{n+2}-1$$

for all integers $n\geq1$.

Question 7 continues on page 9

Question 7 (continued)

(c) The parabola $y = ax^2 + bx + c$ passes through the points $(-h, r)$ and (h, t) as shown. It cuts the y-axis at $(0, s)$.



(i) Show that $r + t = 2ah^2 + 2s$. **2**

(ii) By using integration, NOT Simpson's Rule, show that the shaded area is given by **3**

$$A = \frac{h}{3}(r + 4s + t).$$

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STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Question 1

SOLUTIONS EXT 1 TRIAL 2011

a) $\displaystyle\int \frac{1}{(3+2x)^3}\,dx = \int (3+2x)^{-3}\,dx$

$\displaystyle = \frac{(3+2x)^{-2}}{-2\times 2} + c$

$\displaystyle = -\frac{1}{4(3+2x)^2}$

b) $\displaystyle \frac{d}{dx}\left[2\sin^{-1}(3x)\right] = \frac{2\times 3}{\sqrt{1-(3x)^2}}$

$\displaystyle = \frac{6}{\sqrt{1-9x^2}}$

c) $\displaystyle \lim_{x\to 0} \frac{5}{2}\cdot\frac{\tan 5x}{5x} = \frac{5}{2}$

d) $f(x) = x\log(x-1)$

Domain : $x > 1$

Range : all real y (range $\log(x-1)$ is all real y
range of product is same since $x>1$)

e) $\displaystyle \frac{x-3}{2x} > 1 \qquad x \neq 0$

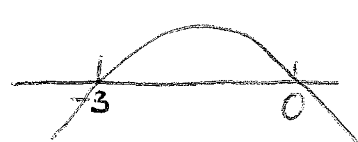
$\displaystyle \frac{(2x)^2(x-3)}{2x} > (2x)^2$

$2x(x-3) - 4x^2 > 0$

$2x[x-3-2x] > 0$

$2x(-x-3) > 0$

$-6x(x+3) > 0$



$\therefore\ -3 < x < 0$

f) $\int_0^1 2x\sqrt{1-x}\,dx$

$= \int_1^0 2(1-u)u^{\frac{1}{2}}(-du)$

$= 2\int_0^1 \left(u^{1/2} - u^{3/2}\right) du$

$= 2\left[\frac{2u^{3/2}}{3} - \frac{2u^{5/2}}{5}\right]_0^1$

$= 2\left(\frac{2}{3} - \frac{2}{5} - 0\right)$

$= \frac{8}{15}$

$u = 1 - x$

$\frac{du}{dx} = -1$

$-du = dx$

When $x = 1$ $u = 0$

When $x = 0$ $u = 1$

QUESTION 2

a) $P(x) = x^3 - 7x^2 + 8x + 16$

(i) $P(4) = 64 - 112 + 32 + 16$

$= 0$

$\therefore x = 4$ is a zero of $P(x)$

(ii)
$$
\begin{array}{r}
x^2 - 3x - 4 \\
x-4 \,\big)\, x^3 - 7x^2 + 8x + 16 \\
\underline{x^3 - 4x^2} \qquad\quad \\
-3x^2 + 8x \qquad \\
\underline{-3x^2 + 12x} \qquad \\
-4x + 16 \\
\underline{-4x + 16} \\
0
\end{array}
$$

$x^2 - 3x - 4 = (x-4)(x+1)$

$\therefore P(x) = (x-4)^2(x+1)$

Alternatively:

Try $x = -1$

$P(-1) = -1 - 7 - 8 + 16$

$= 0$

$\therefore (x+1)$ is a factor

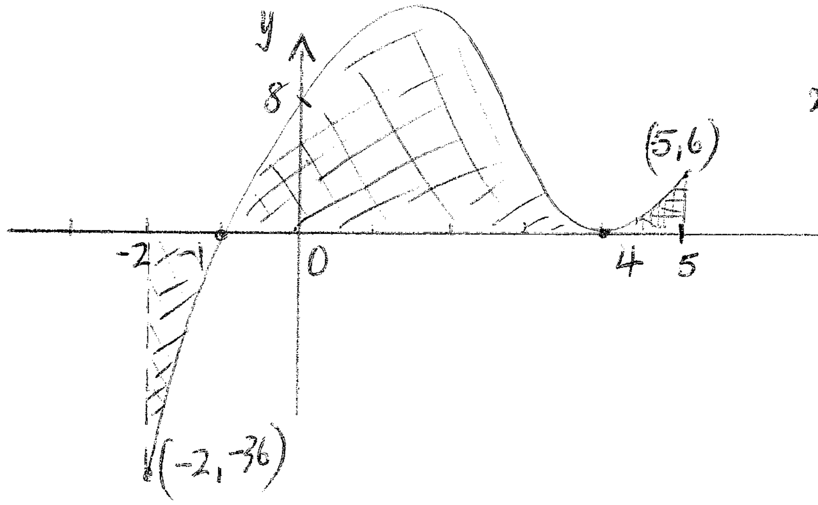
Note $P'(x) = 3x^2 - 14x + 8$

$= (3x-2)(x-4)$

ie $x = 4$ is a double root

$\therefore P(x) = (x-4)^2(x+1)$

(iii)



$P(0) = 8$

$P(-2) = -36$

$P(5) = 6$

(iv) $\text{Area} = \left| \int_{-2}^{-1} P(x)\,dx \right| + \underbrace{\int_{-1}^{4} P(x)\,dx + \int_{4}^{5} P(x)\,dx}_{\text{can be combined}}$

OR

To remove absolute value reverse limits

$$\int_{-1}^{-2} P(x)\,dx + \int_{-1}^{5} P(x)\,dx$$

b) $A\,(3, 4)$ $\quad B\,(0, -5)$

$-3 : 5$ (externally)

$X = \left(x = \dfrac{5 \times 3 + 3 \times 0}{-3+5} \;,\; y = \dfrac{5 \times 4 + 3(-5)}{-3+5} \right)$

$\therefore X \left(\dfrac{15}{2} , \dfrac{35}{2} \right)$

c) General term of $\left(2x + \frac{1}{x^2}\right)^8$ is ${}^8C_k\,(2)^{8-k}\,x^{8-k} \cdot (x^{-2})^k$

ie $T_k = {}^8C_k\,2^{8-k}\,x^{8-3k}$

x^2 term: $\quad x^2 = x^{8-3k}$

$\therefore k = 2$

Coefficient of x^2 is ${}^8C_2\,(2)^{8-2} = {}^8C_2\,2^6$

$= 1792$

Question 3

a) (i) $f(x) = \tan(x^2)$

$\therefore f'(x) = 2x \sec^2(x^2)$

(ii) $\int_{-a}^{a} x \sec^2(x^2)\, dx = \frac{1}{2} \int 2x \sec^2(x^2)\, dx$

$= \frac{1}{2} \left[\tan(x^2)\right]_{-a}^{a}$

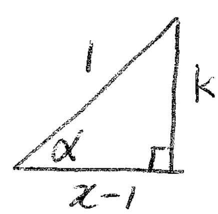
$= \frac{1}{2} \left(\tan(a^2) - \tan a^2\right)$

$= 0$

Alternatively:
State that $x \sec^2(x^2)$ is an odd function $\therefore I = 0$

b) $\sin\left[\cos^{-1}(x-1)\right]$

Let $\cos^{-1}(x-1) = \alpha$



$k^2 = 1^2 - (x-1)^2$
$= 2x - x^2$
$k = \sqrt{2x - x^2}$

$\therefore \sin\alpha = \frac{\sqrt{2x - x^2}}{1}$

$\therefore \sin\left[\cos^{-1}(x-1)\right] = \sqrt{2x - x^2}$

3 c) $T(4t, 2t^2)$ parabola $x^2 = 8y$

$y = \frac{1}{8}x^2$

$\frac{dy}{dx} = \frac{2x}{8}$

$= \frac{x}{4}$

OR

$x = 4t$ $\quad$ $y = 2t^2$

$\frac{dx}{dt} = 4$ $\quad$ $\frac{dy}{dt} = 4t^2$

$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$

$= t$

At T, $\frac{dy}{dx} = \frac{4t}{4}$
$= t$

$\therefore$ Equation of tangent at T: $y - 2t^2 = t(x - 4t)$

$y - 2t^2 = tx - 4t^2$

$y - tx + 2t^2 = 0$

(ii) Solve simultaneously $y = tx - 2t^2$ and $y = \frac{-x^2}{4}$

$tx - 2t^2 = -\frac{x^2}{4}$

$4tx - 8t^2 = -x^2$

$x^2 + 4tx - 8t^2 = 0$ $\quad$ *

The roots of this equation will be the x values of the points P and Q where tangent intersects the parabola

ie x_1 and x_2 are the roots of *

(iii) $\frac{x_1 + x_2}{2} = \left(\frac{\text{sum of roots}}{2}\right) \div 2$

$= -4t \div 2$

$= -2t$

3

(iv) Sub $x = -2t$ into tangent equation to find y value

$$y - t(-2t) + 2t^2 = 0$$
$$y = -4t^2$$

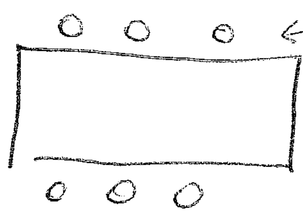
$\therefore$ M is $(-t, -4t^2)$

(v) Locus of M: $x = -2t \Rightarrow t = -\frac{x}{2}$

$$y = -4t^2$$
$$= -4\left(-\frac{x}{2}\right)^2$$
$$= -x^2$$

$\therefore$ Locus is $y = -x^2$ [a concave down parabola vertex the origin]

QUESTION 4



These seats are 'different' from those opposite them. eg they have a different view in the restaurant.

a) (i) No of ways $= 6 \times 5 \times 4 \times 3 \times 2$
$= {}^6P_5$
$= 720$

(ii) Sit X anywhere.
Then Y has 5 seats to choose from, but only one of them is opposite X

$\therefore P(X \text{ opp } Y) = \frac{1}{5}$

Alternatively: $P(X \text{ opp } Y) = \frac{3 \times 2 \times {}^4P_3}{{}^6P_5}$ ← (X + Y) 3 choices, 2 ways; other 3 people have 4 seats to choose from

$= \frac{1}{5}$

b) $f(x) = 1 + 3\cos\frac{x}{2}$

Amplitude $= 3$

Period $= \frac{2\pi}{n}$ where $n = \frac{1}{2}$ [OR $\frac{1}{2}$ wave in 2π then 1 wave in 4π]

$= 4\pi$

(ii) Half a wave in $2\pi \Rightarrow x = 2\pi$

$\therefore y = 1 + 3\cos\left(\frac{2\pi}{2}\right)$

$= 1 + 3(-1)$

$= -2$

$\therefore P(2\pi, -2)$

OR $f'(x) = -\frac{3}{2}\sin\frac{x}{2}$

$f'(x) = 0$ when $x = 2\pi$

$y = 1 + 3\cos\left(\frac{2\pi}{2}\right)$

$= -2$

(iii) Domain: $0 \le x \le 2\pi$

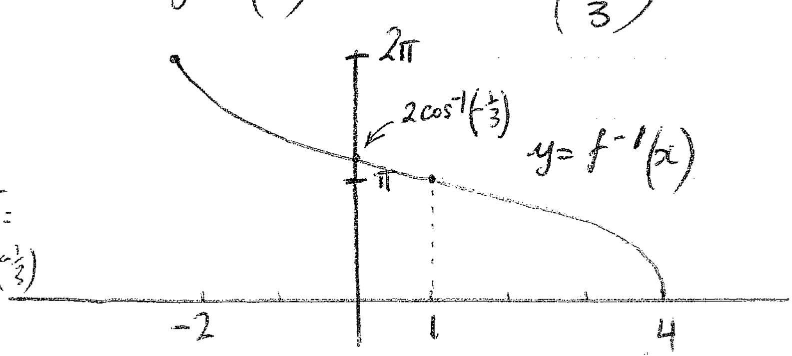
(iv) $x = 1 + 3\cos\left(\frac{y}{2}\right)$

$\frac{x-1}{3} = \cos\frac{y}{2}$

$\therefore y = 2\cos^{-1}\frac{(x-1)}{3}$

$f^{-1}(x) = 2\cos^{-1}\left(\frac{x-1}{3}\right)$

(v)



y intercept: $y = 2\cos^{-1}(-\frac{1}{3})$

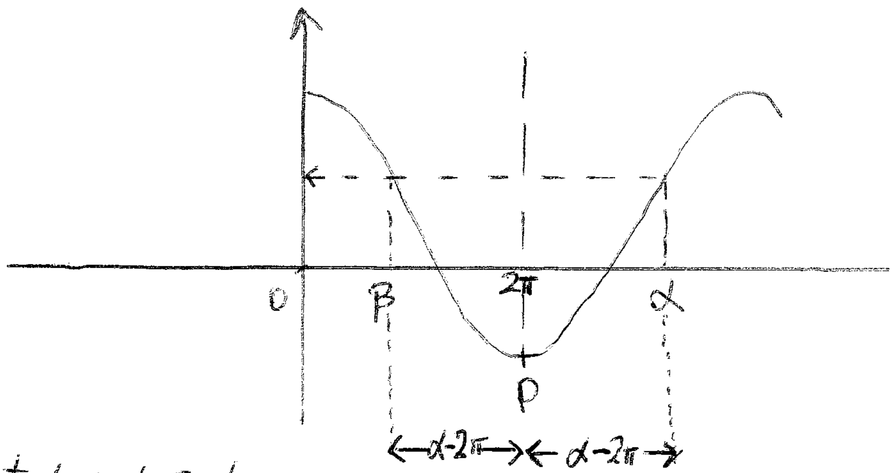
Domain $-1 \le \frac{x-1}{3} \le 1$

$-3 \le x - 1 \le 3$

$-2 \le x \le 4$

Old range of $f(x)$

(v)



Let α and β have same y value :

$\beta = 2\pi - (\alpha - 2\pi)$ by symmetry about P

$\beta = 4\pi - \alpha$

$\therefore f^{-1}[f(\alpha)] = f^{-1}[f(\beta)]$ since $f(\alpha) = f(\beta)$

$= \beta$ since β is in the domain of $f^{-1}(x)$

$= 4\pi - \alpha$

Question 5

a) volume $= \pi \int_0^{\pi/3} \sin^2 2x \, dx$

$\cos 2x = 1 - 2\sin^2 x$

$\sin^2 x = \frac{1}{2}(1 - \cos 2$

$\therefore \sin^2 2x = \frac{1}{2}(1 - \cos$

$= \frac{\pi}{2} \int_0^{\pi/3} (1 - \cos 4x)\, dx$

$= \frac{\pi}{2} \left[x - \frac{1}{4} \sin 4x \right]_0^{\pi/3}$

$= \frac{\pi}{2} \left[\frac{\pi}{3} + \frac{\sqrt{3}}{8} - 0 \right]$

$= \frac{\pi^2}{6} + \frac{\pi\sqrt{3}}{16}$

b) (i) No. Ways $= {}^1C_1\, {}^1C_1\, {}^{19}C_3$ (2 brothers + 3 from 19)

Probability $= \frac{{}^1C_1 \times {}^1C_1 \times {}^{19}C_3}{{}^{21}C_5} = \frac{1}{21}$

(ii) There are two patterns (1, 1, 3) and (1, 2, 2

(1,1,3): One player each from 2 teams, and 3 players from the other team ${}^7C_1\, {}^7C_1\, {}^7C_3$

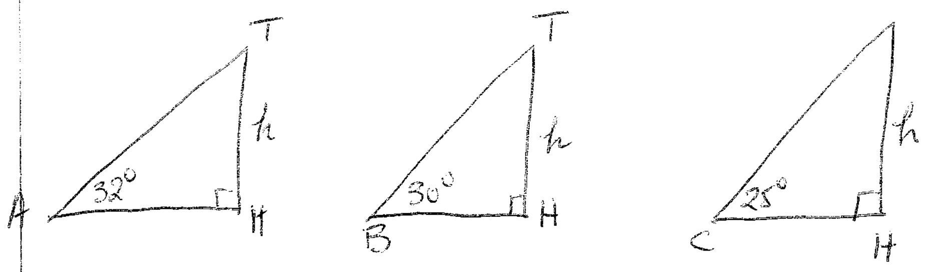
The team with 3 players can be selected in 3 ways $= {}^3C_1$

(1,2,2): One player from one team and 2 players from other two teams $= {}^7C_1\, {}^7C_2\, {}^7C_2$

The team with 1 player can be selected in ${}^3C_1$ ways.

$\therefore$ Total $= {}^3C_1\, {}^7C_1\, {}^7C_1\, {}^7C_3 + {}^3C_1\, {}^7C_1\, {}^7C_2\, {}^7C_2$

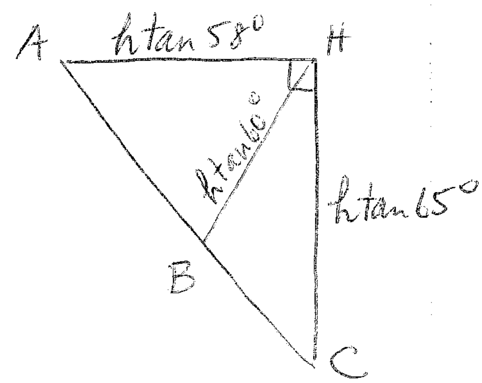
c)



$\tan 32° = \dfrac{h}{AH}$ $\qquad BH = h \tan 60°$ $\qquad CH = h\tan 65°$

(i) $AH = \dfrac{h}{\tan 32}$

$= h \tan 58°$



(ii) $\tan C = \dfrac{h \tan 58°}{h \tan 65°}$

$= \dfrac{\tan 58°}{\tan 65°}$

$\therefore C = 36° 44'$

(iii) By the sine rule in $\triangle HBC$

$$\frac{\sin B}{h\tan 65°} = \frac{\sin C}{h \tan 60°}$$

$$\therefore \sin B = \frac{h\tan 65° \sin C}{h\tan 60°}$$

$$= \frac{\tan 65° \sin 36° 44'}{\tan 60°}$$

$$\therefore B = 47° 46' \text{ or } 132° 13'$$

If $B = 47° 46'$ then $\angle BHC = 180 - (47° 46' + 36° 44')$

$= 95° 29'$

which is impossible

$\therefore \angle B = 132° 13'$

Question 6

a) $y = \tan^{-1}(e^x)$

$$\frac{dy}{dx} = \frac{1}{1+(e^x)^2} \times e^x$$

$$= \frac{e^x}{1+e^{2x}}$$

At $x=0$, $\frac{dy}{dx} = \frac{e^0}{1+e^0}$

$$= \frac{1}{2}$$

(ii) $x=0$, $y = \tan^{-1}(1)$

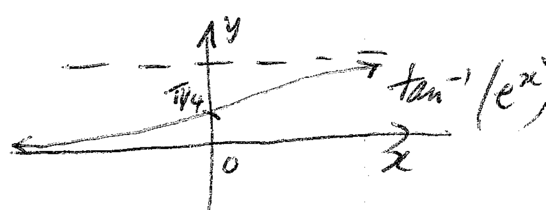
$$= \frac{\pi}{4}$$

Tangent: $y - \frac{\pi}{4} = \frac{1}{2}(x-0)$

$y = \frac{1}{2}x + \frac{\pi}{4}$ or $2x - 4y + \pi = 0$

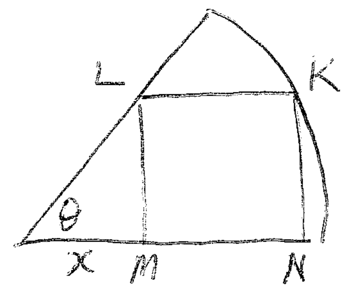
(iii) As $x \to \infty$, $e^x \to \infty$

$\therefore y \to \frac{\pi}{2}$



(Graph not expected in solution)

b)



radius = 5

(i) $\tan\theta = \frac{LM}{x}$

But $\tan\theta = 3 \Rightarrow 3 = \frac{LM}{x}$

$3x = LM$

Now $KN = LM$ (opp sides of a rectangle are equal)

$\therefore KN = 3x$

(ii) By Pythagoras, $(3x)^2 + (x+y)^2 = 5^2$ in ΔOKN

$\therefore (x+y)^2 = 25 - 9x^2$

(iii) $P = 2(3x+y)$ since $P = (KN + MN) \times 2$

$= 6x + 2y$

$= 4x + 2(x+y)$

$= 4x + 2\sqrt{25-9x^2}$

(iv) $\frac{dP}{dx} = 4 + 2 \times \frac{1}{2}(25-9x^2)^{-1/2}(-18x)$

$= 4 - \frac{18x}{\sqrt{25-9x^2}}$

At maximum, $\frac{dP}{dx} = 0$

$\therefore 4\sqrt{25-9x^2} = 18x$

$\sqrt{25-9x^2} = \frac{9}{2}x$

$25 - 9x^2 = \frac{81}{4}x^2$

$100 = 117x^2$

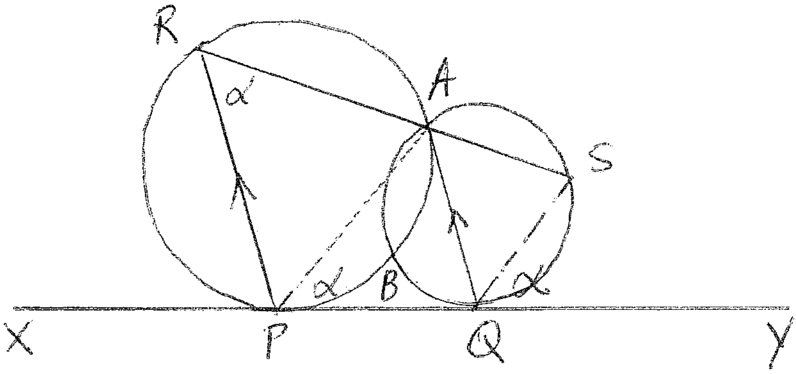
$\therefore x = \sqrt{\frac{100}{117}}$

Test if maximum

x	0·8	$\frac{\sqrt{100}}{\sqrt{117}}=0{\cdot}9$	1
dP/dx	>0	0	<0

The perimeter P is maximised when $x = 0{\cdot}92$ (2 d.p.)

QUESTION 7



(i) Join SQ

Let $\angle SQY = \alpha$

then $\angle QAS = \alpha$ (angle in alternate segment)

$\angle ARP = \alpha$ (corresponding angles in parallel lines $AQ \parallel RP$)

$\therefore \angle SQY = \angle ARP$

$\therefore$ PQSR is a cyclic quad (exterior angle of cyclic quad)

(ii) Join PA

$\angle APQ = \angle ARP$ (angle in alternate segment)

$\therefore \angle APQ = \angle SQY$ (both α)

$\therefore AP \parallel SQ$ (corresponding angles are equal)

b) $$\frac{2\times 1}{2\times 3} + \frac{2^2\times 2}{3\times 4} + \frac{2^3\times 3}{4\times 5} + \cdots + \frac{2^n\times n}{(n+1)(n+2)} = \frac{2^{n+1}}{n+2} - 1$$

for $n \geq 1$

Test $n=1$

$$LHS = \frac{2\times 1}{2\times 3}$$
$$= \frac{1}{3}$$
$$RHS = \frac{2^2}{3} - 1$$
$$= \frac{1}{3}$$

$\therefore$ Result holds for $n=1$

Assume true for $n=k$, where k is an integer, $k\geq 1$

ie assume $$\frac{2\times 1}{2\times 3} + \frac{2^2\times 2}{3\times 4} + \cdots + \frac{2^k\times k}{(k+1)(k+2)} = \frac{2^{k+1}}{k+2} - 1$$

We wish to show that the result holds for $n=k+1$

ie $$\frac{2\times 1}{2\times 3} + \cdots + \frac{2^k\times k}{(k+1)(k+2)} + \frac{2^{k+1}.(k+1)}{(k+2)(k+3)} = \frac{2^{k+2}}{k+3} - 1$$

$$LHS = \frac{2^{k+1}}{k+2} - 1 + \frac{2^{k+1}.(k+1)}{(k+2)(k+3} \quad \text{by assumption}$$
$$= \frac{2^{k+1}(k+3) + 2^{k+1}.(k+1)}{(k+2)(k+3)} - 1$$
$$= \frac{2^{k+1}[k+3+k+1]}{(k+2)(k+3)} - 1$$
$$= \frac{2^{k+1}\cdot 2(k+2)}{(k+2)(k+3)} - 1$$
$$= \frac{2^{k+2}}{k+3} - 1$$

$\therefore$ By mathematical induction the result is true for all integers $n \geq 1$

Question 7

(i) Since parabola passes through $(-h, r)$ and $(0, s)$ and (h, t) then

sub $x=0$: $s = c$ (1)

sub $x=-h$: $r = a(-h)^2 - bh + c$ (2)

$t = a(h)^2 + bh + c$ (3)

(2) + (3) : $r + t = 2ah^2 + 2c$

$= 2ah^2 + 2s$ since $s = c$ from(

(ii) $\int_{-h}^{h} (ax^2 + bx + c)\,dx = \left[\frac{ax^3}{3} + \frac{bx^2}{2} + cx\right]_{-h}^{h}$

$= \left(\frac{ah^3}{3} + \frac{bh^2}{2} + ch\right) - \left(-\frac{ah^3}{3} + \frac{bh^2}{2} - c\right.$

$= 2\left(\frac{ah^3}{3} + ch\right)$

$= \frac{h}{3}\left(2ah^2 + 6c\right)$

$= \frac{h}{3}\left(2ah^2 + 4s + 2s\right)$ since c=

$= \frac{h}{3}\left(r + 4s + t\right)$ from (i)