NORTH
SYDNEY
GIRLS HIGH
SCHOOL

2018 | **HSC Trial Examination**

Mathematics Extension 1

General Instructions

- Reading Time – 5 minutes
- Working Time – 2 hours
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided
- In Questions 11 – 14, show relevant mathematical reasoning and/or calculations

Total marks – 70

Section I – 10 marks (pages 3 – 7)

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 8 – 15)

- Attempt Questions 11 – 14
- Allow about 1 hours 45 minutes for this section

NAME:______________________________ **TEACHER:**____________________________

STUDENT NUMBER:

Question	1-10	11	12	13	14	Total
Mark	/10	/15	/15	/15	/15	/70

Section I

10 marks
Attempt Questions 1−10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 What is the remainder when $P(x) = 2x^3 - x^2 + 5x + k$ is divided by $x - 2$?

A. $-30 + k$

B. -30

C. 22

D. $22 + k$

2 Which expression is equivalent to $\sin x - \sqrt{3}\cos x$?

A. $2\sin\left(x + \frac{2\pi}{3}\right)$

B. $2\sin\left(x - \frac{2\pi}{3}\right)$

C. $2\sin\left(x + \frac{\pi}{3}\right)$

D. $2\sin\left(x - \frac{\pi}{3}\right)$

3 What is the value of $\lim_{x \to 1} \dfrac{3x\sin(x-1)}{2-2x}$?

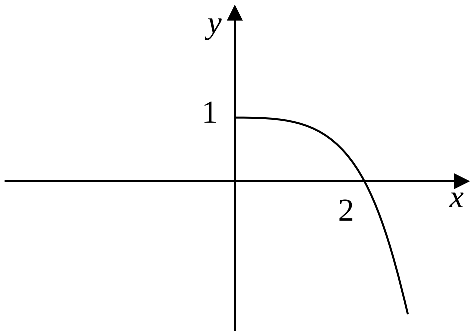
A. $-\dfrac{3}{2}$

B. 0

C. $\dfrac{3}{2}$

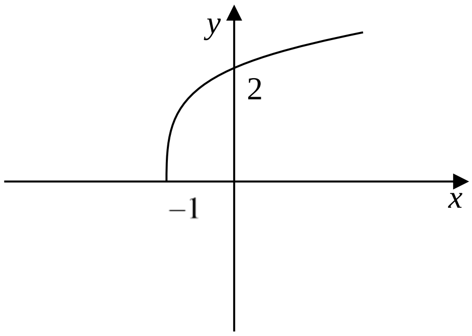
D. Undefined

4 The diagram shows the graph of $y = f(x)$

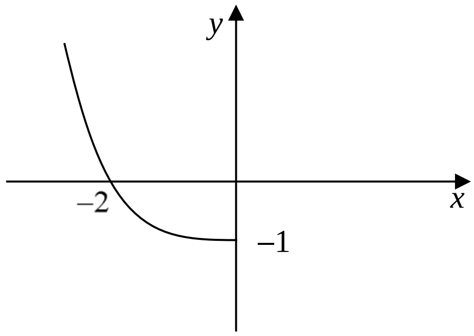


Which of the following is the graph of $y = f^{-1}(x)$

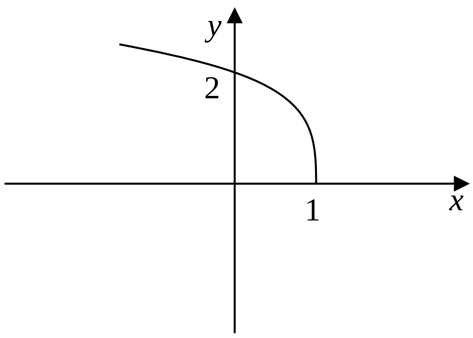
A.



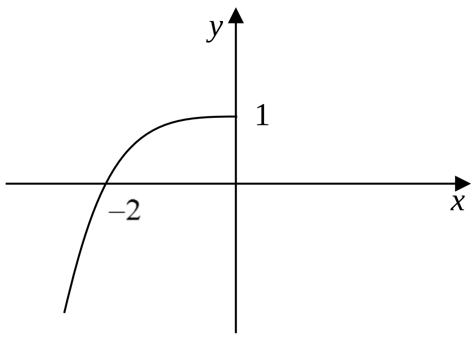
B.



C.



D.



5 The acute angle between the lines $2x-3y+1=0$ and $y=-3x+5$ is θ.
What is the value of $\tan\theta$?

A. $\frac{11}{3}$

B. $\frac{7}{9}$

C. $-\frac{7}{9}$

D. $-\frac{11}{3}$

6 Given that $\alpha+\beta+\gamma=4$, $\alpha\beta+\beta\gamma+\alpha\gamma=2$ and $\alpha\beta\gamma=\frac{1}{2}$, which polynomial equation has roots α, β, and γ?

A. $2x^3-4x^2+2x-1=0$

B. $x^3-4x^2+2x-\frac{1}{2}=0$

C. $2x^3-8x^2+4x+1=0$

D. $x^3+4x^2-2x+\frac{1}{2}=0$

7 What is the domain of the function $y=\cos^{-1}\left(\frac{5}{x}\right)$?

A. $-5\le x\le 5$, $x\ne 0$

B. $0<x\le\frac{5}{\pi}$

C. $x\le -5$, $x\ge 5$

D. $x<0$, $x\ge\frac{5}{\pi}$

8 How many real solutions does the equation $\left|(x+1)^2-4\right|=x+3$ have?

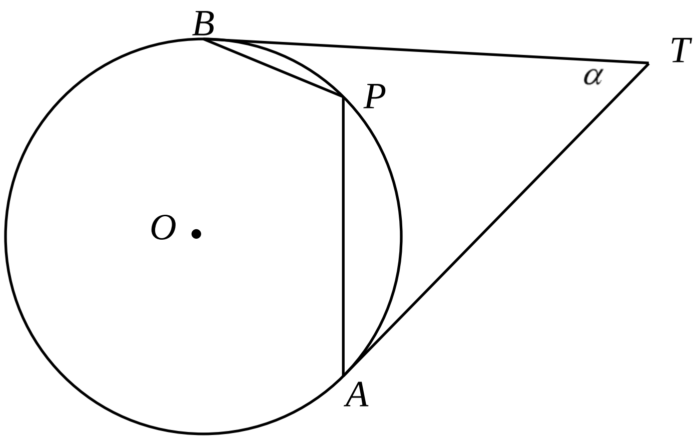
A. 3

B. 2

C. 1

D. 0

9 In the diagram below, *A*, *B* and *P* lie on a circle centred at *O*.
The tangents to the circle at *A* and *B* meet at the point *T*, and $\angle ATB=\alpha$.



What is the size of $\angle APB$ in terms of α ?

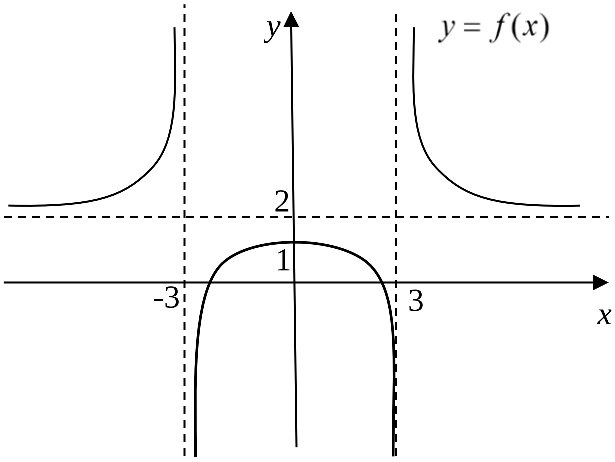
A. $90+\alpha$

B. $90+\dfrac{\alpha}{2}$

C. $180-\alpha$

D. $180-\dfrac{\alpha}{2}$

10 Below is the graph of $y = f(x)$.



Which of the following is a possible equation for the function $f(x)$?

A. $f(x) = \dfrac{2x^2 + 1}{x^2 - 9}$

B. $f(x) = \dfrac{x^2 + 2x - 9}{x^2 - 9}$

C. $f(x) = \dfrac{2x^2 - 9}{x^2 - 9}$

D. $f(x) = 1 + \dfrac{x^2 + 9}{9 - x^2}$

Section II

Total marks − 60
Attempt Questions 11−14
Allow about 1 hour 45 minutes for this section.

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11 to 14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) The point P divides the interval $A(1,-3)$ to $B(5,2)$ externally in the ratio 3:1. **1**

Find the x coordinate of P.

(b) Differentiate $\cos^{-1}(2x)$ with respect to x. **2**

(c) Find $\int \frac{3}{2+x^2}\,dx$. **2**

(d) Find $\int x\sqrt{x-2}\,dx$ using the substitution $x=u^2+2$. **3**

(e) Sketch the graph of the function $y=2\sin^{-1}\frac{x}{3}$. **2**

(f) Solve $\frac{2x+3}{x+3}\geq 1-x$. **3**

(g) Find $\int \sin^2 3x\,dx$. **2**

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) Prove the identity $\cot\theta - \cot x = \dfrac{\sin(x-\theta)}{\sin x \sin\theta}$. **2**

(b) Find a general solution to the equation $2\cos\left(3x - \dfrac{\pi}{4}\right) = 1$. **2**

(c) A particle undergoes simple harmonic motion about the origin O.

Its displacement x centimetres from O at time t seconds, is given by:

$$x = 3\sin\left(2t + \frac{\pi}{3}\right)$$

(i) What is the amplitude of the motion? **1**

(ii) Express the acceleration of the particle in terms of its displacement. **1**

(iii) What is the maximum speed of the particle? **1**

Question 12 continues on page 11

(d) Use mathematical induction to prove that for $n \geq 2$ **3**

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\ldots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}$$

(e) A particular rechargeable battery has a maximum charge of 500 volts.

While recharging, the charge of the battery, in volts, is $V(t)$, where t is the time in minutes after the start of recharging.

At any time t, the rate at which the charge of the battery is increasing is proportional to the difference between 110% of its maximum charge and its current charge.

At the beginning of the recharging process $V = 310$ and $\frac{dV}{dt} = 27$.

(i) Show that $\frac{dV}{dt} = 0.1125(550 - V)$. **1**

(ii) Show that $V = 550 - Pe^{-0.1125t}$ satisfies the equation in part (i). **1**

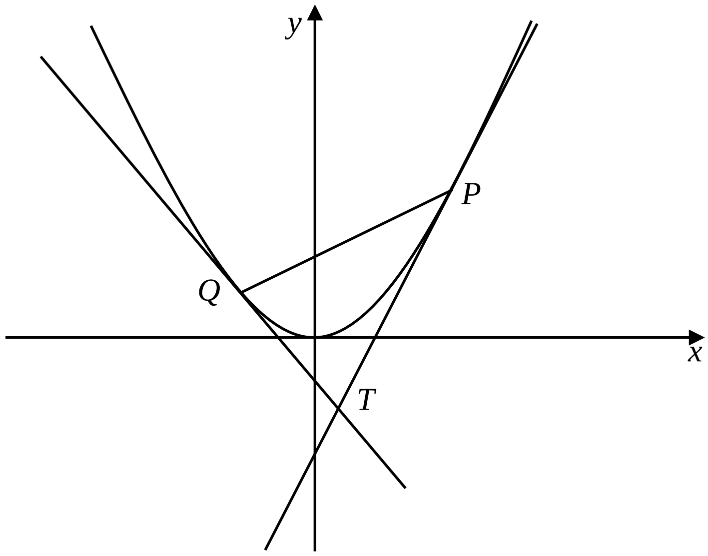
(iii) Find the time taken for the battery to reach its maximum charge. **3**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) The diagram shows a chord PQ that joins two distinct points, $P\left(6p, 3p^2\right)$ and $Q\left(6q, 3q^2\right)$ that lie on the parabola $x^2 = 12y$.

The tangents at P and Q meet at the point T.



The equation of the chord PQ is $y = \frac{1}{2}(p+q)x - apq$ (Do not prove this)

(i) The point $R(-5, 0)$ lies on the straight line that passes through P and Q. Show that $5(p+q) = -6pq$. **1**

(ii) Show that the equation of the locus of T, as P and Q vary is **2**

$$y = -\frac{5}{6}x.$$

(iii) Find any restrictions that apply to this locus. **2**

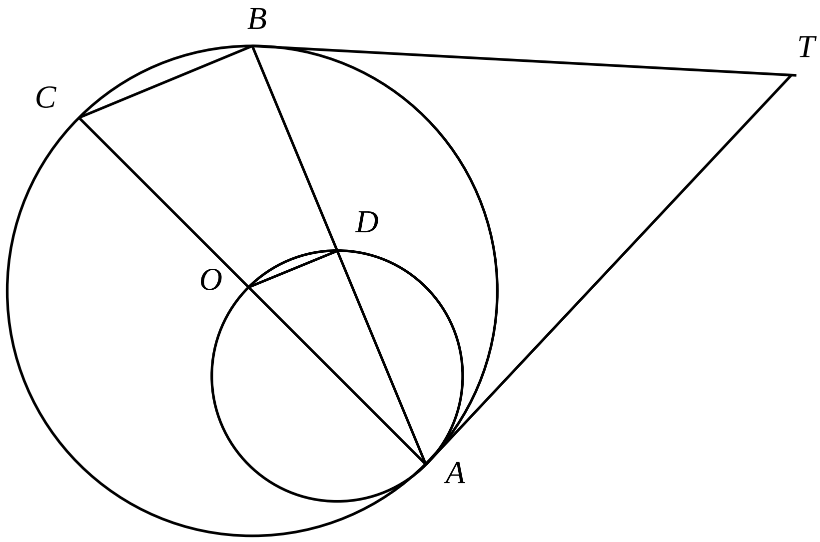
Question 13 continues on page 13

(b) In the diagram, the two circles touch at the point A.

AC is the diameter of the larger circle. The smaller circle also passes through the centre O of the larger circle.

B is a point on the larger circle and the tangents at A and B meet at T.

The chord AB intersects the smaller circle at D.



Copy the diagram into your answer booklet.

(i) Show that CB is parallel to OD. **2**

(ii) Show that $BD = DA$. **1**

(iii) Show that O, D and T are collinear. **3**

(c) The velocity v m/s of an object that is moving along the x-axis and undergoing simple harmonic motion is given by:

$$v^2 = 24 + 8x - 2x^2.$$

(i) Find the amplitude and centre of motion. **2**

(ii) Initially, the particle is at the centre of motion and moving towards the right. **2**

Find when the particle is 2 m to the right of the centre of the motion for the second time.

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) It is known that $2x-1$ is a factor of the polynomial $P(x)=2x^3-x^2-32x+16$. **2**

Sketch the graph of $y=2x^3-x^2-32x+16$. (Do not find any turning points)

(b) A particle is moving horizontally. Initially it is at rest 0.5 m to the right of the origin.

The acceleration of the particle as it moves in a straight line is given by $\ddot{x}=3x^2-x-16$ where x is its displacement at time t.

(i) In which direction will the particle first move? Briefly explain your answer. **1**

(ii) Show that the velocity of the particle can be expressed as $v^2=2x^3-x^2-32x+16$. **2**

You may refer to your graph from part (a) when answering the following questions.

(iii) Where is the particle when it is at its maximum speed? **2**

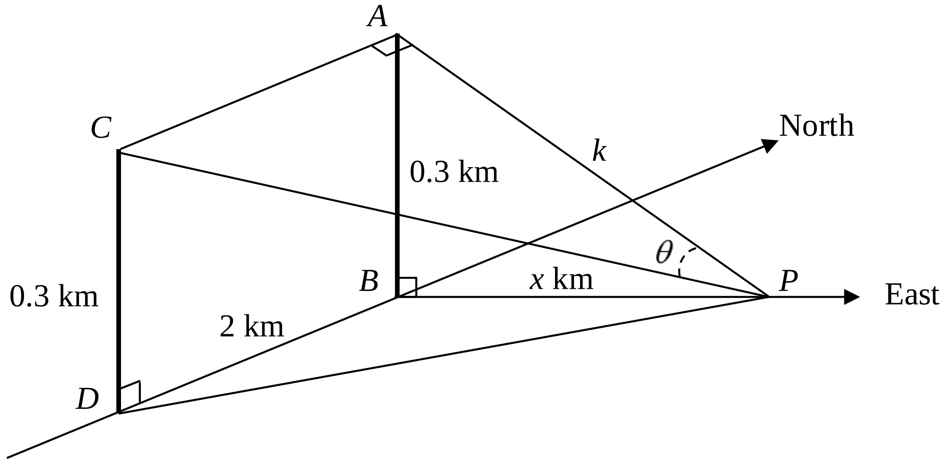
(iv) Describe the motion of the particle. **2**

Question 14 continues on page 15

(c) *AB* and *CD* are two towers of height 0.3 km. *B* is 2 km due North of *D*.

A vehicle at *P* is travelling due East away from *B* at a constant speed of 10 km/h.

Let the distance *BP* be x and the distance *AP* be k.



(i) Show that $\dfrac{dk}{dt} = \dfrac{10x}{\sqrt{x^2 + 0.09}}$. **2**

(ii) By first finding an expression for k in terms of θ, show that: **1**

$$\frac{dk}{d\theta} = -2\,\text{cosec}^2\,\theta$$

(iii) Find the rate at which θ is changing when the vehicle is 0.4 km from *B*. **3**

End of paper

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Mathematics Extension 1
Solutions

1.	A	B	C	●
2.	A	B	C	●
3.	●	B	C	D
4.	A	B	●	D
5.	●	B	C	D
6.	A	●	C	D
7.	A	B	●	D
8.	●	B	C	D
9.	A	●	C	D
10.	A	B	●	D

Section I

1 What is the remainder when $P(x)=2x^3-x^2+5x+k$ is divided by $x-2$?

A. $-30+k$

B. -30

C. 22

D. $22+k$

Solution

$$P(2)=2(2)^3-(2)^2+5(2)+k$$
$$=16-4+10+k$$
$$=22+k$$

Answer

D. $22+k$

2 Which expression is equal to $\sin x - \sqrt{3}\cos x$?

A. $2\sin\left(x + \frac{2\pi}{3}\right)$

B. $2\sin\left(x - \frac{2\pi}{3}\right)$

C. $2\sin\left(x + \frac{\pi}{3}\right)$

D. $2\sin\left(x - \frac{\pi}{3}\right)$

Solution

$$\begin{aligned}\sin x - \sqrt{3}\cos x &= R\sin(x+\alpha)\\ &= R\sin x\cos\alpha + R\cos x\sin\alpha\end{aligned}$$

$$\begin{aligned}R &= \sqrt{1^2 + \left(\sqrt{3}\right)^2}\\ &= \sqrt{4}\\ &= 2\end{aligned}$$

$\cos\alpha = \frac{1}{2}$ $\quad \therefore \alpha = \frac{\pi}{3}$ $\quad$ *or* $\quad \alpha = \frac{5\pi}{3}$

$\sin\alpha = -\frac{\sqrt{3}}{2}$ $\quad \therefore \alpha = \frac{4\pi}{3}$ $\quad$ *or* $\quad \alpha = \frac{5\pi}{3}$

$\alpha = \frac{5\pi}{3}$ which is equivalent to $\alpha = -\frac{\pi}{3}$

Answer

D. $2\sin\left(x - \frac{\pi}{3}\right)$

3 What is the value of $\lim_{x \to 1} \frac{3x\sin(x-1)}{2-2x}$?

A. $-\frac{3}{2}$

B. 0

C. $\frac{3}{2}$

D. Undefined

Solution

$$\lim_{x \to 1} \frac{3x\sin(x-1)}{(2-2x)} = \lim_{x \to 1} \frac{3x\sin(x-1)}{-2(x-1)}$$

$$= \lim_{x \to 1} \left(\frac{3x}{-2} \times \frac{\sin(x-1)}{(x-1)} \right)$$

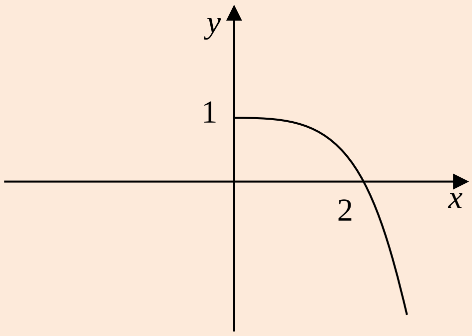
$$= \frac{3(1)}{-2}(\times 1)$$

$$= -\frac{3}{2}$$

Answer

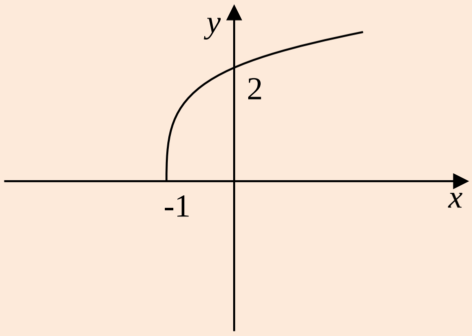
A. $-\frac{3}{2}$

4 *The diagram shows the graphs of* $y = f(x)$

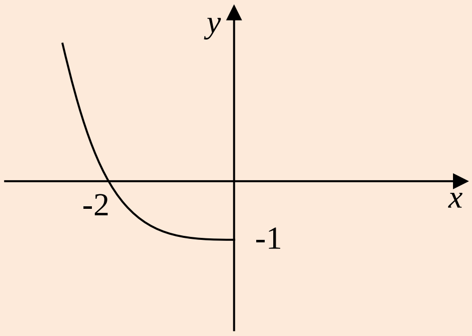


Which of the following is the graph of $y = f^{-1}(x)$

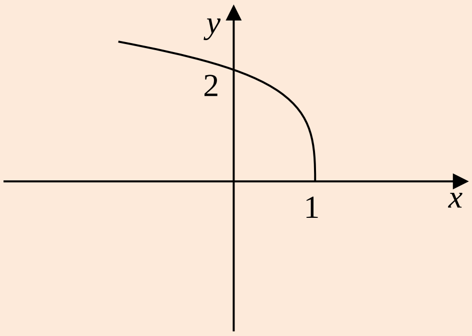
A.



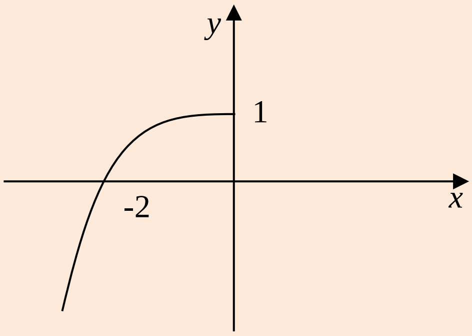
B.



C.

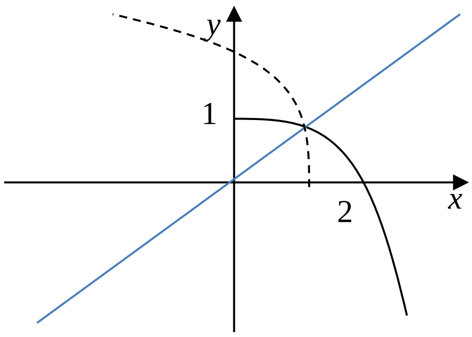


D.



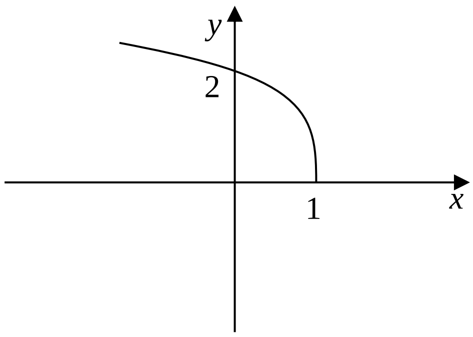
Solution

The inverse is the reflection in the line $y = x$



Answer

C.



5 The acute angle between the lines $2x-3y+1=0$ and $y=-3x+5$ is θ.
What is the value of $\tan\theta$?

A. $\frac{11}{3}$

B. $\frac{7}{9}$

C. $-\frac{7}{9}$

D. $-\frac{11}{3}$

Solution

$$
\begin{aligned}
2x-3y+1&=0 \\
-3y&=-2x-1 \\
y&=\frac{2}{3}x+\frac{1}{3} \\
\therefore m_1&=\frac{2}{3}
\end{aligned}
$$

$$
\begin{aligned}
y&=-3x+5 \\
\therefore m_2&=-3
\end{aligned}
$$

$$
\begin{aligned}
\tan\theta&=\left|\frac{m_1-m_2}{1+m_1m_2}\right| \\
&=\left|\frac{\frac{2}{3}-(-3)}{1+\frac{2}{3}(-3)}\right| \\
&=\left|\frac{\frac{2}{3}+3}{1-2}\right| \\
&=\left|\frac{\frac{11}{3}}{-1}\right| \\
&=\frac{11}{3}
\end{aligned}
$$

Answer

A. $\frac{11}{3}$

6 Given that $\alpha+\beta+\gamma=4$, $\alpha\beta+\beta\gamma+\alpha\gamma=2$ and $\alpha\beta\gamma=\frac{1}{2}$, which polynomial equation has roots α, β, and γ?

A. $2x^3-4x^2+2x-1=0$

B. $x^3-4x^2+2x-\frac{1}{2}=0$

C. $2x^3-8x^2+4x+1=0$

D. $x^3+4x^2-2x+\frac{1}{2}=0$

Solution

Let $a=1$

$$\alpha+\beta+\gamma=-\frac{b}{a}=4$$

$$\alpha\beta+\beta\gamma+\alpha\gamma=\frac{c}{a}=2$$

$$\alpha\beta\gamma=-\frac{d}{a}=\frac{1}{2}$$

Let $a=1$

$b=-4$

$c=2$

$d=-\frac{1}{2}$

Answer

B. $x^3-4x^2+2x-\frac{1}{2}=0$

7 What is the domain of the function $y = \cos^{-1}\left(\frac{5}{x}\right)$?

A. $-5 \le x \le 5\ ,\ x \ne 0$

B. $0 < x \le \frac{5}{\pi}$

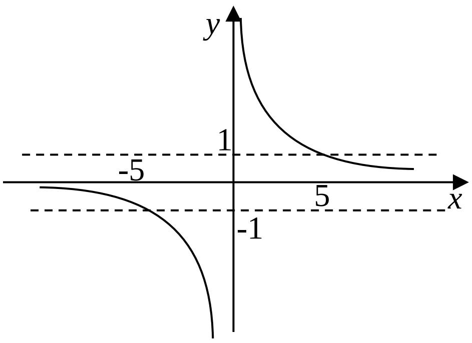
C. $x \le -5,\ x \ge 5$

D. $x < 0,\ x \ge \frac{5}{\pi}$

Solution

$-1 \le \frac{5}{x} \le 1$

Using a graph to solve the inequality

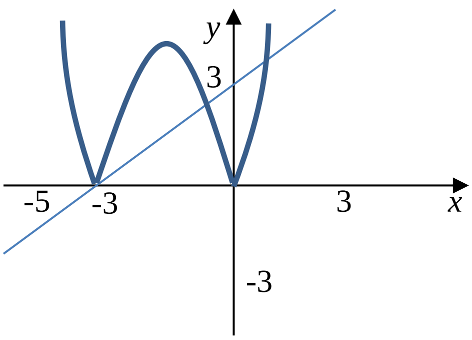


Answer

C. $x \le -5,\ x \ge 5$

8	How many real solutions does the equation $\left|(x+1)^2-4\right| = x+3$ have? A. 3 B. 2 C. 1 D. 0

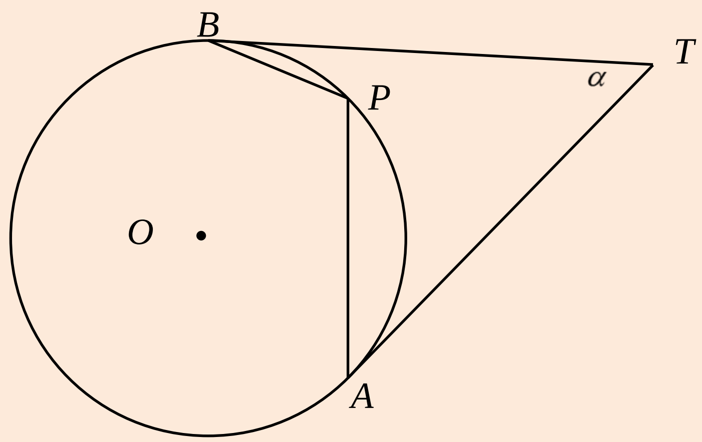
Solution



Answer is

A. 3

9 In the diagram below *A*, *B* and *P* lie on a circle centred at *O*.
The tangents to the circle at *A* and *B* meet at the point *T*, and $\angle ATB = \alpha$.



What is the size of $\angle APB$ in terms of α

A. $90 + \alpha$

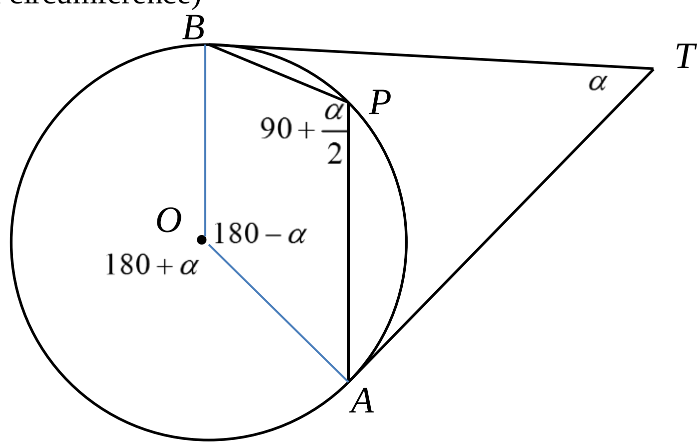
B. $90 + \dfrac{\alpha}{2}$

C. $180 - \alpha$

D. $180 - \dfrac{\alpha}{2}$

Solution

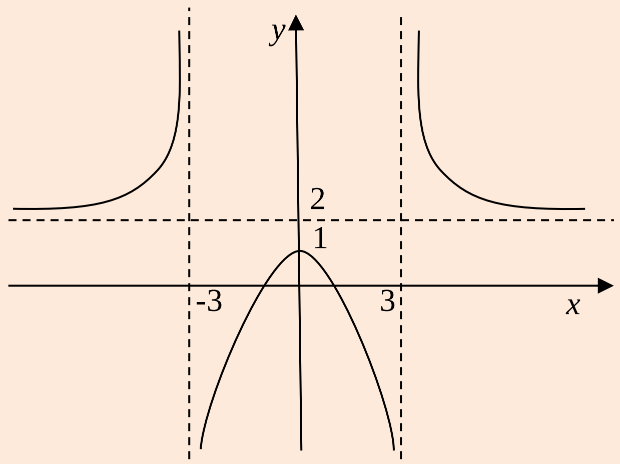
construct two radii *OB* and *OA*.
Therefor, $\angle OAT = \angle OBT = 90$
Therefor $\angle BOA = 180 - \alpha$ (Angle sum of quadrilateral *OATB*)
Therefor $\text{ref}\angle BOA = 180 + \alpha$ (Angle at a point)
$\angle BPA = \dfrac{180 + \alpha}{2} = 90 + \dfrac{\alpha}{2}$ (Angle at centre and circumference)



Answer

B. $90 + \dfrac{\alpha}{2}$

10 Below is the graph of $y = f(x)$.



Which is a possible equation for the function $f(x)$?

A. $f(x) = \dfrac{2x^2+1}{x^2-9}$

B. $f(x) = \dfrac{x^2+2x-9}{x^2-9}$

C. $f(x) = \dfrac{2x^2-9}{x^2-9}$

D. $f(x) = 1 + \dfrac{x^2+9}{9-x^2}$

Solution

This is best completed by eliminating incorrect options. The Provided graph has vertical asymptotes od $x = \pm 3$, horizontal asymptote of $y = 2$ and y intercept of $y = 2$.

Option A is not a solution since when $x = 0$ $y = \dfrac{2(0)^2+1}{(0)^2-9} = \dfrac{-1}{9}$

Option B is not a solution since $\lim\limits_{x\to\infty} f(x) = \dfrac{1+\frac{2}{x}-\frac{9}{x^2}}{1-\frac{9}{x^2}} = 1$ (Horizontal asymptote is $y = 1$)

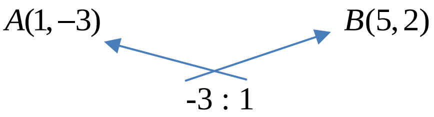
Option D is not a solution since $\lim\limits_{x\to\infty} f(x) = 1 + \dfrac{1+\frac{9}{x^2}}{\frac{9}{x^2}-1} = 1-1 = 0$ (Horizontal asymptote is $y = 0$)

Section II

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) The point P divides the interval $A(1,-3)$ to $B(5,2)$ externally in the ratio 3:1. **1**

Find the x coordinate of P.



$$x = \frac{(1)(1)+(5)(-3)}{-3+1}$$

$$= \frac{1-15}{-2}$$

$$= 7$$

(b) Differentiate $\cos^{-1}(2x)$ with respect to x. **2**

$$\frac{d}{dx}\left(\cos^{-1} 2x\right) = 2 \times \frac{-1}{\sqrt{1-(2x)^2}}$$

$$= \frac{-2}{\sqrt{1-4x^2}}$$

(c) Find $\int \frac{3}{2+x^2}\,dx$. **2**

$$\int \frac{3}{2+x^2}\,dx = 3\int \frac{1}{\left(\sqrt{2}\right)^2 + x^2}\,dx$$

$$= \frac{3}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C$$

(d)	Find $\int x\sqrt{x-2}\,dx$ using the substitution $x = u^2 + 2$.	**3**

$$\int x\sqrt{x-2}\,dx = \int (u^2+2)\sqrt{u^2+2-2}\ (2u)\,du$$
$$= \int (u^2+2)u\ (2u)\,du$$
$$= 2\int (u^4+2u^2)\ du$$
$$= 2\left(\frac{u^5}{5}+\frac{2u^3}{3}\right)+C$$
$$= 2\left(\frac{u^5}{5}+\frac{2u^3}{3}\right)+C$$
$$= 2\left(\frac{(u^2+2)^5}{5}+\frac{2(u^2+2)^3}{3}\right)+C$$
$$= 2\left(\frac{x^5}{5}+\frac{2x^3}{3}\right)+C$$

$x = u^2 + 2$

$\frac{dx}{du} = 2u$

$dx = 2u\,du$

(e)	Sketch the graph of the function $y = 2\sin^{-1}\frac{x}{3}$.	**2**

Domain is:

$-1 \le \frac{x}{3} \le 1$

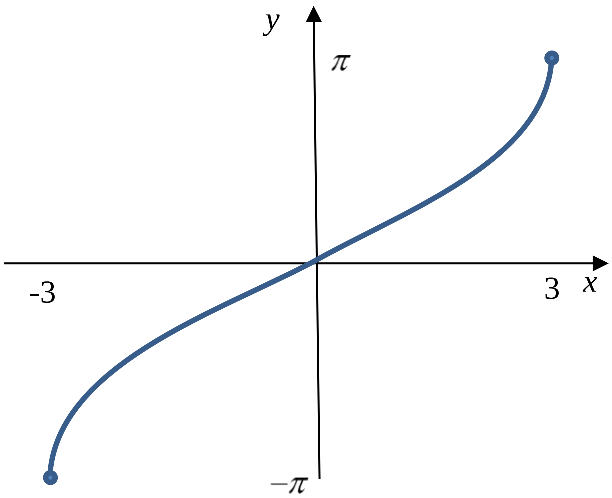
$-3 \le x \le 3$

Range is :

$-\frac{\pi}{2} \le \sin^{-1}\frac{x}{3} \le \frac{\pi}{2}$

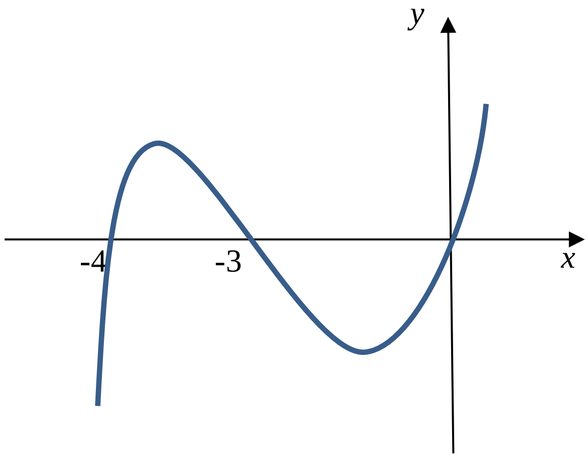
$-\pi \le 2\sin^{-1}\frac{x}{3} \le \pi$

$-\pi \le y \le \pi$



(f)	Solve $\dfrac{2x+3}{x+3} \geq 1-x.$	**3**

$$\frac{2x+3}{x+3} \geq 1-x$$
$$(x+3)(2x+3) \geq (1-x)(x+3)^2$$
$$(x+3)(2x+3)-(1-x)(x+3)^2 \geq 0$$
$$(x+3)\left[(2x+3)-(1-x)(x+3)\right] \geq 0$$
$$(x+3)\left[(2x+3)-\left(x-x^2-3x+3\right)\right] \geq 0$$
$$(x+3)\left[(2x+3)-\left(x-x^2-3x+3\right)\right] \geq 0$$
$$(x+3)\left[x^2+4x\right] \geq 0$$
$$x(x+3)(x+4) \geq 0$$



From the graph
$-4 \leq x < -3 \qquad x \geq 0$

(g)	Find $\int \sin^2 3x \; dx$.	**2**

$$\int \sin^2 3x \; dx = \frac{1}{2}\int (1-\cos 6x) \; dx$$
$$= \frac{1}{2}\left(x - \frac{\sin 6x}{6}\right) + C$$

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a)	Prove the identity $\cot\theta - \cot x = \dfrac{\sin(x-\theta)}{\sin x \sin\theta}$	**2**

$$\begin{aligned} RHS &= \frac{\sin(x-\theta)}{\sin x \sin\theta} \\ &= \frac{\sin x\cos\theta - \cos x \sin\theta}{\sin x \sin\theta} \\ &= \frac{\sin x \cos\theta}{\sin x \sin\theta} - \frac{\cos x \sin\theta}{\sin x \sin\theta} \\ &= \frac{\cos\theta}{\sin\theta} - \frac{\cos x}{\sin x} \\ &= \cot\theta - \cot x \\ &= LHS \end{aligned}$$

(b)	Find a general solution to the equation $2\cos\left(3x - \dfrac{\pi}{4}\right) = 1$	**2**

$$2\cos\left(3x - \frac{\pi}{4}\right) = 1$$

$$\cos\left(3x - \frac{\pi}{4}\right) = \frac{1}{2}$$

$$3x - \frac{\pi}{4} = 2n\pi \pm \cos^{-1}\frac{1}{2}$$ Where n is a an integer

$$3x - \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{3}$$

$$3x = 2n\pi + \frac{\pi}{4} \pm \frac{\pi}{3}$$

$$3x = \frac{24n\pi + 3\pi \pm 4\pi}{12}$$

$$x = \frac{24n\pi + 3\pi \pm 4\pi}{36}$$

$x = \dfrac{24n\pi + 7\pi}{36}$ *OR* $x = \dfrac{24n\pi - \pi}{36}$ Where n is a an integer

(c) A particle undergoes simple harmonic motion about the origin *O*.

Its displacement *x* centimetres from *O* at time *t* seconds, is given by:

$$x = 3\sin\left(2t + \frac{\pi}{3}\right)$$

(i) What is the amplitude of the motion? **1**

Amplitude is 3 cm.

(ii) Express the acceleration of the particle in terms of its displacement. **1**

Since $\ddot{x} = -n^2 x$ and $n = 2$
Then
$\ddot{x} = -2^2 x$
$= -4x$

(iii) What is the maximum speed of the particle? **1**

Method 1

$$x = 3\sin\left(2t + \frac{\pi}{3}\right)$$

$$v = \frac{dx}{dt} = 6\cos\left(2t + \frac{\pi}{3}\right)$$

Max value of $\cos\left(2t + \frac{\pi}{3}\right)$ is 1 so max speed is:

$v = 6(1)$
$= 6$ m/s

Method 2
Maximum speed occurs at the centre of motion, i.e. $x = 0$

$v^2 = n^2\left(a^2 - x^2\right)$
$= 2^2\left(3^2 - 0^2\right)$
$= 36$ m/s
$v = 6$ m/s

(d) Use mathematical induction to prove that for $n \geq 2$ $$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\ldots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}$$	**3**

Test for $n = 2$

$LHS = \left(1-\frac{1}{2^2}\right)$ $RHS = \frac{(2)+1}{2(2)}$

$= 1-\frac{1}{4}$ $= \frac{3}{4}$

$= \frac{3}{4}$ $= LHS$

True for $n = 2$

Assume true for $n = k$

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\ldots\left(1-\frac{1}{k^2}\right)=\frac{k+1}{2k}$$

Prove true for $n = k+1$

Required to prove

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\ldots\left(1-\frac{1}{(k+1)^2}\right)=\frac{(k+1)+1}{2(k+1)}$$

$$LHS = \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\ldots\left(1-\frac{1}{(k)^2}\right)\left(1-\frac{1}{(k+1)^2}\right)$$

$$= \left(\frac{k+1}{2k}\right)\left(1-\frac{1}{(k+1)^2}\right) \quad \text{By Assumption}$$

$$= \left(\frac{k+1}{2k}\right)\left(\frac{(k+1)^2-1}{(k+1)^2}\right)$$

$$= \left(\frac{k+1}{2k}\right)\left(\frac{[(k+1)-1][(k+1)+1]}{(k+1)^2}\right)$$

$$= \left(\frac{1}{2k}\right)\left(\frac{k(k+2)}{(k+1)}\right)$$

$$= \frac{(k+2)}{2(k+1)}$$

$$= RHS$$

By Mathematical Induction, $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\ldots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}$ is true for $n \geq 2$

(e) A particular rechargeable battery has a maximum charge of 500 volts.

While recharging, the charge of the battery, in volts, is $V(t)$, where t is the time in minutes after the start of recharging.

At any time t, the rate at which the charge of the battery is increasing is proportional to the difference between 110% of its maximum charge and its current charge.

At the beginning of the recharging process $V = 310$ and $\frac{dV}{dt} = 27$.

(i) Show that $\frac{dV}{dt} = 0.1125(550 - V)$. **1**

110% of its maximum charge is 550 Volts

$$\frac{dV}{dt} = (\text{Proportional to})(\text{The difference between 110\% of max charge and its current charge})$$
$$= k(550 - V).$$

$V = 310$ and $\frac{dV}{dt} = 27$.

$$27 = k(550 - 310)$$
$$27 = k(240)$$
$$k = \frac{27}{240}$$
$$= 0.1125$$
$$\therefore \frac{dV}{dt} = 0.1125(550 - V).$$

(ii) Show that $V = 550 - Pe^{-0.1125t}$ satisfies the equation in part i). **1**

$$V = 550 - Pe^{-0.1125t}$$
$$\frac{dV}{dt} = 0.1125Pe^{-0.1125t}$$

Since $Pe^{-0.1125t} = 550 - V$

Then $\frac{dV}{dt} = 0.1125(550 - V)$

(iii)	Find the time taken to reach its maximum charge to the nearest second.	**3**

When $t = 0$ $V = 310$ and $\frac{dV}{dt} = 27.$

$$310 = 550 - Pe^{-0.1125(0)}$$
$$310 = 550 - P$$
$$P = 550 - 310$$
$$P = 240$$

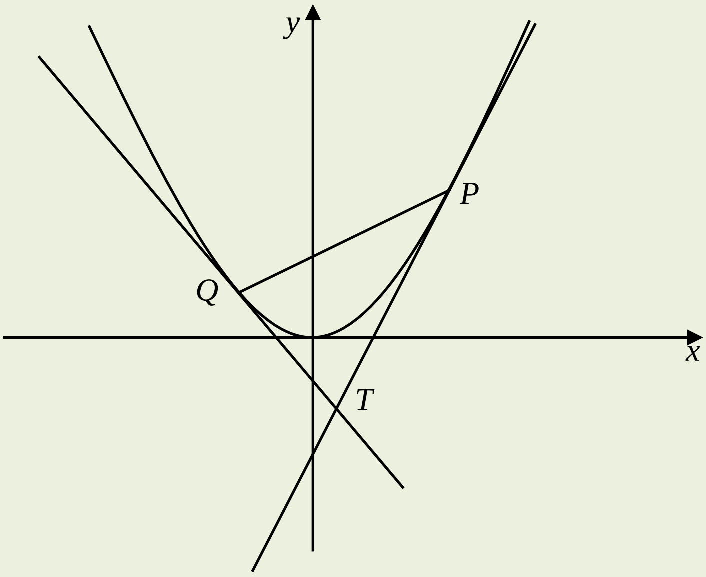
Find t when $V = 500$

$$500 = 550 - 240e^{-0.1125t}$$
$$-50 = -240e^{-0.1125t}$$
$$\frac{5}{24} = e^{-0.1125t}$$
$$-0.1125t = \ln\left(\frac{5}{24}\right)$$
$$t = -\frac{\ln\left(\frac{5}{24}\right)}{0.1125}$$

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) The diagram shows a chord PQ that joins two distinct points, $P\left(6p, 3p^2\right)$ and $Q\left(6q, 3q^2\right)$ that lie on the parabola $x^2 = 12y$.

The tangents at P and Q meet at the point T.



The equation of the chord PQ is $y = \frac{1}{2}\left(p+q\right)x - 3pq$ (Do not prove this)

(i) The point $R(-5,0)$ lies on the straight line that passes through P and Q. **1**

Show that $5(p+q) = -6pq$.

Solution

Sub $R(-5,0)$ into the equation of the chord

$$(0) = \frac{1}{2}\left(p+q\right)(5) - 3pq$$

$$-3pq = \frac{5}{2}\left(p+q\right)$$

$$-6pq = 5\left(p+q\right)$$

(ii)	Show that the equation of the locus of *T*, as *P* and *Q* vary is $$y = -\frac{5}{6}x$$	**2**

Solution

Equation of the tangents at P and Q are $y = px - 3p^2$ and $y = qx - 3q^2$

To find the coordinates of T sole the two equations simultaneously.

$$px - 3p^2 = qx - 3q^2$$
$$px - qx = 3p^2 - 3q^2$$
$$x(p-q) = 3(p-q)(p+q)$$
$$x = 3(p+q)$$

$$y = 3p(p+q) - 3p^2$$
$$y = 3p^2 + 3pq - 3p^2$$
$$y = 3pq$$

$$T\left(3(p+q), 3pq\right)$$

Eliminate the parameters by using part i)

$$5(p+q) = -6pq$$
$$(p+q) = \frac{-6pq}{5}$$

$$x = 3(p+q)$$
$$x = 3 \times \frac{-6pq}{5}$$
$$-\frac{5x}{18} = pq$$

$$y = 3pq$$
$$y = 3\left(-\frac{5x}{18}\right)$$
$$y = -\frac{5x}{6}$$

(iii)	Find any restrictions that apply to this locus.	**2**

The point T must lie outside of the parabola as it the intersection of two tangents. So we must find where the line $y=-\frac{5}{6}x$ intersects the parabola.

Solve $y=-\frac{5}{6}x$ and $x^2=12y.$ simultaneously

$$x^2=12\left(-\frac{5}{6}x\right)$$
$$x^2=-10x$$
$$x^2+10x=0$$
$$x(x+10)=0;$$
$$x=0 \qquad\qquad x=-10$$

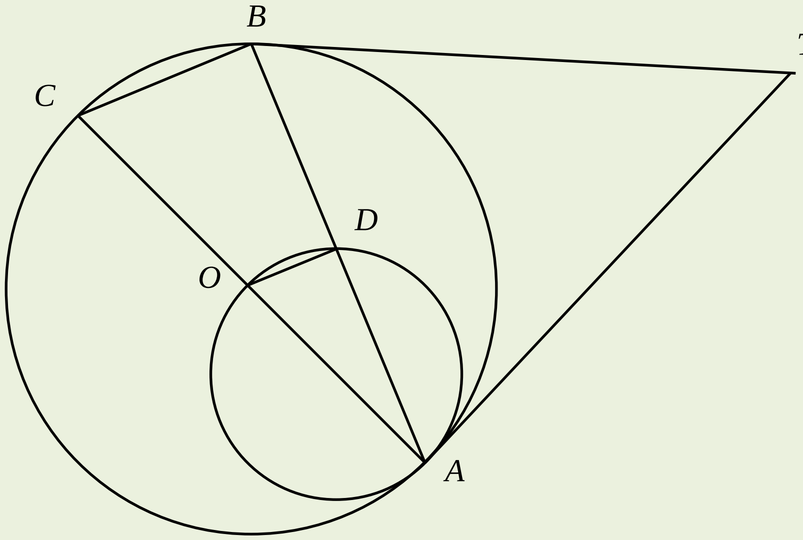
Thus the locus of T is for values of x :
$x>0 \quad x<-10$

(b) In the diagram, the two circles touch at the point A.

AC is the diameter of the larger circle. The smaller circle also passes through the centre O of the larger circle.

B is a point on the larger circle and the tangents at A and B meet at T.

The chord AB intersects the smaller circle at D.



Copy the diagram into your booklet.

(i)	Show that CB is parallel to OD.	**2**

Method 1

The centres of touching circles are collinear with point of contact.

Thus OA passes through the centre of the smaller circle.

$\angle ADO = 90°$ **(**Angle in a semi circle)

$\angle ABC = 90°$ **(**Angle in a semi circle)

$\therefore \angle ABC = \angle AOD$

$\therefore CB \| OD$ (Corresponding angles are equal)

Method 2

$\therefore \angle BAT = \angle AOD$ (Angle in alternate segment)

$\therefore \angle BAT = \angle ACB$ (Angle in alternate segment)

$\therefore \angle AOD = \angle ACB$

$\therefore CB \| OD$ (Corresponding angles are equal)

(ii)	Show that $BD = DA$.	**1**

Method 1

$\frac{CO}{OA} = \frac{BD}{DA}$ (Parallel lines preserve ratios, $CB \parallel OD$)

Since $CO = OA$ **(**radii)
Then $BD = DA$

Method 2 (If used Method 1 in part i)

$\angle ADO = 90°$ **(**part i)
$\therefore BD = DA$ (Perpendicular from centre bisects the chord AB)

(iii)	Show that *O*, *D* and *T* are collinear.	**3**

Aim: prove that *ODT* is straight, hence O, D and T are collinear.

Construct *DT*

In ΔDTB and ΔDTA
$BD = DA$ (Part ii)
DT is common.
$BT = TA$ (Tangents from an external point)

$\Delta DTB \equiv \Delta DTA$ (SSS)

$\angle BDT = \angle ADT$ (Matching angles in congruent triangles)
$\angle BDT + \angle ADT = 180$ (Straight angle)
$\angle BDT = \angle ADT = 90°$

$\angle ODA + \angle ADT = 90 + 90 = 180$

$\therefore \angle ODT$ is straight, hence O, D and T are collinear.

(c) The velocity v m/s of an object that is moving along the x-axis and undergoing simple harmonic motion is given by:

$$v^2 = 24 + 8x - 2x^2.$$

(i) Find the amplitude and centre of motion. **2**

Complete the square to write in standard form.

$$v^2 = n^2\left(a^2 - \left(x - x_0\right)^2\right)$$

$$v^2 = 24 + 8x - 2x^2$$
$$v^2 = -2\left(x^2 - 4x - 12\right)$$
$$v^2 = -2\left(x^2 - 4x + 4 - 16\right)$$
$$v^2 = -2\left(\left(x - 2\right)^2 - 4^2\right)$$
$$v^2 = 2\left(4^2 - \left(x - 2\right)^2\right)$$

Centre of motion is $x_0 = 2$ and amplitude is 4 m.

(ii)	Initially, the particle is at the centre of motion and moving towards the right. Find when the particle is 2 m to the right of the centre of the motion for the second time.	**2**

Since Initially, the particle is at the centre of motion and moving towards the right.

Then $x = x_0 + a\sin(nt)$

$n = \sqrt{2}$, $x_0 = 2$ and $a = 2$

$x = 2 + 4\sin(\sqrt{2}t)$

2 m to the right is $x = 4$

$4 = 2 + 4\text{s in}(\sqrt{2}t)$

$2 = 4\sin(\sqrt{2}t)$

$\frac{1}{2} = \sin(\sqrt{2}t)$

$\sqrt{2}t = \sin^{-1}\left(\frac{1}{2}\right)$

$\sqrt{2}t = \frac{\pi}{6}, \frac{5\pi}{6}, ...etc$

$t = \frac{\pi}{6\sqrt{2}}, \frac{5\pi}{6\sqrt{2}}, ...etc$

The second time that the particle will be 2 m to the right of the centre is $t = \frac{5\pi}{6\sqrt{2}}$ secsonds

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) It is known that $2x-1$ is a factor of the polynomial $P(x)=2x^3-x^2-32x+16.$ **2**

Sketch the graph of $y=2x^3-x^2-32x+16.$

$2x^3-x^2-32x+16=(2x-1)(ax^2+bx+c)$

By considering he expansion of the RHS and equating coefficients.

x^3 term $a=1$

Constant term $c=-16$

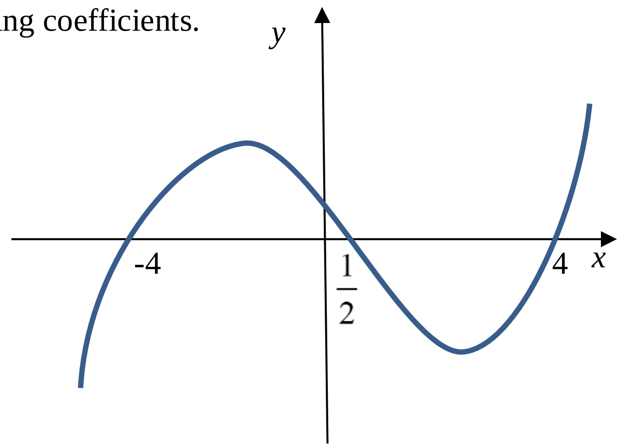
x^2 term

$-x^2=-ax^2+2bx^2$

$-1=-a+2b$

$-1=-1+2b$

$0=b$



$$2x^3-x^2-32x+16=(2x-1)(x^2-16)$$
$$=(2x-1)(x-4)(x+4)$$

(b) A particle is moving horizontally. Initially it is at rest 0.5 m to the right of the origin.

The acceleration of the particle as it moves in a straight line is given by $\ddot{x}=3x^2-x-16$ where *x* is its displacement at time *t*.

(i) In which direction will the particle first move? Briefly explain your answer. **1**

Inititaly the particle is at $x=0.5$ when $t=0$.

$$\ddot{x}=3\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)-16$$

$$\ddot{x}=\left(\frac{3}{4}\right)-\left(\frac{1}{2}\right)-16$$

$$\ddot{x}=-\frac{63}{4}$$

The particle will first move in the negative direction as the acceleration is negative when $v=0$

(ii)	Show that the velocity of the particle can be expressed as $v^2 = 2x^3 - x^2 - 32x + 16.$	**2**

$$\ddot{x} = 3x^2 - x - 16$$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 3x^2 - x - 16$$

$$\frac{1}{2}v^2 = \int\left(3x^2 - x - 16\right)dx$$

$$= x^3 - \frac{x^2}{2} - 16x + C$$

When $x = 0.5$ when $v = 0$.

$$\frac{1}{2}v^2 = x^3 - \frac{x^2}{2} - 16x + C$$

$$0 = \left(\frac{1}{2}\right)^3 - \frac{\left(\frac{1}{2}\right)^2}{2} - 16\left(\frac{1}{2}\right) + C$$

$$0 = \frac{1}{8} - \frac{1}{8} - 8 + C$$

$$0 = -8 + C$$

$$C = 8$$

$$\frac{1}{2}v^2 = x^3 - \frac{x^2}{2} - 16x + 8$$

$$v^2 = 2x^3 - x^2 - 32x + 16$$

You may refer to your graph from part (a) when answering the following questions.

(iii) Where is the particle when it is at its maximum speed? **2**

Let $\ddot{x} = 0$

$$0 = 3x^2 - x - 16$$

$$x = \frac{1 \pm \sqrt{1^2 - 4(3)(-16)}}{2(3)}$$

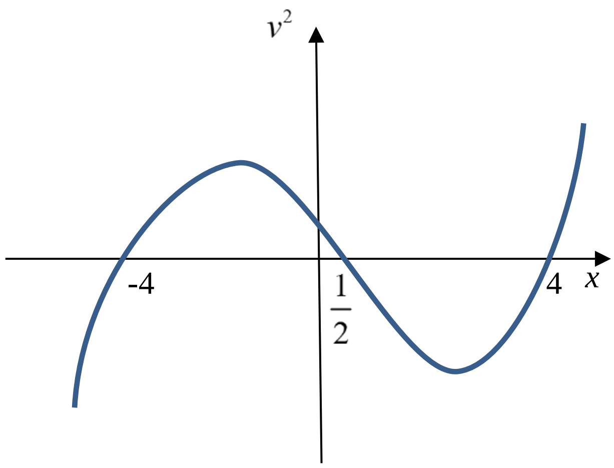
$$x = \frac{1 \pm \sqrt{1+192}}{6}$$

$\therefore$

$x = \frac{1+\sqrt{193}}{6}$ *OR* $x = \frac{1-\sqrt{193}}{6}$

$x \approx \frac{1+\sqrt{193}}{6}$ *OR* $x = \frac{1-\sqrt{193}}{6}$

$x \approx 2.48..$ $\quad$ $x \approx -2.149...$



From the graph in part (a)

Since v^2 is negative when $\frac{1}{2} < x < 4$, Which is not possible, then the max speed is at

$x = \frac{1-\sqrt{193}}{6}$ $\quad$ $x \approx -2.149...$ m

(iv)	Describe the motion of the particle.	**2**

Since the particle is initially at $x = 0.5$ and $v = 0$ and the particle then moves to the leftm it will again have zero velocity when $x = -4$.

The acceleration at $x = -4$ is

$$\begin{aligned}\ddot{x} &= 3(-4)^2 - (-4) - 16 \\ &= 48 + 4 - 16 \\ &= 36 \text{ ms}^{-2}\end{aligned}$$

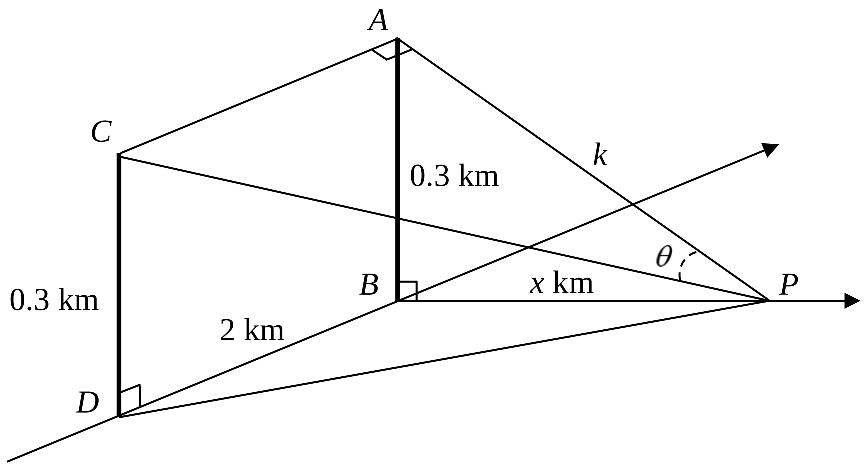
Thus the particle will oscillate between $x = 0.5$ and $x = -4$. Max speed will be at

$x = \dfrac{1-\sqrt{193}}{6}$ $\quad\quad$ $x \approx -2.149...$ m

(c) *AB* and *CD* are two towers of height 300 m. *DB* is 2 km and *B* is due North of *D*.

A vehicle at *P* is travelling due east away from *B* at a constant speed of 10 km/h.

Let the distance *BP* be *x* and the distance *AP* be *k*.

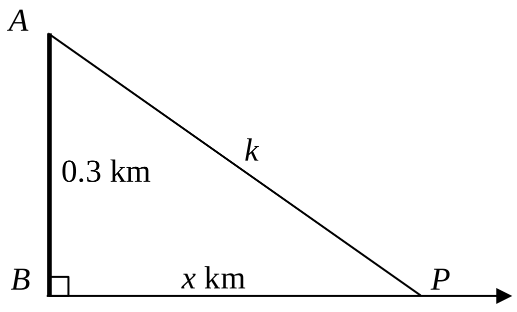


(i)	Show that $\dfrac{dk}{dt} = \dfrac{10x}{\sqrt{x^2 + 0.09}}$.	2

Since a vehicle at *P* is travelling due east away from *B* at a constant speed of 10 km/h.

Then let $\dfrac{dx}{dt} = 10$

Using Pythagoras in Triangle *ABP*



$$k^2 = 0.3^2 + x^2$$

$$k = \sqrt{0.3^2 + x^2} = \left(0.3^2 + x^2\right)^{\frac{1}{2}}$$

$$\frac{dk}{dx} = \frac{1}{2}(2x)\left(0.3^2 + x^2\right)^{-\frac{1}{2}}$$

$$\frac{dk}{dx} = \frac{x}{\sqrt{0.3^2 + x^2}}$$

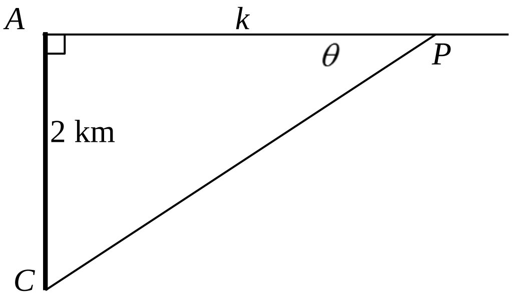
$$\frac{dk}{dt} = \frac{dk}{dx} \times \frac{dx}{dt}$$

$$= \frac{x}{\sqrt{0.3^2 + x^2}} \times 10$$

$$= \frac{10x}{\sqrt{0.09 + x^2}}$$

(ii)	By first finding an expression for k in terms of θ, show that: $$\frac{dk}{d\theta} = -2\operatorname{cosec}^2\theta$$	**1**

In triangle *CAP*



$$\tan\theta = \frac{2}{k}$$

$$k = \frac{2}{\tan\theta}$$

$$k = 2(\tan\theta)^{-1}$$

$$\frac{dk}{d\theta} = -2(\tan\theta)^{-2}(\sec^2\theta)$$

$$\frac{dk}{d\theta} = -2(\cot^2\theta)(\sec^2\theta)$$

$$\frac{dk}{d\theta} = -2\left(\frac{\cos^2\theta}{\sin^2\theta}\right)\left(\frac{1}{\cos^2\theta}\right)$$

$$\frac{dk}{d\theta} = -2\left(\frac{1}{\sin^2\theta}\right)$$

$$\frac{dk}{d\theta} = -2\operatorname{cosec}^2\theta$$

(iii)	Find the rate at which θ is changing when the vehicle is 0.4 km from *B*.	**3**

When $x = 0.4$

$k^2 = 0.3^2 + 0.4^2$

$k^2 = 0.25$

$k = 0.5$

When $k = 0.5$

$\tan\theta = \dfrac{2}{0.5}$

$\tan\theta = 4$

$\theta = \tan^{-1} 4$

$$\frac{d\theta}{dt} = \frac{d\theta}{dk} \times \frac{dk}{dt}$$

$$= \frac{1}{-2\operatorname{cosec}^2\left(\tan^{-1} 4\right)} \times \frac{10x}{\sqrt{0.5^2 + 0.09}}$$

$$= \frac{1}{-2\operatorname{cosec}^2\left(\tan^{-1} 4\right)} \times \frac{10(0.5)}{\sqrt{(0.5)^2 + 0.09}}$$

$$= -\frac{5}{2\sqrt{0.34}} \times \frac{1}{\left(\frac{\sqrt{17}}{4}\right)^2}$$

$$= -\frac{35}{17\sqrt{0.34}}$$

End of paper