NORTH SYDNEY GIRLS HIGH SCHOOL

2014

TRIAL HSC EXAMINATION

Mathematics Extension 1

GENERAL INSTRUCTIONS

- Reading Time – 5 minutes
- Working Time – 2 hours
- Write using black or blue pen
- Board approved calculators may be used
- A table of standard integrals is provided at the back of this booklet
- Show all necessary working in questions 11 – 14

Total Marks – 70

Section 1 10 marks

- Attempt Questions 1 -10
- Allow about 15 minutes for this section.

Section 2 60 marks

- Attempt Questions 11 – 14
- Allow about 1 hour 45 minutes for this section.

NAME:______________________________ **TEACHER:**___________________

NUMBER:___________________________

QUESTION	MARK
1 – 10	/10
11	/15
12	/15
13	/15
14	/15
TOTAL	/70

Section I
Total marks – 10
Attempt Questions 1−10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10

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1 Which of the following is an expression for $\int \cos^2 2x\, dx$?

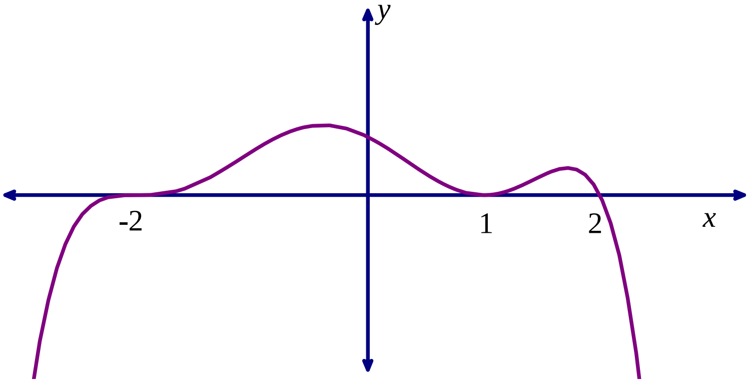
(A) $x - \frac{1}{4}\sin 4x + C$

(B) $x + \frac{1}{4}\sin 4x + C$

(C) $\frac{x}{2} - \frac{1}{8}\sin 4x + C$

(D) $\frac{x}{2} + \frac{1}{8}\sin 4x + C$

2 Below is the graph of the polynomial $y = P(x)$



Which of the following is a possible equation for $P = (x)$?

(A) $P(x) = (x-2)^3(1-x)^2(x+2)$

(B) $P(x) = (x-2)(1-x)^2(x+2)^3$

(C) $P(x) = (2-x)^3(1-x)^2(x+2)$

(D) $P(x) = (2-x)(1-x)^2(x+2)^3$

3 What is the maximum value of $\sqrt{5}\cos x - 2\sin x$?

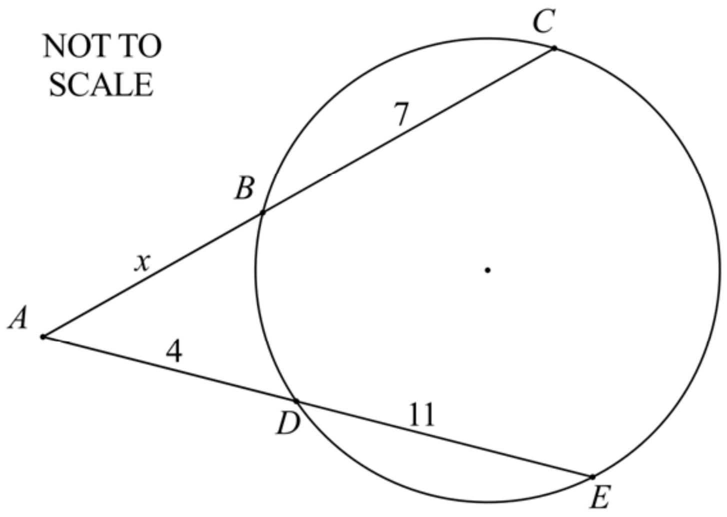
(A) $\sqrt{5}$

(B) 3

(C) 5

(D) $\sqrt{29}$

4 In the diagram below, *BC* and *DE* are chords of a circle. *CB* and *ED* produced meet at *A*.



What is the value of *x*?

(A) $\frac{11}{28}$

(B) $\frac{28}{11}$

(C) 5

(D) 12

5 What is the domain and range of the equation $y = 4\cos^{-1} 3x$?

(A) Domain: $-\frac{1}{3} \le x \le \frac{1}{3}$ and Range: $-2\pi \le y \le 2\pi$

(B) Domain: $-3 \le x \le 3$ and Range: $-2\pi \le y \le 2\pi$

(C) Domain: $-\frac{1}{3} \le x \le \frac{1}{3}$ and Range: $0 \le y \le 4\pi$

(D) Domain: $-3 \le x \le 3$ and Range: $0 \le y \le 4\pi$

6 What is the solution of the inequation $3x+2<|2x-1|$?

(A) $x<-\frac{1}{5}$

(B) $-3<x<\frac{1}{5}$

(C) $x<-\frac{1}{5}$ or $x>3$

(D) $x<-3$

7 What is the value of $\lim_{x\to 0}\frac{3x\cos 4x}{\sin 2x}$?

(A) 0

(B) $\frac{3}{2}$

(C) $\frac{2}{3}$

(D) Undefined

8 Which of the following is the derivative x of $\tan^{-1}\left(e^{-x}\right)$?

(A) $\frac{e^{x}}{1+e^{2x}}$

(B) $\frac{-e^{-x}}{1+e^{2x}}$

(C) $\frac{-e^{-x}}{1+e^{-2x}}$

(D) $\frac{e^{x}}{1+e^{-2x}}$

9 A particle undergoing simple harmonic motion and has acceleration according to the equation:

$$v^2 = 9(5 - x^2)$$

What is the amplitude A and Period T of this motion?

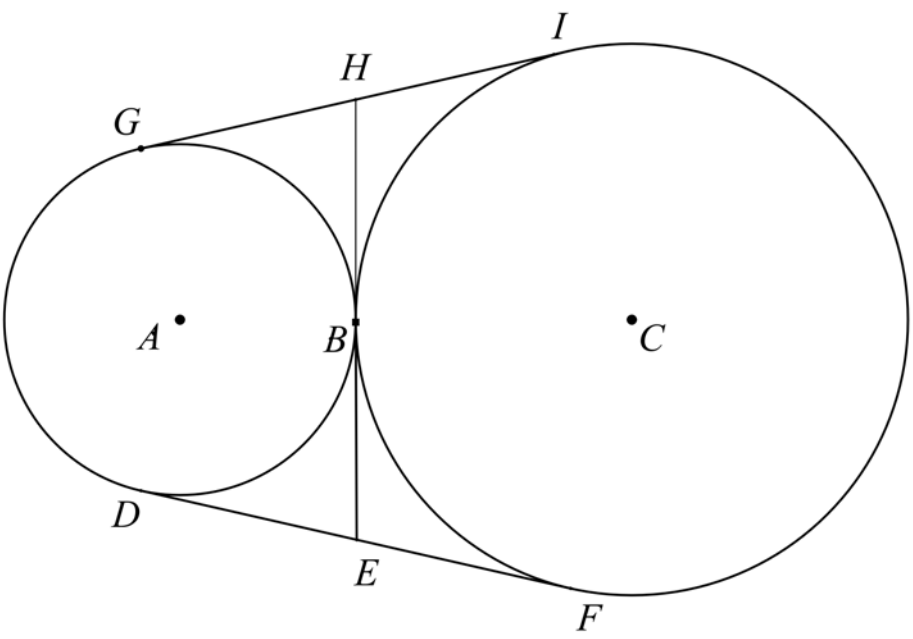
(A) $A = \sqrt{5}$ and $T = \frac{2\pi}{9}$

(B) $A = 5$ and $T = \frac{2\pi}{9}$

(C) $A = \sqrt{5}$ and $T = \frac{2\pi}{3}$

(D) $A = 5$ and $T = \frac{2\pi}{3}$

10 In the diagram below DF, GI and HE are common tangents to the two unequal circles that have centres at A and C.



Which of the following quadrilaterals are **<u>NOT</u>** cyclic?

(A) *ABED*

(B) *HICB*

(C) *GIFD*

(D) *ACFD*

Section II
Total marks – 60
Attempt Questions 11−14
Allow about 1 hour and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Factorise the polynomial $P(x) = x^3 - 7x + 6$. **2**

(b) Evaluate $\int_0^1 \frac{1}{\sqrt{4-3x^2}}\,dx$. **2**

(c) Solve $\frac{2x+3}{x} \geq x$. **3**

(d) The point P divides the interval from $A(-1,3)$ to $B(7,-4)$ externally in the ratio $2:5$. Find the coordinates of P. **2**

(e) Use the substitution $u = x - 1$ to evaluate: **3**

$$\int_2^4 \frac{3x}{(x-1)^2}\,dx.$$

(f) The two curves $y = 2 - x^2$ and $y = x^3$ intersect at $(1,0)$. **3**

Find the acute angle between the two curves at $(1,0)$.

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) Given the function $f(x)=\dfrac{2x+1}{x-1}$

(i) Write down the equation of any vertical and horizontal asymptotes. **1**

(ii) Given that $f(x)$ is a hyperbola, state the domain of the inverse function $f^{-1}(x)$. **1**

(iii) Sketch the graph of the inverse function $y=f^{-1}(x)$. Clearly label all important features of the graph. **2**

(b) A can of soft drink at temperature *T* degrees is removed from a fridge and placed in a room that has a constant temperature of *A* degrees. The rate at which the can of soft drink warms can be expressed using the equation:

$$\frac{dT}{dt}=-k(T-A)$$

where *t* is the time in minutes after the can is placed in the room, and *k* is a positive constant.

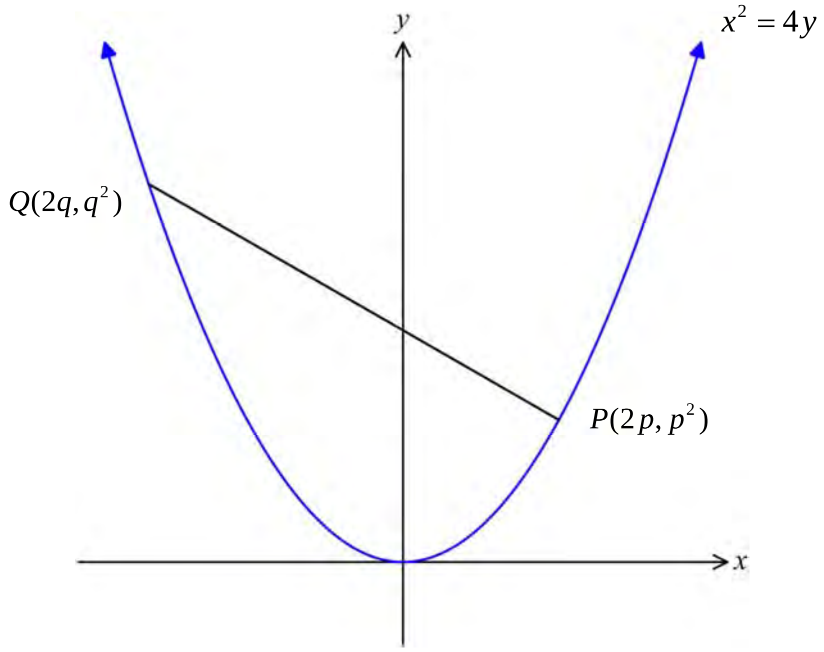
(i) Show that $T=A+Pe^{-kt}$ satisfies the equation, where *P* is a constant. **1**

(ii) If the room temperature is $30°$ and the soft drink warms from $3°$ to $15°$ in the first 10 minutes find the exact value of *k*. **2**

(iii) Find the time taken to the nearest second for temperature of the can to increase by another 12 degrees. **2**

Question 12 continues on page 8

(c) The diagram shows two distinct points $P(2p, p^2)$ and $Q(2q, q^2)$ that lie on the parabola $x^2 = 4y$.



(i) Show that the line *PQ* has the equation $y = \frac{p+q}{2}x - pq$. **2**

(ii) The point $R(8, 0)$ lies on the straight line that passes through *P* and *Q*.

(1) Show that. $4(p+q) = pq$ **1**

(2) Given that *M* is the midpoint of *PQ*, show that the equation of the locus of *M*, as *P* and *Q* vary is **1**

$$2y = x^2 - 8x$$

(3) State any restrictions on the locus. **2**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) Use mathematical induction to prove that if n is an integer and $n \geq 1$ and $a > 0$ **3**

$$\frac{1}{a} - \frac{1}{a+1} - \frac{1}{(a+1)^2} - \ldots - \frac{1}{(a+1)^n} = \frac{1}{a(a+1)^n}$$

(b) A particle moves in simple harmonic motion such that its displacement x centimetres after t seconds is given by $x = 2\sqrt{3}\cos 2t - 2\sin 2t - 7$.

(i) Express x in the form $x = 4\cos(2t + \alpha) - 7$, where $0 \leq \alpha \leq \frac{\pi}{2}$ **2**

(ii) What is the period of the motion? **1**

(iii) What is the range of possible values for x? **1**

(iv) How much time elapses from when the particle first starts until it is first at its minimum displacement? **2**

(c) By sketching appropriate graphs or otherwise solve: **3**

$$\sin^{-1}(3x+1) = \cos^{-1} x$$

(d) The polynomial $P(x) = x^3 + ax^2 - 6x - 4$, where $a > 0$, has three zeroes. **3**
One zero is the product of the other two.

Find the value of a.

Question 14 (15 marks) Use a SEPARATE writing booklet.

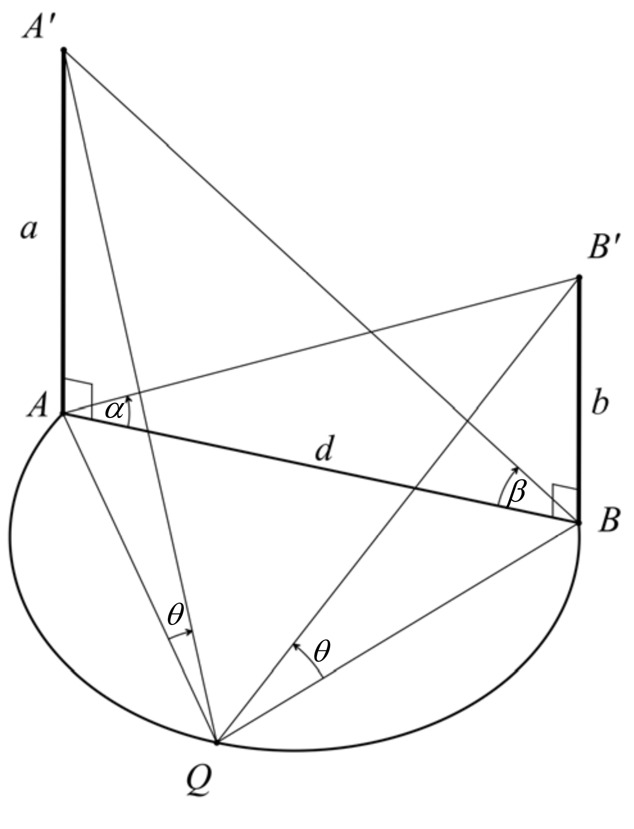
(a) (i) An object is initially 2 m to the right of the origin travelling with velocity 6 m/s. The acceleration $\ddot{x}$ ms^{-2} of the object is given by: **2**

$$\ddot{x} = 2x^3 + 4x$$

Find an expression for v^2 in terms of x.

(ii) Find the minimum speed of the object giving a reason. **2**

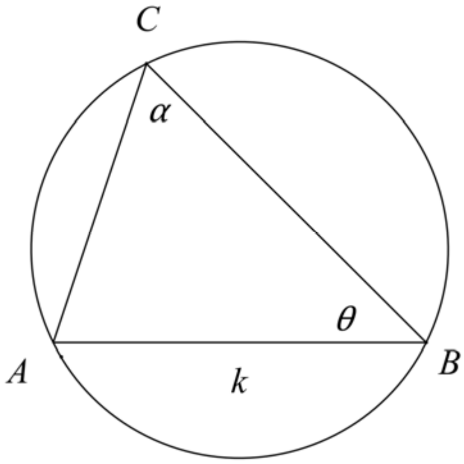
(b) AQB is a semi-circle in the horizontal plane with diameter AB of length d metres. There are two vertical posts AA' and BB' of heights a and b respectively. From Q, the angle of elevation to the tops of both posts A' and B' is θ. From A the angle of elevation to B' is α and from B the angle of elevation to A' is β.



(i) Prove that $d^2 = \dfrac{a^2 + b^2}{\tan^2 \theta}$. **3**

(ii) Show that $\tan^2 \alpha + \tan^2 \beta = \tan^2 \theta$. **1**

(c) The points A, B and C all lie on a circle. A and B are fixed, AB has a constant length k cm and C lies on the major arc. $\angle ABC = \theta$ and $\angle ACB = \alpha$ as shown in the diagram below. C moves on the major arc so that θ increases at 0.1 radians per second.



(i) If P is the perimeter of the triangle ABC then show that: **2**

$$P = \frac{k\left[\sin\alpha + \sin\theta + \sin(\theta+\alpha)\right]}{\sin\alpha}$$

(ii) Find the value of $\frac{dP}{dt}$ in terms of k and α when $\theta = \alpha$. **2**

(iii) Show that $\frac{dP}{dt} = 0$ when $\theta = \frac{\pi}{2} - \frac{\alpha}{2}$. **1**

(iv) Find the value of θ that will maximise P, given $\alpha = \frac{3\pi}{8}$. **2**

End of paper

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

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1 Which of the following is an expression for $\int \cos^2 2x\,dx$?

Since

$\cos 2\theta = 2\cos^2\theta - 1$

$\cos^2\theta = \dfrac{\cos 2\theta + 1}{2}$

$\cos^2 2\theta = \dfrac{\cos 4\theta + 1}{2}$

$$\int \cos^2 2x\,dx = \int \frac{1+\cos 4x}{2}\,dx$$

$$= \frac{x}{2} + \frac{\sin 4x}{8} + C$$

ANSWER is D

2 ANSWER is D

$P(x) = (2-x)(1-x)^2(x+2)^3$

3 What is the maximum value of $\sqrt{5}\cos x - 2\sin x$?

$\sqrt{5}\cos x - 2\sin x = R\cos(x+\alpha)$ so thus the maximum value is when $\cos(x+\alpha) = 1$.

$R = \sqrt{(\sqrt{5})^2 + 2^2}$

$R = 3$

ANSWER is (C) 3

4

$\dfrac{AB}{AE} = \dfrac{AD}{AC}$

$\dfrac{x}{15} = \dfrac{4}{x+7}$

$x^2 + 7x = 60$

$x^2 + 7x - 60 = 0$

$(x+12)(x-5) = 0$

Since $x > 0$

$x = 5$

ANSWER is C

5 What is the domain and range of the equation $y = 4\cos^{-1} 3x$?

Domain

$-1 \le 3x \le 1$

$-\dfrac{1}{3} \le x \le \dfrac{1}{3}$

Range

$0 \le \cos^{-1} 3x \le \pi$

$0 \le 4\cos^{-1} 3x \le 4\pi$

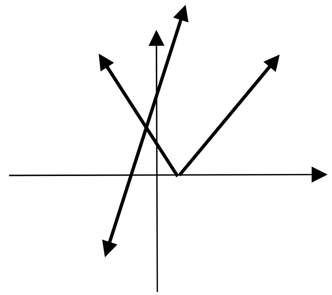
ANSWER is (C)

Domain: $-\dfrac{1}{3} \le x \le \dfrac{1}{3}$ and Range: $0 \le y \le 4\pi$

6 What is the solution of the inequation $3x + 2 < |2x-1|$?

Best method is to sketch the two functions

$y = 3x + 2$ and $y = |2x-1|$



Only one intersection where

$3x + 2 = -[2x-1]$

$5x = -1$

$x = -\dfrac{1}{5}$

ANSWER is (A) $x < -\dfrac{1}{5}$

7 What is the value of $\lim_{x\to 0}\frac{3x\cos 4x}{\sin 2x}$?

$$\lim_{x\to 0}\frac{3x\cos 4x}{\sin 2x}=\lim_{x\to 0}\frac{3x}{\sin 2x}\times\lim_{x\to 0}\cos 4x$$
$$=\frac{3}{2}\lim_{x\to 0}\frac{2x}{\sin 2x}\times\lim_{x\to 0}\cos 4x$$
$$=\frac{3}{2}\times 1\times\cos 0=\frac{3}{2}$$

ANSWER is (B) $\frac{3}{2}$

8 Which of the following is the derivative of $\tan^{-1}\left(e^{-x}\right)$?

$$\frac{d}{dx}\tan e^{-x}=\frac{1}{1+(e^{-x})^2}\times\frac{d}{dx}\left(e^{-x}\right)$$
$$=\frac{-e^{-x}}{1+e^{-2x}}$$

ANSWER is (C) $\frac{-e^{-x}}{1+e^{-2x}}$

9

$$\ddot{x}=-n^2x$$
$$\frac{d}{dx}\left(\frac{1}{2}v^2\right)=-n^2\left(x-x_0\right)$$
$$\left(\frac{1}{2}v^2\right)=-\frac{n^2x^2}{2}+C$$
$$v^2=-\frac{n^2x^2}{2}+D$$

When $v=0 \quad x=A$

$$0=-n^2A^2+D$$
$$D=n^2A^2$$

$$v^2=-n^2x^2+n^2A^2$$
$$v^2=n^2\left(A^2-x^2\right)$$

Thus $n=3$ and $A=\sqrt{5}$

$$T=\frac{2\pi}{3}$$

ANSWER is (C) $A=\sqrt{5}$ and $T=\frac{2\pi}{3}$

10

$\angle ADF=90°$ Angle between a radius and tangent

$\angle CFD=90°$ Angle between a radius and tangent

If $ACFD$ is cyclis then $\angle CADF=90°$ and $\angle ACF=90°$ opposite angle are supplementary.

So $ACFD$ must be a rectangle if it is cyclic.

But the circles are not equal and so AD and CF are not equal.

$ACFD$ is not cyclic

ANSWER is (D) $ACFD$

Question 11

(a) **Factorise the polynomial**

$P(x) = x^3 - 7x + 6$.

$$P(x) = x^3 - 7x + 6$$
$$P(1) = 1^3 - 7(1) + 6$$
$$= 0$$

$\therefore (x-1)$ is a factor of P(x)

$$P(x) = x^3 - 7x + 6$$
$$= (x-1)(x^2 + bx - 6)$$

x term is

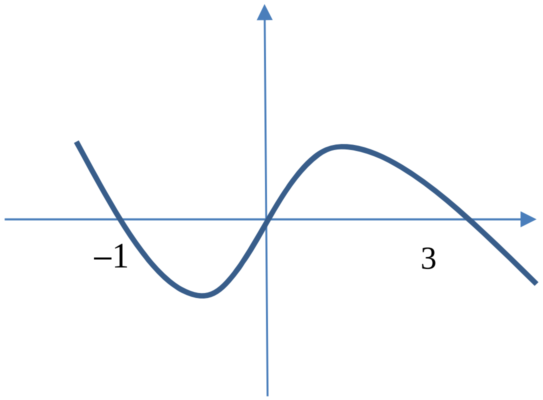
$$-7x = -bx - 6x$$
$$7 = b + 6$$
$$b = 1$$
$$P(x) = (x-1)(x^2 + x - 6)$$
$$= (x-1)(x-2)(x+3)$$

(b) **Evaluate** $\int_0^1 \frac{1}{\sqrt{4-3x^2}}\,dx$. **2**

$$\int_0^1 \frac{1}{\sqrt{4-3x^2}}dx = \int_0^1 \frac{1}{\sqrt{4-(x)^2}}$$
$$= \int_0^1 \frac{1}{\sqrt{3}\sqrt{\frac{4}{3}-(x)^2}}dx$$
$$= \frac{1}{\sqrt{3}}\sin^{-1}\frac{x\sqrt{3}}{2}dx$$

(c)

$$x(2x+3) \ge x^3 \qquad x \ne 0$$
$$x(2x+3) - x^3 \ge 0$$
$$x(2x+3-x^2) \ge 0$$
$$x(x+1)(3-x) \ge 0$$



From the graph of $y = x(x+1)(3-x)$

$x \le -1$ or $0 < x \le 3$ since $x \ne 0$

(d)

$A(-1,3)$ to $B(7,-4)$

$-2:5$

$$x = \frac{(5)(-1)+(-2)(7)}{5-2} \qquad y = \frac{(5)(3)+(-2)(-4)}{5-2}$$
$$= \frac{-5-14}{3} \qquad = \frac{15+8}{3}$$
$$= \frac{-19}{3} \qquad = \frac{23}{3}$$

P has Coordinates $\left(\frac{-19}{3}, \frac{23}{3}\right)$

(e) **Evaluate:** $\int_2^4 \frac{3x}{(x-1)^2}\,dx$.

$u = x - 1$

$x = u + 1$

$\frac{du}{dx} = 1$

$du = dx$

When $x = 4$, $u = 4 - 1 = 3$

$x = 2$, $u = 2 - 1 = 1$

$$\int_2^4 \frac{3x}{(x-1)^2}dx = \int_1^3 \frac{3(u+1)}{(u)^2}du$$
$$= \int_1^3 \frac{3u}{u^2} + \frac{3}{u^2}du$$
$$= \int_1^3 \frac{3}{u} + \frac{3}{u^2}du$$
$$= \left[3\ln u - 3u^{-1}\right]_1^3$$
$$= \left(3\ln 3 - \frac{3}{3}\right) - \left(3\ln 1 - \frac{3}{1}\right)$$
$$= 3\ln 3 + 2$$

(f) The two curves $y = 2 - x^2$ and $y = x^3$ intersect at $(1,0)$. Find the acute angle between the two curves at $(1,0)$.

$\tan\theta = \left|\frac{m_1 - m_2}{1 - m_1 m_2}\right|$ where m_1 and m_2 are the gradients of the two functions at $x = 1$

$y = 2 - x^2$ $\quad$ $y = x^3$

$\frac{dy}{dx} = -2x$ $\quad$ $\frac{dy}{dx} = 3x^2$

at $x = 1$ $\quad$ at $x = 1$

$m_1 = -2$ $\quad$ $m_2 = 3$

$$\tan\theta = \left|\frac{-2-3}{1-(-2)(3)}\right|$$

$$\tan\theta = \left|\frac{-5}{-5}\right|$$

$\tan\theta = 1$

$\theta = 45°$

Question 12

(a) **(i)**

Vertical Asymptote: $x = 1$

Horizontal Asymptote: $y = 2$

(ii) $y \neq 2$

(iii)

x intercept of $f(x)$ is when

$f(x) = 0$

$$0 = \frac{2x+1}{x-1}$$

$0 = 2x + 1$

$x = -\frac{1}{2}$

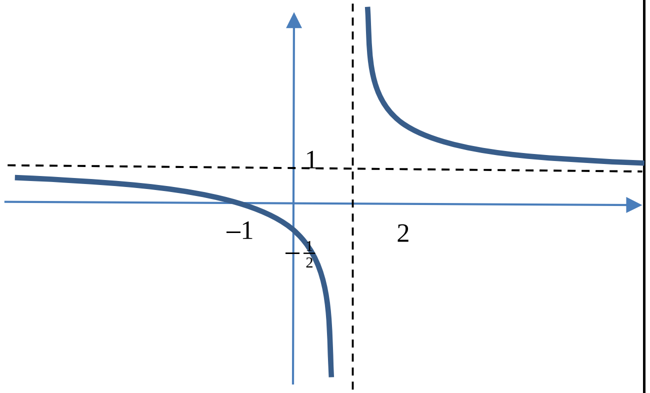
y intercept of $f(x)$ is

$$y = \frac{2(0)+1}{(0)-1}$$

$= -1$

So the x intercept of $f^{-1}(x)$ is $x = -1$

So the y intercept of $f^{-1}(x)$ is $y = -\frac{1}{2}$



(b) (i) Show that $T = A + Pe^{-kt}$ satisfies the equation, where *P* is a constant.

$T = A + Pe^{-kt}$

$$\frac{dT}{dt} = -kPe^{-kt}$$

$$= -k\left(A + Pe^{-kt} - A\right)$$

$$\frac{dT}{dt} = -k(T - A)$$

(ii) When $t = 0$ $T = 3$ and When $t = 10$ $T = 15$

$3 = 30 + Pe^0$

$P = -27$

$15 = 30 - 27e^{-10k}$

$15 = 27e^{-10k}$

$$\frac{15}{27} = e^{-10k}$$

$$\ln\left(\frac{15}{27}\right) = -10k$$

$$-\frac{1}{10}\ln\left(\frac{15}{27}\right) = k$$

OR

$$\frac{1}{10}\ln\left(\frac{27}{15}\right) = k$$

iii)

$27 = 30 - 27e^{-kt}$

$3 = 27e^{-kt}$

$$\frac{3}{27} = e^{-kt}$$

$$\ln\left(\frac{1}{9}\right) = -kt$$

$$-\frac{\ln\left(\frac{1}{9}\right)}{k} = t$$

$$\frac{\ln\left(\frac{1}{9}\right)}{\frac{1}{10}\ln\left(\frac{15}{27}\right)} = t$$

$t = 37.38132742$

Time Taken is
$37.38132742 - 10 = 27.38132742$ minutes
$= 27`23``$

c) (i) Show that the line PQ has the equation $y = \frac{p+q}{2}x - pq$.

$$m_{CHORD} = \frac{p^2 - q^2}{2p - 2q}$$

$$m_{CHORD} = \frac{p+q}{2}$$

Equation

$$y - p^2 = \frac{p+q}{2}(x - 2p)$$

$$y - p^2 = \frac{p+q}{2}x - \frac{2p^2 - 2pq}{2}$$

$$y = \frac{p+q}{2}x - p^2 + 2pq + p^2$$

$$y = \frac{p+q}{2}x - p^2 + 2pq + p^2$$

$$y = \frac{p+q}{2}x - pq$$

(ii) The point $R(8,0)$ lies on the straight line that passes through P and Q.

(1) Show that $4(p+q) = pq$.

$$y = \frac{p+q}{2}x - pq$$

$$0 = \frac{p+q}{2} \times 8 - pq$$

$$0 = 4(p+q) - pq$$

$$4(p+q) = pq$$

(2) Given that M is the midpoint of PQ, show that as P and Q vary the

$$2y = x^2 - 8x$$

Midpoint

$$x = \frac{2p + 2q}{2} \qquad y = \frac{p^2 + q^2}{2}$$

$$= p + q$$

Locus of M

$$x^2 - 8x = (p+q)^2 - 8x$$

$$= (p^2 + 2pq + q^2) - 8(p+q)$$

From part (ii) (1) $4(p+q) = pq$

$$= (p^2 + 2pq + q^2) - 2 \times 4(p+q)$$

$$= (p^2 + 2pq + q^2) - 2 \times pq$$

$$= p^2 + q^2$$

$$= 2y$$

(3) State any restrictions on the locus.

The y-value of M must always be greater than or equal to the y-value of the parabola $x^2 = 4y$.

$$\frac{x^2 - 8x}{2} \geq \frac{x^2}{4}$$

$$2x^2 - 16x \geq x^2$$

$$x^2 - 16x \geq 0$$

$$x(x - 16) \geq 0$$

$$x \leq 0 \qquad x \geq 16$$

Question 13

(a) Use mathematical induction to prove that if *n* is an integer and $n \geq 1$ and $a > 0$

$$\frac{1}{a}-\frac{1}{a+1}-\frac{1}{(a+1)^2}-\dots-\frac{1}{(a+1)^n}=\frac{1}{a(a+1)^n}$$

When $n=1$

$$LHS=\frac{1}{a}-\frac{1}{a+1} \quad RHS=\frac{1}{a(a+1)^1}$$

$$=\frac{a+1-a}{a(a+1)} \quad =\frac{1}{a(a+1)}$$

$$=\frac{1}{a(a+1)}$$

True for $n=1$

Assume true when $n=k$

$$\frac{1}{a}-\frac{1}{a+1}-\frac{1}{(a+1)^2}-\dots-\frac{1}{(a+1)^k}=\frac{1}{a(a+1)^k}$$

Prove true when $n=k+1$ if true for $n=k$

Aim : Prove that

$$\frac{1}{a}-\frac{1}{a+1}-\frac{1}{(a+1)^2}-\dots-\frac{1}{(a+1)^{k+1}}=\frac{1}{a(a+1)^{k+1}}$$

$$LHS=\frac{1}{a}-\frac{1}{a+1}-\frac{1}{(a+1)^2}-\dots-\frac{1}{(a+1)^{k+1}}$$

$$=\frac{1}{a}-\frac{1}{a+1}-\frac{1}{(a+1)^2}-\dots-\frac{1}{(a+1)^k}-\frac{1}{(a+1)^{k+1}}$$

$$=\frac{1}{a(a+1)^k}-\frac{1}{(a+1)^{k+1}}$$ By Assumption

$$=\frac{a+1-a}{a(a+1)^{k+1}}$$

$$=\frac{1}{a(a+1)^{k+1}}=RHS$$

By Mathematical Induction

$$\frac{1}{a}-\frac{1}{a+1}-\frac{1}{(a+1)^2}-\dots-\frac{1}{(a+1)^n}=\frac{1}{a(a+1)^n}$$ is

true for all integers $n \geq 1$

(b) (i) Express x in the form

$x=4\cos(2t+\alpha)-7$, where $0 \leq \alpha \leq \frac{\pi}{2}$

$$2\sqrt{3}\cos 2t-2\sin 2t=4\cos(2t+\alpha)$$

$$=4\cos 2t\cos\alpha-4\sin 2t\sin\alpha$$

$$2\sqrt{3}\cos 2t-2\sin 2t=4\cos(2t+\alpha)$$

$$2\sqrt{3}=4\cos\alpha \qquad -2=-4\sin\alpha$$

$$\frac{\sqrt{3}}{2}=\cos\alpha \qquad \frac{1}{2}=\sin\alpha$$

$$\therefore \tan\alpha=\frac{1}{\sqrt{3}}$$ where $0 \leq \alpha \leq \frac{\pi}{2}$

$$\alpha=\tan^{-1}\frac{1}{\sqrt{3}}=\frac{\pi}{6}$$

(ii) What is the period of the motion?

$$T=\frac{2\pi}{n}=\frac{2\pi}{2}=\pi \text{ seconds}$$

(iii) What is the range of possible values for *x*?

Amplitude is 4 and centre of motion is at $x=-7$. Range of possible x- values is

$-11 \leq x \leq -3$

(iv) How much time elapses from when the particle first starts until it is first at its minimum displacement?

When $x=-11$

$$-11=4\cos\left(2t+\frac{\pi}{6}\right)-7$$

$$-4=4\cos\left(2t+\frac{\pi}{6}\right)$$

$$-1=\cos\left(2t+\frac{\pi}{6}\right)$$

$$-1=\cos\left(2t+\frac{\pi}{6}\right)$$

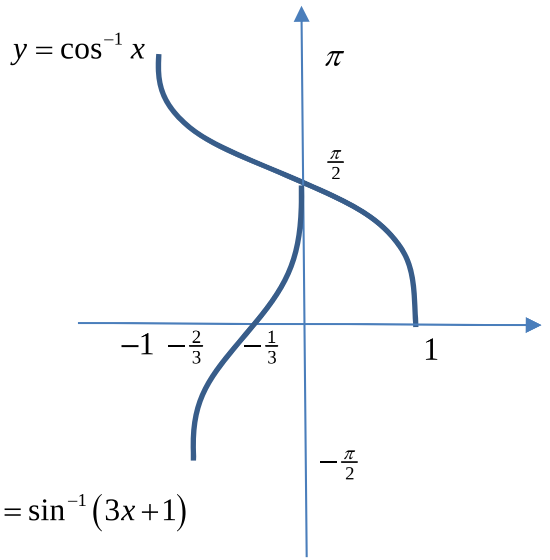
$$2t+\frac{\pi}{6}=\cos^{-1}(-1)$$

$$2t+\frac{\pi}{6}=\pi$$

$$2t=\frac{5\pi}{6}$$

$$t=\frac{5\pi}{12}$$

(c) Sketch both functions $y=\cos^{-1}x$ and $y=\sin^{-1}(3x+1)$ on the same plane.



From the graph, there is only one solution, $x=0$

ALTERNATIVE :

Solve algebraically:

Let $\alpha=\sin^{-1}(3x+1)$ and thus $\alpha=\cos^{-1}x$

Let $\sin\alpha=3x+1$ and $\cos\alpha=x$

By Pythagoras using $\cos\alpha=x$ then $\sin\alpha=\sqrt{1-x^2}$

$$(3x+1)=\sqrt{1-x^2}$$

$$(3x+1)^2=(\ )$$

$$9x^4+6x^3+1=1-x^2$$

$$10x^2+6x=0$$

$$2x(5x+3)=0$$

$$x=0$$

OR

$$x=\frac{-3}{5}$$

Check solutions in original equation.

$x=0\quad \sin^{-1}(1)=\cos^{-1}(0)$

$x=\frac{-3}{5}\quad \sin^{-1}\left(-\frac{9}{5}+1\right)\neq\cos^{-1}\left(\frac{-3}{5}\right)$

Solution is $x=0$

(d) Let the zeroes be α , β and $\alpha\beta$

Product of Roots

$$(\alpha)(\beta)(\alpha\beta)=4$$

$$\alpha^2\beta^2=4$$

$$\alpha\beta=\pm2$$

Sum in Pairs

$$\alpha\beta+\alpha^2\beta+\alpha\beta^2=-6$$

$$\alpha\beta(1+\alpha+\beta)=-6 \qquad I$$

Sum of Roots

$$\alpha+\beta+\alpha\beta=-a \qquad II$$

If $\alpha\beta=2$ then

In I

$$\alpha+\beta+2=-a$$

$$\alpha+\beta+1=-a-1$$

And in II

$$\alpha\beta(1+\alpha+\beta)=-6$$

$$(2)(-a-1)=-6$$

$$-a-1=-3$$

$$a=2$$

If $\alpha\beta=-2$ then

In I

$$\alpha+\beta-2=-a$$

$$\alpha+\beta+1=-a+3$$

And in II

$$\alpha\beta(1+\alpha+\beta)=-6$$

$$(-2)(-a+3)=-6$$

$$-a+3=3$$

$$-a=0$$

$$a=0$$

Since $a>0\quad a=2$

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) (i)

$\ddot{x} = 2x^3 + 4x$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 2x^3 + 4x$

$\frac{1}{2}v^2 = \frac{2x^4}{4} + \frac{4x^2}{2} + C$

$v^2 = x^4 + 4x^2 + D$

When $x = 2 \quad v = 6$

$(6)^2 = (2)^4 + 4(2)^2 + D$

$36 = 16 + 16 + D$

$4 = D$

$v^2 = x^4 + 4x^2 + 4$

$v^2 = (x^2 + 2)^2$

(ii) Acceleration $\ddot{x} = 2x^3 + 4x$ is positive when x is positive.

Since the particle is initially at $x = 2$ with positive velocity the velocity of the particle will always be increasing and so the minimum velocity is its initial velocity of 6m/s

(b) (i) Prove that $d^2 = \frac{a^2 + b^2}{\tan^2\theta}$.

$\angle AQB = 90°$ (Angle in a semi circle)

Let AQ and BQ be x and y respectively.

In ΔAQB

$d^2 = x^2 + y^2$ By Pythagoras

In $\Delta AQA'$

$\tan\theta = \frac{a}{x}$

$x = \frac{a}{\tan\theta}$

In $\Delta BQB'$

$\tan\theta = \frac{b}{y}$

$y = \frac{b}{\tan\theta}$

$d^2 = \left(\frac{a}{\tan\theta}\right)^2 + \left(\frac{b}{\tan\theta}\right)^2$

$d^2 = \frac{a^2 + b^2}{\tan^2\theta}$

(ii) Show that $\tan^2\alpha + \tan^2\beta = \tan^2\theta$.

In $\Delta ABA'$

$\tan\beta = \frac{a}{d}$

In $\Delta BAB'$

$\tan\alpha = \frac{b}{d}$

$\text{LHS} = \tan^2\alpha + \tan^2\beta$

$= \left(\frac{a}{d}\right)^2 + \left(\frac{b}{d}\right)^2$

$= \frac{a^2 + b^2}{d^2}$

$= (a^2 + b^2) \div d^2$

$= (a^2 + b^2) \div \frac{a^2 + b^2}{\tan^2\theta}$

$= \tan^2\theta$

$= RHS$

(c) (i) If P is the perimeter of the triangle ABC then show that:

$P = \frac{k[\sin\alpha + \sin\theta + \sin(\theta + \alpha)]}{\sin\alpha}$

$P = AB + AC + BC$

$AB = k$

$\frac{AC}{\sin\theta} = \frac{AB}{\sin\alpha}$

$AC = \frac{k\sin\theta}{\sin\alpha}$

$$\frac{BC}{\sin\left(\pi-(\theta+\alpha)\right)}=\frac{AB}{\sin\alpha}$$

$$AC=\frac{k\sin\left(\pi-(\theta+\alpha)\right)}{\sin\alpha}$$

$$AC=\frac{k\sin(\theta+\alpha)}{\sin\alpha}$$

$$P=k+\frac{k\sin(\theta)}{\sin\alpha}+\frac{k\sin(\theta+\alpha)}{\sin\alpha}$$

$$P=\frac{k\left[\sin\alpha+\sin\theta+\sin(\theta+\alpha)\right]}{\sin\alpha}$$

(ii) Find the value of $\frac{dP}{dt}$ in terms of k and α when $\theta=\alpha$.

Since $\angle ACB$ is an angle subtended by AB in the major segment α is a constant.

$$\frac{dP}{dt}=\frac{dP}{d\theta}\times\frac{d\theta}{dt}$$

$$\frac{d\theta}{dt}=0.1$$

$$P=\frac{k\left[\sin\alpha+\sin\theta+\sin(\theta+\alpha)\right]}{\sin\alpha}$$

$$\frac{dP}{d\theta}=\frac{k\left[\cos\theta+\cos(\theta+\alpha)\right]}{\sin\alpha}$$

$$\frac{dP}{dt}=\frac{dP}{d\theta}\times\frac{d\theta}{dt}$$

$$=\frac{k\left[\cos\theta+\cos(\theta+\alpha)\right]}{\sin\alpha}\times\frac{1}{10}$$

When $\theta=\alpha$

$$=\frac{k\left[\cos\alpha+\cos(2\alpha)\right]}{10\sin\alpha}$$

(iii) Show that $\frac{dP}{dt}=0$ when $\theta=\frac{\pi}{2}-\frac{\alpha}{2}$.

$\frac{dP}{dt}=0$ when $\frac{dP}{d\theta}=0$

Let $\theta=\frac{\pi}{2}-\frac{\alpha}{2}$

$$\frac{dP}{d\theta}=\frac{k\left[\cos\left(\frac{\pi}{2}-\frac{\alpha}{2}\right)+\cos\left(\frac{\pi}{2}-\frac{\alpha}{2}+\alpha\right)\right]}{\sin\alpha}$$

$$\frac{dP}{d\theta}=\frac{k\left[\cos\left(\frac{\pi}{2}-\frac{\alpha}{2}\right)+\cos\left(\frac{\pi}{2}+\frac{\alpha}{2}\right)\right]}{\sin\alpha}$$

NOTE:

$$\cos\left(\frac{\pi}{2}-\frac{\alpha}{2}\right)=-\cos\left(\frac{\pi}{2}+\frac{\alpha}{2}\right)$$

$$\frac{dP}{d\theta}=\frac{k\left[\cos\left(\frac{\pi}{2}-\frac{\alpha}{2}\right)-\cos\left(\frac{\pi}{2}-\frac{\alpha}{2}\right)\right]}{\sin\alpha}=0$$

(iv) Find the value of θ that will maximise P, given $\alpha=\frac{3\pi}{8}$.

Since $\frac{dP}{d\theta}=0$ when $\theta=\frac{\pi}{2}-\frac{\alpha}{2}$ then

$\theta=\frac{\pi}{2}-\frac{3\pi}{16}$ is a possible value åthat can maximise P

Find the second derivative and test $\theta=\frac{5\pi}{16}$

$$\frac{d^2P}{d\theta^2}=\frac{k\left[-\sin\theta-\sin(\theta+\alpha)\right]}{\sin\alpha}$$

Let $\theta=\frac{5\pi}{16}$

$$\frac{d^2P}{d\theta^2}=\frac{-k\left[\sin\frac{5\pi}{16}+\sin\left(\frac{5\pi}{16}+\alpha\right)\right]}{\sin\alpha}$$

Since $0\le\alpha\le\frac{\pi}{2}$

$\sin\frac{5\pi}{16}+\sin\left(\frac{5\pi}{16}+\alpha\right)>0$ and $\sin\alpha>0$

Therefore $\frac{d^2P}{d\theta^2} < 0$ when $\theta = \frac{5\pi}{16}$ thus P is a maximum when $\theta = \frac{5\pi}{16}$

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x$$

$$\int \frac{1}{x}dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a <$$