NORTH SYDNEY GIRLS HIGH SCHOOL

2019 | **HSC Trial Examination**

Mathematics Extension 1

General Instructions

- Reading Time – 5 minutes
- Working Time – 2 hours
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided
- In Questions 11 – 14, show relevant mathematical reasoning and/or calculations

Total marks – 70

Section I – 10 marks (pages 3 – 7)

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 8 – 15)

- Attempt Questions 11 – 14
- Allow about 1 hours 45 minutes for this section

NAME:____________________________ **TEACHER:**__________________________

STUDENT NUMBER:

Question	1-10	11	12	13	14	Total
Mark	/10	/15	/15	/15	/15	/70

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 The point P divides the interval from $A(-1,-3)$ to $B(5,4)$ externally in the ratio 2 : 3. What is the x-coordinate of P?

A. -13

B. $\dfrac{13}{5}$

C. -3

D. $-\dfrac{7}{5}$

2 What is size of the acute angle between the lines $y = 2x - 5$ and $y = 1 - 3x$?

A. $\tan^{-1}(1)$

B. $\tan^{-1}\left(\dfrac{1}{7}\right)$

C. $\tan^{-1}\left(\dfrac{4}{7}\right)$

D. $\tan^{-1}\left(\dfrac{5}{7}\right)$

3 The polynomial $P(x) = 3x^3 - 2x^2 + 5x - 1$ has roots $\alpha,\ \beta,$ and γ.

What is the value of $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$?

A. 2

B. 5

C. –5

D. –2

4 The angle θ satisfies $\sin\theta = \dfrac{a}{b}$ where $\dfrac{\pi}{2} < \theta < \pi$, $a > 0$ and $b > 0$.

What is the value of $\sin 2\theta$?

A. $\dfrac{a\sqrt{b^2 - a^2}}{b^2}$

B. $\dfrac{-a\sqrt{b^2 - a^2}}{b^2}$

C. $\dfrac{2a\sqrt{b^2 - a^2}}{b^2}$

D. $\dfrac{-2a\sqrt{b^2 - a^2}}{b^2}$

5 The velocity v of a particle moving along the x-axis in terms of its displacement x is given by

$$v = 1 + x^2$$

Initially the particle is at the origin.

Which of the following is the equation for displacement x in terms of time t?

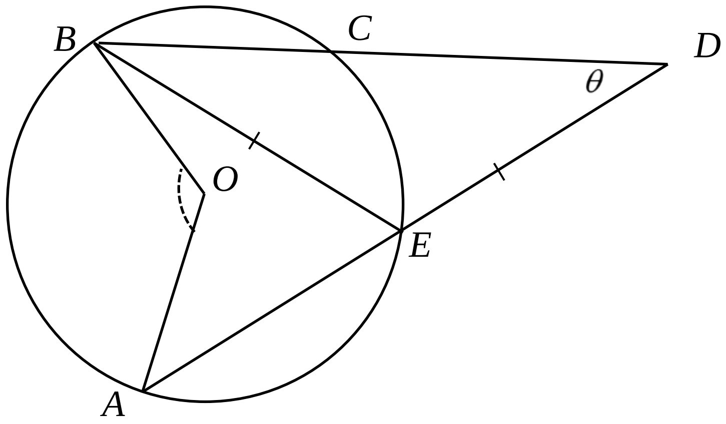
A. $x = \sqrt[3]{3t}$

B. $t = \dfrac{3x + x^3}{3}$

C. $x = \tan t$

D. $x = \tan^{-1} t$

6 In the diagram below, A, B, C and E lie on a circle centred at O.
The chords BC and AE, when produced, meet at D.
$BE = ED$ and $\angle ADB = \theta$.



What is the size of $\angle AOB$ in terms of θ?

A. θ

B. $90+\theta$

C. $180-\theta$

D. 4θ

7 Which expression is equal to $\int \sin^2 3x \, dx$?

A. $\frac{1}{2}\left(x-\frac{1}{6}\sin 6x\right)+C$

B. $\frac{1}{2}\left(x+\frac{1}{6}\sin 6x\right)+C$

C. $\frac{1}{2}\left(x-\frac{1}{3}\sin 3x\right)+C$

D. $\frac{1}{2}\left(x+\frac{1}{3}\sin 3x\right)+C$

8 A particle's acceleration $\ddot{x}$ ms^{-2} is defined by $\ddot{x} = -2x$, where x is the displacement in metres. How long, in seconds, does it take to travel from its maximum displacement to its minimum displacement?

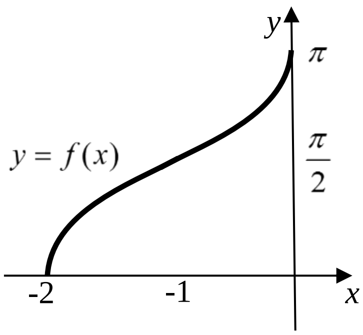
A. $\dfrac{\pi\sqrt{2}}{2}$

B. $\pi\sqrt{2}$

C. $2\pi\sqrt{2}$

D. $4\pi\sqrt{2}$

9 Below is the graph of $y = f(x)$.



Which of the following is a possible equation for the function $f(x)$?

A. $f(x) = \dfrac{\pi}{2} - \sin^{-1}(x+1)$

B. $f(x) = \sin^{-1}(x-1) + \dfrac{\pi}{2}$

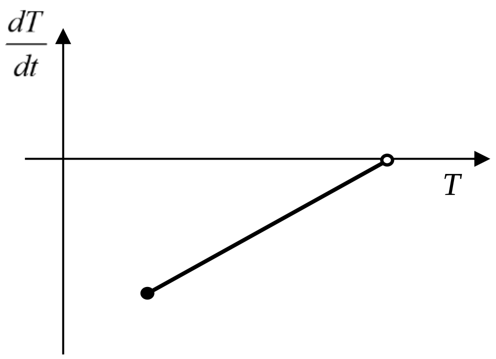
C. $f(x) = \cos^{-1}\left(-(x+1)\right)$

D. $f(x) = -\cos^{-1}(x-1)$

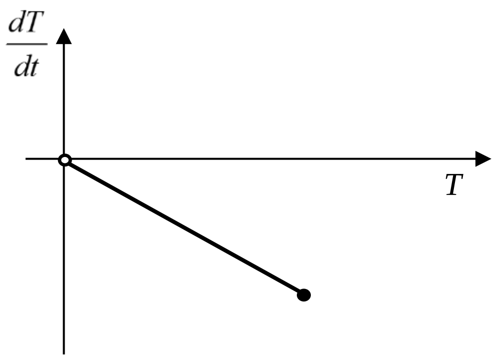
10 A hot metal rod is placed in a room with temperature $25°C$ and cools according to the equation $T = S + Ae^{-kt}$, where T is the temperature of the metal rod after t minutes and k, A and S are positive constants.

Which of the following is a possible graph of $\frac{dT}{dt}$ as a function of the temperature T of the rod?

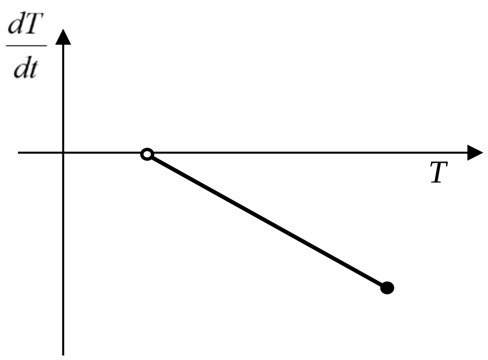
A.



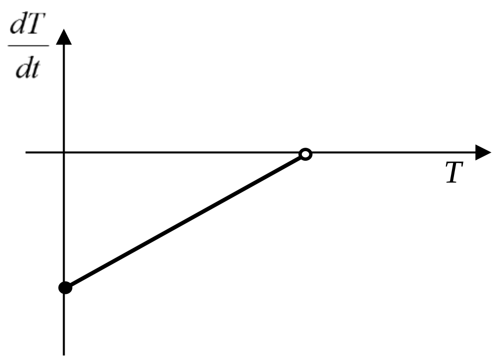
B.



C.



D.



End of Section I

Section II

Total marks − 60
Attempt Questions 11−14
Allow about 1 hour 45 minutes for this section.

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11 to 14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Show that $x = 2$ is a root of $P(x) = 2x^3 - 9x^2 + 7x + 6$. **1**

(ii) Hence factorise $P(x) = 2x^3 - 9x^2 + 7x + 6$. **2**

(b) Evaluate $\lim_{x \to 0} \dfrac{\sin 2x}{x \cos x}$. **1**

(c) Differentiate $\cos^{-1}(x-3)$. **1**

(d) Find $\displaystyle\int \frac{dx}{3+4x^2}$. **2**

(e) Solve $\dfrac{x-4}{x-3} \le x$. **3**

(f) Find $\displaystyle\int \cos 2x \sin^3 2x \; dx$. **2**

(g) Evaluate $\displaystyle\int_1^{\sqrt{5}} \frac{x}{\sqrt{1+3x^2}}\, dx$ using the substitution $u = 1 + 3x^2$. **3**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) A particle is undergoing simple harmonic motion about $x = 0$.

At time t seconds the displacement x is given by

$$x = \sqrt{3}\sin 3t - \cos 3t \,.$$

(i) Write $x = \sqrt{3}\sin 3t - \cos 3t$ in the form $x = A\sin\left(nt - \alpha\right)$, where $A > 0$. **2**

(ii) Find the period of the motion. **1**

(iii) When does the particle first reach maximum speed after time $t = 0$? **2**

(iv) How long will it take for the particle to return to its original position? **2**

(b) The population P of a city increases at a rate proportional to the number by which the city's population exceeds 2 million. This can be expressed by the differential equation

$$\frac{dP}{dt} = k\left(P - 2\times 10^{6}\right),$$

where t is the time in years and k is a constant.

(i) Show that $P = 2\times 10^{6} + Ae^{kt}$, where A is a constant, is a solution to this differential equation. **1**

(ii) The population of the city was 2.5 million at the start of 2000, and 3 million at the start of 2019. **2**

Find the values of A and k.

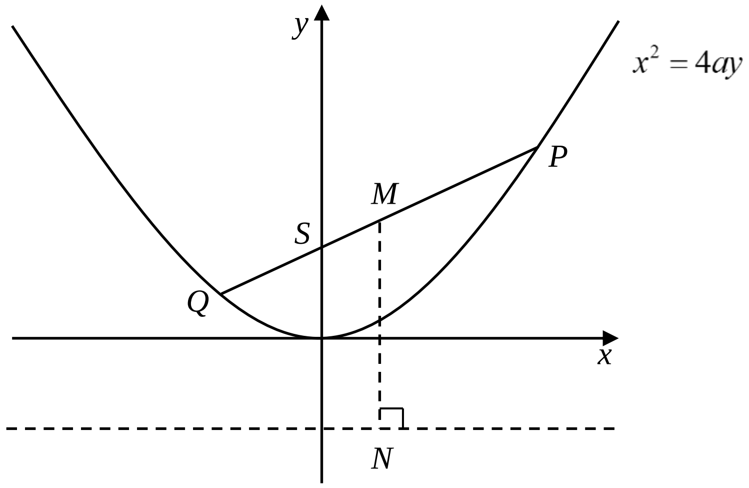
Question 12 continues on page 11

Question 12 (continued)

(c) The diagram shows a chord PQ that joins two distinct points, $P\left(2ap, ap^2\right)$ and $Q\left(2aq, aq^2\right)$ that lie on the parabola $x^2 = 4ay$.

The chord PQ passes through the focus S and M is the midpoint of PQ.

N lies on the directrix such that MN is perpendicular to the directrix.



The equation of the chord PQ is $y = \frac{1}{2}(p+q)x - apq$. (Do **NOT** prove this)

(i) Show that $pq = -1$. **1**

(ii) T is the midpoint of MN. Show that the coordinates of T are **2**

$$\left(a(p+q),\ \frac{a}{4}\left(p^2+q^2-2\right)\right).$$

(iii) Find the Cartesian equation of the locus of T. **2**

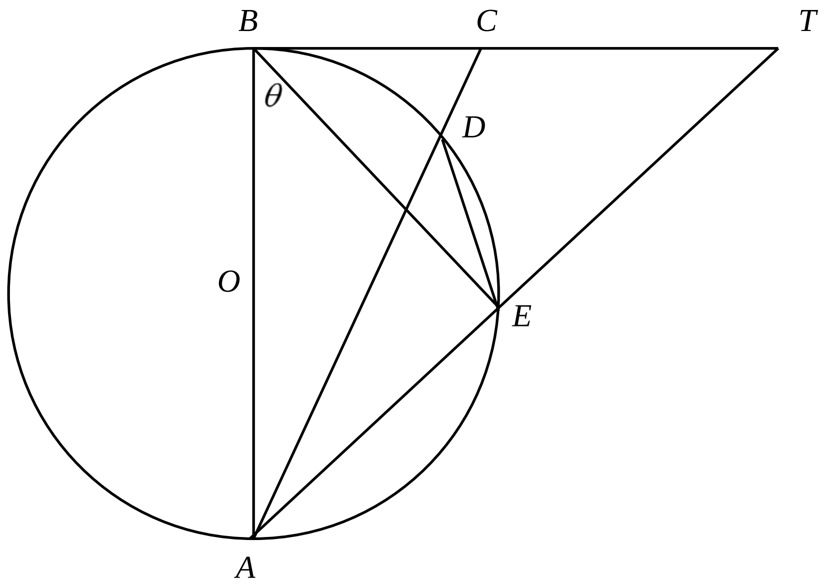
End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) Prove by mathematical induction that, for $n \geq 1$, **3**

$$\frac{2\times 1}{2\times 3}+\frac{2^2\times 2}{3\times 4}+\frac{2^3\times 3}{4\times 5}+\ldots+\frac{2^n\times n}{(n+1)(n+2)}=\frac{2^{n+1}}{n+2}-1.$$

(b) In the diagram AB is a diameter of the circle.
The tangent at B meets the secants AE and AD at T and C respectively.
Let $\angle ABE = \theta$.



(i) Explain why $\angle EAB = 90^\circ - \theta$. **1**

(ii) Prove that quadrilateral $CDET$ is cyclic. **3**

Question 13 continues on page 13

Question 13 (continued)

(c) Consider the function $f\left(x\right)=\log_e\left(\dfrac{2x}{x+2}\right)$.

(i) Show that the domain of $f\left(x\right)$ is $x>0$, $x<-2$. **1**

(ii) Find the *x*-intercept. **1**

(iii) Show that $f\left(x\right)$ is increasing for all values of *x* in its domain. **2**

(iv) By considering $\lim\limits_{x\to\infty}\left(\log_e\left(\dfrac{2x}{x+2}\right)\right)$, write down the equation of the horizontal asymptote. **1**

(v) Sketch the graph of $y=f(x)$, showing all intercepts and asymptotes. **2**

(vi) Write down the domain of $y=f^{-1}(x)$. **1**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) By differentiating, show that **2**

$$\tan^{-1} x + \tan^{-1} \frac{1-x}{1+x} = \frac{\pi}{4} \text{ for } x > -1 .$$

(b) A particle is moving along the x-axis. The square of its velocity is given by

$$v^2 = -2x^4 + 14x^2 - 24 ,$$

where v m/s is the velocity and x is the displacement in metres from the origin.

Initially the particle is stationary at $x = 2$.

(i) Show that **1**

$$a = -4x\left(x^2 - \frac{7}{2} \right),$$

where a is the acceleration of the particle.

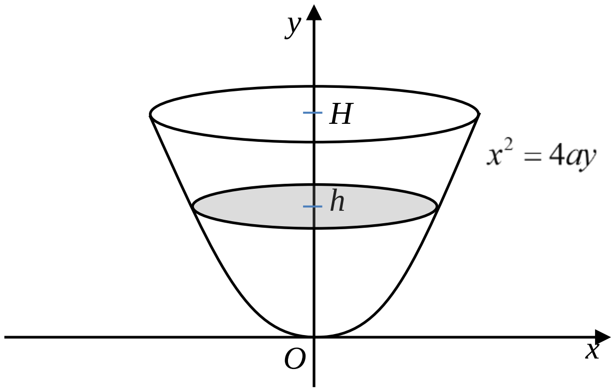
(ii) Explain why the object starts moving in the negative direction. **1**

(iii) Where does the particle next come to rest? **2**

(iv) Describe the motion of the particle. Include information about the greatest speed and where that speed occurs. **3**

Question 14 continues on page 15

(c) The diagram below shows a paraboloid that is created by rotating the parabola $x^2 = 4ay$ around the y-axis.



(i) Find the volume of water in the paraboloid when the water is at height h. **1**

The paraboloid is filled with water up to a height H. Water is then released from a small hole at point O. It takes T seconds for the water to fully drain out.

It is known that the volume V of water in the paraboloid decreases at a rate given by

$$\frac{dV}{dt} = -k\sqrt{h^3},$$

where k is a positive constant and h is the height of water.

(ii) Show that **2**

$$\frac{dh}{dt} = \frac{-k\sqrt{h}}{4a\pi}.$$

(iii) Hence or otherwise show that **3**

$$h = H\left(\frac{t}{T} - 1\right)^2 \quad \text{for} \quad 0 \le t \le T.$$

End of paper

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Mathematics Extension 1 Solutions

1.	A
2.	A
3.	B
4.	D
5.	C
6.	D
7.	A
8.	A
9.	C
10.	C

Section II

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a)	(i)	Show that $x = 2$ is a root of $P(x) = 2x^3 - 9x^2 + 7x + 6$	**1**

$$
\begin{aligned}
P(2) &= 2(2)^3 - 9(2)^2 + 7(2) + 6 \\
&= 16 - 36 + 14 + 6 \\
&= 0
\end{aligned}
$$

$x = 2$ is a root

	(ii)	Hence factorise $P(x) = 2x^3 - 9x^2 + 7x + 6$	**2**

$$P(x) = 2x^3 - 9x^2 + 7x + 6 = \left((x-2)\left(2x^2 + bx - 3\right)\right)$$

Consider the x^2 term on both sides

$$9x^2 = -4x^2 + bx^2$$

$$b = -5$$

$$
\begin{aligned}
2x^3 - 9x^2 + 7x + 6 &= (x-2)\left(2x^2 - 5x - 3\right) \\
&= (x-2)(2x+1)(x-3)
\end{aligned}
$$

Marker's Comment

a) (i) Showing that $x = 2$ is a zero of $P(x)$ just requires showing that $P(2) = 0$.

You were not required to show that $(x-2)$ is a factor so you should not be stating 'by the factor theorem' or 'by the remainder theorem' in your working.

(b)	Evaluate $\lim_{x \to 0} \dfrac{\sin 2x}{x \cos x}$.	**1**

$$
\begin{aligned}
\lim_{x \to 0} \frac{\sin 2x}{x \cos x} &= 2\left(\lim_{x \to 0} \frac{\sin 2x}{2x \cos x}\right) \\
&= 2\left(\lim_{x \to 0}\left(\frac{1}{\cos x}\right)\left(\frac{\sin 2x}{2x}\right)\right) \\
&= 2\left(\left(\frac{1}{\cos 0}\right)(1)\right) \\
&= 2
\end{aligned}
$$

Marker's Comment

b) Students should show their working here.

(c)	Differentiate $\cos^{-1}(x-3)$.	**1**

$$\frac{d}{dx}\left(\cos^{-1}(x-3)\right)=\frac{-1}{\sqrt{1-(x-3)^2}}$$

(d)	Find $\int \frac{dx}{3+4x^2}$.	**2**

$$\begin{aligned}\int \frac{dx}{3+4x^2} &= \frac{1}{4}\int \frac{dx}{\frac{3}{4}+x^2}\\ &= \frac{1}{4}\int \frac{1}{\left(\frac{\sqrt{3}}{2}\right)^2+x^2}dx\\ &= \frac{1}{4}\times\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x}{\sqrt{3}}\right)+C\\ &= \frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{2x}{\sqrt{3}}\right)+C\end{aligned}$$

(e)	Solve $\dfrac{x-4}{x-3} \le x$.	**3**

$$\frac{x-4}{x-3} \le x. \qquad x \ne 3$$

$$(x-3)(x-4) \le x(x-3)^2$$

$$0 \le x(x-3)^2 - (x-3)(x-4)$$

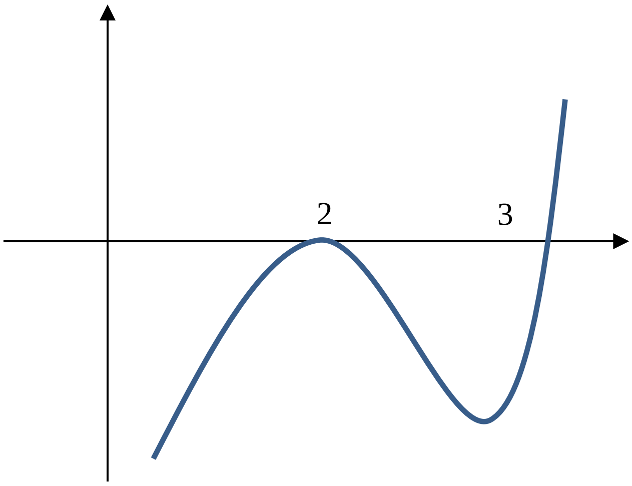
$$0 \le (x-3)\left[x(x-3)-(x-4)\right]$$

$$0 \le (x-3)\left[x^2-3x-x+4\right]$$

$$0 \le (x-3)\left[x^2-4x+4\right]$$

$$0 \le (x-3)\left[x-2\right]^2$$

From the graph of $y \le (x-3)[x-2]^2$



The solution is $x \ge 3$ and $x = 2$ but $x \ne 3$, So the solution is $x > 3$ and $x = 2$

Marker's Comment

e) The vast majority of students used the method of multiplying by the square of the denominator. Those who then expanded and then factorised the cubic wasted precious time and often made errors. Students are encouraged to extract a common factor as per the solution.

Many students forgot to exclude $x = 3$ from their solution (write this down at the beginning of your working so you don't forget) and many also forgot to include $x = 2$.

(f) Find $\int \cos(2x)\sin^3(2x)\ dx$. **2**

$$\int \cos(2x)\sin^3(2x)\ dx = \int \cos 2x[\sin 2x]^3\ dx$$

$$= \frac{1}{2}\int (2\cos 2x)[\sin 2x]^3\ dx$$

$$= \frac{1}{2} \times \frac{[\sin 2x]^4}{4} + C$$

$$= \frac{[\sin 2x]^4}{8} + C$$

Marker's Comment

f) Students who recognised that this question required the application of the reverse chain rule provided a much more efficient solution.

Others used various substitutions such as $u = \cos 2x$. This method was also successful. The final answer looked different to the solutions but effectively can be shown to only differ by a constant.

(g) Evaluate $\int_1^{\sqrt{5}} \frac{x}{\sqrt{1+3x^2}}\,dx$ using the substitution $u = 1+3x^2$. **3**

$$\int_1^{\sqrt{5}} \frac{x}{\sqrt{1+3x^2}}\,dx$$

$$u = 1+3x^2$$

$$\frac{du}{dx} = 6x$$

$$dx = \frac{du}{6x}$$

When $x = \sqrt{5}$ $u = 1+3(5) = 16$

When $x = 1$ $u = 1+3(1) = 4$

$$\begin{aligned}
\int_1^{\sqrt{5}} \frac{x}{\sqrt{1+3x^2}}\,dx &= \int_4^{16} \frac{1}{6\sqrt{u}}\,dx \\
&= \frac{1}{6}\int_4^{16} u^{-\frac{1}{2}}\,dx \\
&= \frac{1}{6}\left[2u^{\frac{1}{2}}\right]_4^{16} \\
&= \frac{1}{3}\left[\sqrt{u}\right]_4^{16} \\
&= \frac{1}{3}(4-2) \\
&= \frac{2}{3}
\end{aligned}$$

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) A particle moves in undergoing simple harmonic motion about $x = 0$.

At time t seconds the displacement x is given by

$$x = \sqrt{3}\sin 3t - \cos 3t .$$

(i) Write $x = \sqrt{3}\sin 3t - \cos 3t$ in the form $x = A\sin(nt - \alpha)$, where $A > 0$. **2**

$$\sqrt{3}\sin 3t - \cos 3t = A\sin(3t - \alpha)$$
$$= A\sin 3t\cos\alpha - A\cos 3t\sin\alpha$$

By equating co-efficients

$1 = A\sin\alpha \quad I \qquad \sqrt{3} = A\cos\alpha \quad II$

$A^2 = 1^1 + \left(\sqrt{3}\right)^2$

$A = 2$

$1 = 2\sin\alpha \quad I \qquad \sqrt{3} = 2\cos\alpha \quad II$

α is in the 1st quadrant.

$\frac{1}{2} = \sin\alpha$

$\alpha = \frac{\pi}{6}$

$$x = 2\sin\left(3t - \frac{\pi}{6}\right)$$

A majority of students did not justify their choice of quadrant for α. Otherwise well done.

(ii) Find the period of the motion. **1**

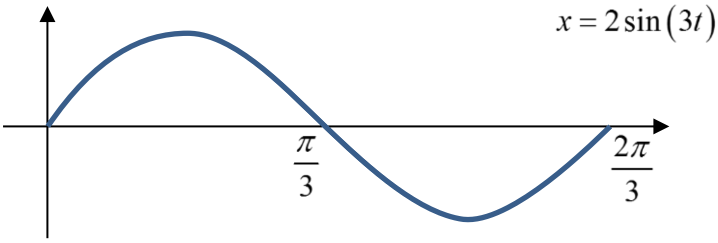
Period is $T = \frac{2\pi}{3}s$

The period is a time, measured in seconds. Please write units. (Not penalised)

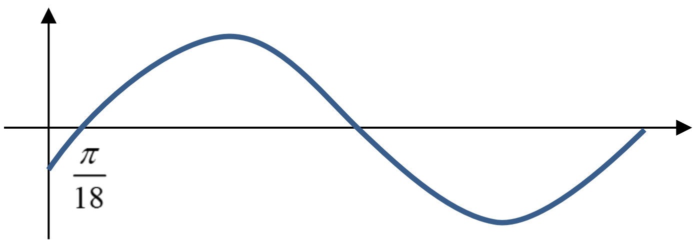
(iii)	When does the particle first reach maximum speed after time $t=0$?	**2**

Method 1 – Via Graphing

Sketch the graph



$x=2\sin\left(3\left(t-\frac{\pi}{18}\right)\right)$ By writing the equation in this form we are easily able to see $x=2\sin(3t)$ has been shifted $\frac{\pi}{18}$ to the right.



Maximum speed occurs at the centre of motion, which is when $x=0$

Maximum speed first occurs at

$t=\frac{\pi}{18}$

Method 2 - Algebraic

Maximum speed is at centre of motion, which is $x=0$.

$$0=2\sin\left(3\left(t-\frac{\pi}{18}\right)\right)$$

$$0=\sin\left(3\left(t-\frac{\pi}{18}\right)\right)$$

$$0=3\left(t-\frac{\pi}{18}\right)$$

$$0=t-\frac{\pi}{18}$$

$$t=\frac{\pi}{18}$$

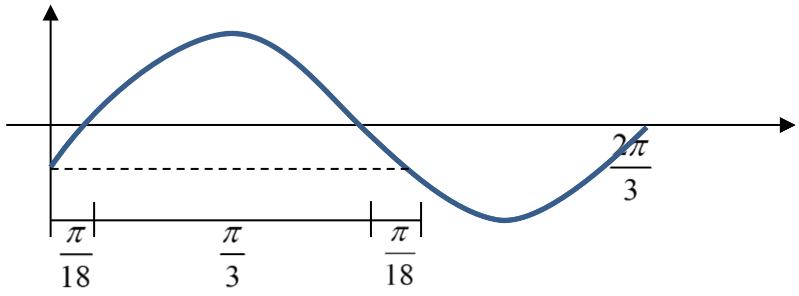
Generally well done. However the maximum speed corresponds with a velocity of 6 or −6. Students who solved $\dot{x}=6$ should also have considered $\dot{x}=-6$.

(iv)	How long will it take for the particle to return to its original position?	**2**

Method 1 – Via Graphing

From the graph the next time that particle returns to its original position is

$$t = \frac{\pi}{3} + 2\times\frac{\pi}{18} = \frac{4\pi}{9}$$



Method 2 - Algebraic

When $t = 0$

$$x = 2\sin\left(3(0) - \frac{\pi}{6}\right)$$

$$x = 2\sin\left(-\frac{\pi}{6}\right)$$

$$x = 2\left(-\frac{1}{2}\right)$$

$$x = -1$$

Let $x = -1$

$$-1 = 2\sin\left(3t - \frac{\pi}{6}\right)$$

$$\frac{-1}{2} = \sin\left(3t - \frac{\pi}{6}\right)$$

$$3t - \frac{\pi}{6} = -\frac{\pi}{6}$$

$t = 0$ First time at $x = -1$

$$3t - \frac{\pi}{6} = \frac{7\pi}{6}$$

$$3t = \frac{7\pi}{6} + \frac{\pi}{6}$$

$$t = \frac{8\pi}{18}$$

$t == \frac{4\pi}{9}$ Second time at $x = -1$

(iv)	A large number of students believed that the starting position was $x = 0$. As this led to solving exactly the same equation as in part (iii), no marks could be awarded for this.

(b) The population P of a city increases at a rate proportional to the number by which the city's population exceeds 2 million. This can be expressed by the differential equation

$$\frac{dP}{dt} = k\left(P - 2\times 10^6\right),$$

where t is the time in years and k is a constant.

(i) Show that $P = 2\times 10^6 + Ae^{kt}$, where A is a constant, is a solution to this differential equation. **1**

$$\begin{aligned} LHS &= \frac{dP}{dt} \\ &= k\left(Ae^{kt}\right) \\ &= k\left(P - 2\times 10^6\right) = RHS \end{aligned}$$

This is a standard proof and an easy mark. Please learn it.

(ii) The population of the city was 2.5 million at the start of 2000, and 3 million at the start of 2019. **2**

Find the values of A and k.

When $t = 0$, $P = 2.5\times 10^6$

$$\begin{aligned} 2\cdot 5\times 10^6 &= 2\times 10^6 + Ae^0 \\ 0\cdot 5\times 10^6 &= A(1) \\ A &= 5\times 10^5 \end{aligned}$$

When $t = 19$, $P = 3\times 10^6$

$$\begin{aligned} 3\times 10^6 &= 2\times 10^6 + 5\times 10^5 e^{19k} \\ 10^6 &= 5\times 10^5 e^{19k} \\ 2 &= e^{19k} \\ \ln 2 &= 19k \\ k &= \frac{\ln 2}{19} = 0.03648 \end{aligned}$$

The question did not state which year corresponds to $t = 0$. It was intended that this be 2000, and most students assumed this. Those who substituted $t = 2000$ and $t = 2019$ made the question difficult and were prone to errors.
A disturbingly large number of students (about 20%) thought 2019 was 9 years after 2000.
Otherwise well done.

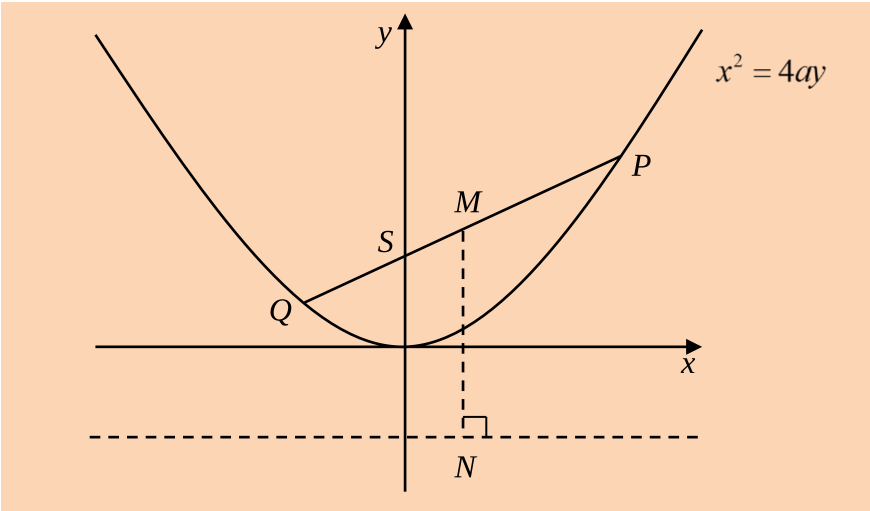
Question 12 (continued)

(c) The diagram shows a chord PQ that joins two distinct points, $P\left(2ap, ap^2\right)$ and $Q\left(2aq, aq^2\right)$ that lie on the parabola $x^2 = 4ay$.

The chord PQ passes through the focus S and M is the midpoint of PQ.

N lies on the directrix such that MN is perpendicular to the directrix

.



The equation of the chord PQ is $y = \frac{1}{2}(p+q)x - apq$. (Do **NOT** prove this)

(i) Show that $pq = -1$. **1**

Sub in the coordinates of the focus $S(0, a)$ into the equation of the chord $y = \frac{1}{2}(p+q)x - apq$

$$(a) = \frac{1}{2}(p+q)(0) - apq$$
$$a = -apq$$
$$-1 = pq$$

Well done. However a few students assumed that $a = 1$. This is not stated or implied anywhere.

(ii) T is the midpoint of MN. Show that the coordinates of T are **2**

$$\left(a(p+q),\ \frac{a}{4}\left(p^2+q^2-2\right)\right).$$

The coordinates of M are:

$$M\left(\frac{2ap+2aq}{2},\frac{ap^2+aq^2}{2}\right)$$

$$M\left(a(p+q),\frac{a}{2}\left(p^2+q^2\right)\right)$$

The coordinates of N are:

$$N\left(a(p+q),-a\right)$$

The coordinates of T are:

$$T\left(a(p+q),\frac{\frac{a}{2}\left(p^2+q^2\right)-a}{2}\right)$$

$$T\left(a(p+q),\frac{\frac{a}{2}\left[\left(p^2+q^2\right)-2\right]}{2}\right)$$

$$T\left(a(p+q),\frac{a}{4}\left[\left(p^2+q^2\right)-2\right]\right)$$

This is a SHOW question. Don't take shortcuts in the algebra.

(iii)	Find the Cartesian equation of the locus of *T*.	**2**

$$\begin{aligned}(p+q)^2 &= p^2+q^2+2pq \\ &= p^2+q^2+2(-1) \qquad \text{From part } i) \\ &= p^2+q^2-2\end{aligned}$$

Using the point $T\left(a(p+q), \frac{a}{4}\left[\left(p^2+q^2\right)-2\right]\right)$

$x = a(p+q)$ $\quad A$

$y = \frac{a}{4}\left[\left(p^2+q^2\right)-2\right]$

$y = \frac{a}{4}\left[(p+q)^2\right]$ $\quad B$

From A

$\frac{x}{a} = (p+q)$

The sub into B

$y = \frac{a}{4}\left[\left(\frac{x}{a}\right)^2\right]$

$y = \frac{x^2}{4a}$

Well done. However some students tried to eliminate a. a is not a parameter.

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) Prove by mathematical induction that, for $n \geq 1$, **3**

$$\frac{2\times1}{2\times3}+\frac{2^2\times2}{3\times4}+\frac{2^3\times3}{4\times5}+\ldots+\frac{2^n\times n}{(n+1)(n+2)}=\frac{2^{n+1}}{n+2}-1.$$

Step 1 – prove true for $n=1$

$$LHS=\frac{2^1\times1}{(1+1)(1+2)}=\frac{2}{(2)(3)}=\frac{1}{3}$$

$$RHS=\frac{2^{1+1}}{1+2}-1=\frac{2^2}{3}-1=\frac{4}{3}-1=\frac{1}{3}=LHS$$

Step 1 – Assume true for $n=k$

$$\frac{2\times1}{2\times3}+\frac{2^2\times2}{3\times4}+\frac{2^3\times3}{4\times5}+\ldots+\frac{2^k\times k}{(k+1)(k+2)}=\frac{2^{k+1}}{k+2}-1$$

Step 1 – If true for $n=k$ then prove true for $n=k+1$

Required to prove "

$$\frac{2\times1}{2\times3}+\frac{2^2\times2}{3\times4}+\frac{2^3\times3}{4\times5}+\ldots+\frac{2^k\times k}{(k+1)(k+2)}+\frac{2^{k+1}\times(k+1)}{(k+2)(k+3)}=\frac{2^{k+2}}{k+3}-1$$

$$\begin{aligned}
LHS&=\frac{2\times1}{2\times3}+\frac{2^2\times2}{3\times4}+\frac{2^3\times3}{4\times5}+\ldots+\frac{2^k\times k}{(k+1)(k+2)}+\frac{2^{k+1}\times(k+1)}{(k+2)(k+3)}\\
&=\frac{2^{k+1}}{k+2}-1+\frac{2^{k+1}\times(k+1)}{(k+2)(k+3)} \qquad \text{From Assumption}\\
&=\frac{2^{k+1}}{k+2}+\frac{2^{k+1}\times(k+1)}{(k+2)(k+3)}-1\\
&=\frac{2^{k+1}(k+3)+2^{k+1}\times(k+1)}{(k+2)(k+3)}-1\\
&=\frac{2^{k+1}\left[(k+3)+(k+1)\right]}{(k+2)(k+3)}-1\\
&=\frac{2^{k+1}(2k+4)}{(k+2)(k+3)}-1\\
&=\frac{2^{k+2}(k+2)}{(k+2)(k+3)}-1\\
&=\frac{2^{k+2}}{(k+3)}-1=RHS
\end{aligned}$$

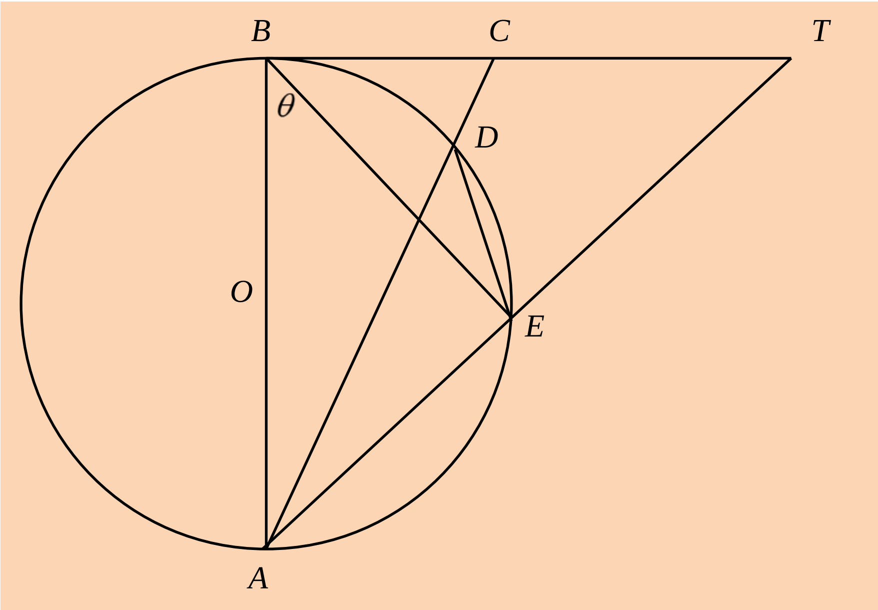
By mathematical induction the result is true for all values $n\geq1$

Marker's Comments

Very well done. You are advised to show substitution and simplification when verifying the initial case – this is an easy mark and it would be a shame to lose it because of insufficient working. It helps to pay attention to the RHS of the result to be proved. This will help you notice that the "-1" does not need to be combined with the other two terms.

This is a proof, so if you leave steps out, you risk not being awarded full credit. Some students write just the first and last terms of the series in the inductive step– this is not ideal. You should be showing the term for $n = k$ before the term for $n = k+1$, so that it is easy to see that you are substituting the assumption for the terms up to and including the $n = k$ term.

(b) In the diagram AB is the diameter of the circle.
The tangent at B meets the secants AE and AD at T and C respectively.
Let $\angle ABE = \theta$.



(i) Explain why $\angle EAB = 90° - \theta$. **1**

Since AB is a diameter then $\angle AEB$ is a right angle since it will be an angle in a semi circle. Therefor $\angle EAB = 90° - \theta$ as the angle sum of triangle AEB is $180°$.

$$\angle EAB + \angle AEB + \angle EBA = 180°$$
$$\angle EAB + 90° + \theta = 180°$$
$$\angle EAB = 90° - \theta$$

(ii) Prove that quadrilateral $CDET$ is cyclic. **3**

ΔABT

$\angle ABT = 90°$	Angle between tangent BT and radii BO
$\angle TAB = \angle EAB = 90° - \theta$	Part i)
$\angle TAB + \angle ATB + \angle TBA = 180°$	Angle sum of ΔABT
$(90° - \theta) + \angle ATB + 90° = 180°$	
$\angle ATB = \theta$	

$\angle ADE = \angle ABE = \theta$ Angle subtended in the same segment by chord AD

$\angle ADE = \angle CTE = \theta$

Therefor $CDET$ is cyclic Exterior angle is equal to the interior opposite angle

Marker's Comments

Most students knew how to approach the proof. However, the communication needs to be more precise. Students should learn the suggested abbreviations for the circle geometry result. More practice with writing out proofs is recommended.

- An "explain why" should be answered using full statements of the results instead of the abbreviations. An "angle sum of triangle result" quoted without reference to the angle names and/or the triangle name was not awarded full credit.
- Most results are about relationships between two angles – so both angle names should be referenced eg $\angle ADE = \angle ABE = \theta$ (angles subtended in the same segment by AE)

  NOT $\angle ADE = \theta$ (angles subtended in the same segment by AE)
- Name the triangle you are working in and the angles.
  $\angle BTA = 180 - 90 - 90 + \theta$ (angle sum of Δ) is not very meaningful
  $\angle BTA = 180 - \angle BAT - \angle ATB$ (angle sum ΔBAT)
- The difference between the forward theorem "Opposite angles of a cyclic quadrilateral are supplementary" and the converse "As the opposite angles are supplementary, then the quadrilateral is cyclic" is subtle but important. This was a common error and was penalised.

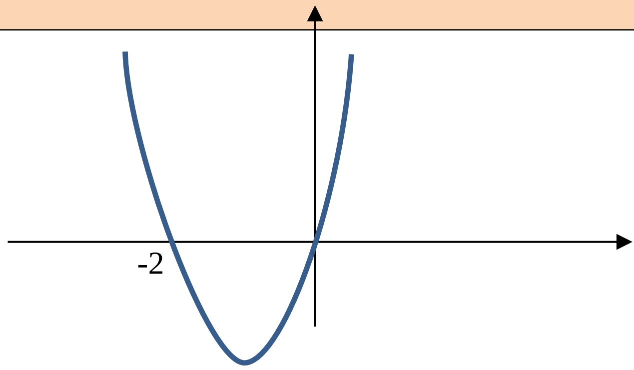
Question 13 (continued)

(c) Consider the function $f(x) = \log_e\left(\frac{2x}{x+2}\right)$.

(i) Show that the domain of $f(x)$ is $x > 0$, $x < -2$.

1

$$\frac{2x}{x+2} > 0$$
$$2x(x+2) > 0(x+2)^2$$
$$2x(x+2) > 0$$
$$\therefore x > 0, \quad x < -2$$



(ii) Find the x-intercept. **1**

$$0 = \log_e\left(\frac{2x}{x+2}\right)$$
$$\frac{2x}{x+2} = 1$$
$$2x = x+2$$
$$x = 2$$

(iii) Show that $f(x)$ is increasing for all values of x in its domain. **2**

$$f(x) = \log_e\left(\frac{2x}{x+2}\right)$$
$$f(x) = \log_e(2x) - \log_e(x+2)$$

$$f'(x) = \frac{2}{2x} - \frac{1}{x+2}$$
$$f'(x) = \frac{(x+2)-x}{x(x+2)}$$
$$f'(x) = \frac{2}{x(x+2)}$$

When $x > 0 \quad x+2 > 0$ $\qquad \therefore \frac{2}{x(x+2)} > 0$

When $x < -2 \quad x+2 < 0$ $\qquad \therefore \frac{2}{x(x+2)} > 0$

$f'(x) > 0$ and thus $f(x)$ is increasing for all values of x in its domain.

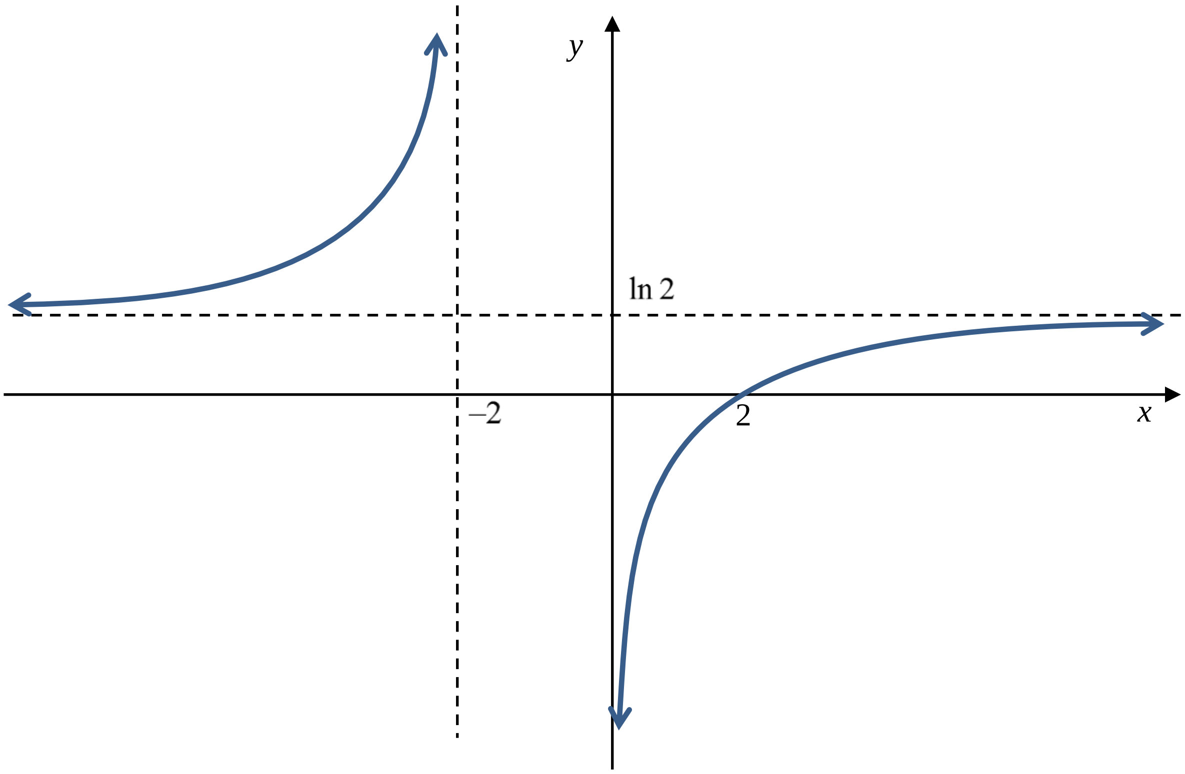
(iv)	Find $\lim_{x\to\infty}\left(\log_e\left(\frac{2x}{x+2}\right)\right)$.	**1**

Consider:

$$\lim_{x\to\infty}\left(\frac{2x}{x+2}\right)=\lim_{x\to\infty}\left(\frac{2(x+2)-4}{x+2}\right)$$
$$=\lim_{x\to\infty}\left(2-\frac{4}{x+2}\right)$$
$$=2$$

$$\lim_{x\to\infty}\left(\log_e\left(\frac{2x}{x+2}\right)\right)=\lim_{x\to\infty}\left(\log_e(2)\right)$$
$$=\log_e(2)$$

(v)	Sketch the graph of $y=f(x)$, showing all intercepts and asymptotes.	**2**



(vi)	Write down the domain of $y=f^{-1}(x)$.	**1**

All real x, $x \neq \ln 2$

Marker's Comments

Most parts were done well. In Part (iii) you are asked to "show that" the function is increasing for all values in the domain. Most students knew they needed to find $f'(x)$ but some made errors in finding the derivative. It is easier to use log laws to simplify the expression before differentiating. You then had to make the argument that $f'(x) > 0$ for all values in the domain found in part (i). Some students did not bother to justify this, instead just stating that it was positive, possibly because a similar expression results when finding the domain in part (i).. To be awarded credit, you either had to justify again or make reference to the result proved in part (i). Always make a conclusion – the last line of a "show that" response should be the result you have been asked to "show".

In part (iv) most students found the limit but setting out was at times questionable. You need to answer the question – some stated that the horizontal asymptote was at $y = \ln 2$ instead of offering a value for the limit asked. Students are advised to take note of the setting out modelled in the solutions.

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) (i) By differentiating, show that **2**

$$\tan^{-1} x + \tan^{-1}\frac{1-x}{1+x} = \frac{\pi}{4} \text{ for } x > -1.$$

$$\frac{d}{dx}\left(\tan^{-1} x + \tan^{-1}\frac{1-x}{1+x}\right) = \frac{1}{1+x^2} + \frac{1}{1+\left(\frac{1-x}{1+x}\right)^2} \times \left(\frac{-(1+x)-(1-x)}{(1+x)^2}\right)$$

$$= \frac{1}{1+x^2} + \frac{1}{1+\left(\frac{1-x}{1+x}\right)^2} \times \left(\frac{-2}{(1+x)^2}\right)$$

$$= \frac{1}{1+x^2} + \frac{-2}{(1+x)^2+(1-x)^2}$$

$$= \frac{1}{1+x^2} + \frac{-2}{(1+2x+x^2)+(1-2x+x^2)}$$

$$= \frac{1}{1+x^2} + \frac{-2}{(2+2x^2)}$$

$$= \frac{1}{1+x^2} - \frac{1}{1+x^2}$$

$$= 0$$

$\therefore \tan^{-1} x + \tan^{-1}\frac{1-x}{1+x}$ is a constant function in its continuous sections.

Let $x = 1$

$$\tan^{-1}1 + \tan^{-1}\frac{1-1}{1+1} = \tan^{-1}1 + \tan^{-1}0$$

$$= \frac{\pi}{4} + 0$$

$$= \frac{\pi}{4}$$

Since the gradient is always zero then $\tan^{-1} x + \tan^{-1}\frac{1-x}{1+x} = \frac{\pi}{4}$ for $x > -1$.

Marker's Comments

- This question required students to use differentiation to prove this result, some students did not and thus were only able to receive a maximum of 1 mark if they used compound angle results and made no further errors.
- Many students were not able to differentiate the LHS and thus were unable to obtain any marks.
- The most common error was to believe that if $\frac{d}{dx}(LHS) = \frac{d}{dx}(RHS)$ then it follows that $LHS = RHS$.

(b) A particle is moving along the x-axis. The square of its velocity is given by

$$v^2 = -2x^4 + 14x^2 - 24,$$

where v is velocity and x is the displacement from the origin.

Initially the particle is stationary at $x = 2$.

(i) Show that $a = -4x\left(x^2 - \frac{7}{2}\right)$ where a is the acceleration of the particle. **1**

$$v^2 = -2x^4 + 14x^2 - 24$$

$$\frac{1}{2}v^2 = -x^4 + 7x^2 - 12$$

$$a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = -4x^3 + 14x$$

$$a = -4x\left(x^2 - \frac{7}{2}\right)$$

(ii) Explain why the object starts moving in the negative direction. **1**

$$a = -4(2)\left((2)^2 - \frac{7}{2}\right)$$

$$= -8\left(\frac{1}{2}\right)$$

$$= -4$$

When $x = 2$ The particle is stationary and the acceleration is negative, therefor the particle moves in the negative direction

Many students were able to show that acceleration was negative at $x = 2$ but it failed to mention that the particle was stationary in their explanation of why it moves in the negative direction.

(iii)	Where does the particle next come to rest?	**2**

Let $v = 0$

$$v^2 = -2x^4 + 14x^2 - 24$$
$$0 = -2\left(x^4 - 7x^2 + 12\right)$$
$$0 = -2\left(x^2 - 4\right)\left(x^2 - 3\right)$$

$$x = 2, \quad -2, \quad \sqrt{3}, \quad -\sqrt{3}$$

Since the object starts at $x = 2$ and moves in the negative direction the next time it will come to rest will be at $x = \sqrt{3}$

> Most students were able to find algebraically, where $v = 0$, but unfortunately some were unable to interpret where it would **<u>next</u>** come to rest.

(iv)	Describe the motion of the particle. Include information about the greatest speed and where that speed occurs.	**3**

At $x = \sqrt{3}$ the acceleration is

$$a = -4\sqrt{3}\left(\left(\sqrt{3}\right)^2 - \frac{7}{2}\right)$$
$$= -4\sqrt{3}\left(3 - 3\frac{1}{2}\right)$$
$$= \frac{4\sqrt{3}}{2}$$

Which is positive. So at $x = \sqrt{3}$ the object will move in the positive direction.

This means that the particle will oscillate between $x = \sqrt{3}$ and $x = 2$.

The maximum velocity will occur when acceleration is zero

$$0 = -4x\left(x^2 - \frac{7}{2}\right)$$
$$x = 0, \quad \sqrt{\frac{7}{2}}, \quad -\sqrt{\frac{7}{2}}$$

Since the object is only between $x = \sqrt{3}$ and $x = 2$ .then max velocity will occur at $x = \sqrt{\frac{7}{2}}$

The max speed at $x=\sqrt{\frac{7}{2}}$ is

$$v^2=-2\left(\frac{7}{2}-4\right)\left(\frac{7}{2}-3\right)=$$

$$v^2=-2\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)$$

$$v^2=\frac{1}{2}$$

$$v=\frac{1}{\sqrt{2}}$$

Most students were able to find the max value of speed and where it occurs, this was worth 1 mark.

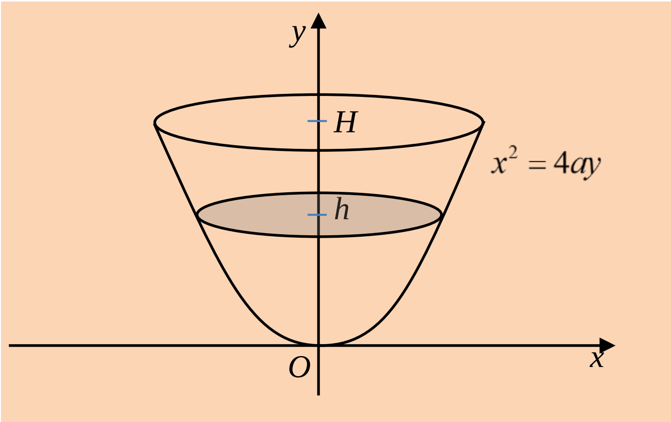
A second mark was awarded for realising that the particle was oscillating between $x=\sqrt{3}$ and $x=2$. and $x=2$.

A third mark was awarded if students showed that the particle would start to move the positive direction after it stopped at $x=\sqrt{3}$.

Question 14 continues on page 15

Question 14 (continued)

(c) The diagram below shows a paraboloid that is created by rotating the parabola $x^2 = 4ay$ around the y-axis.



(i) Find the volume of water in the paraboloid when the water is at height h. **1**

$$V = \pi\int_0^h x^2\ dy$$

$$= \pi\int_0^h 4ay\ dy$$

$$= 4a\pi\left[\frac{y^2}{2}\right]_0^h$$

$$= 4a\pi\left(\frac{h^2}{2} - 0\right)$$

$$V = 2a\pi h^2$$

Very well done

The paraboloid is filled with water up to a height H. Water is then released from a small hole at point O. It takes T seconds for the water to fully drain out.

It is known that the volume V of water in the paraboloid decreases at a rate given by

$$\frac{dV}{dt} = -k\sqrt{h^3},$$

where k is a positive constant and h is the height of water.

(ii) Show that **2**

$$\frac{dh}{dt} = \frac{-k\sqrt{h}}{4a\pi}.$$

From part i)

$V = 2a\pi h^2$

$\frac{dV}{dh} = 4a\pi h$

$\frac{dh}{dV} = \frac{1}{4a\pi h}$

$\frac{dV}{dt} = -k\sqrt{h^3}$ (Given)

$$\begin{aligned} LHS &= \frac{dh}{dt} \\ &= \frac{dh}{dV} \times \frac{dV}{dt} \\ &= \left(\frac{1}{4a\pi h}\right) \times \left(-k\sqrt{h^3}\right) \\ &= \frac{-k\sqrt{h^3}}{4a\pi h} \\ &= \frac{-kh\sqrt{h}}{4a\pi h} \\ &= \frac{-k\sqrt{h}}{4a\pi} \\ &= RHS \end{aligned}$$

Very well done

(iii)	Hence or otherwise show that $$y = H\left(1-\frac{t}{T}\right)^2 \quad \text{for} \quad 0 \le t \le T.$$	**3**

$\frac{dh}{dt} = \frac{-k\sqrt{h}}{4a\pi}$

By taking the reciprocal:

$\frac{dt}{dh} = \frac{-4a\pi}{k\sqrt{h}}$

$= -\frac{4a\pi}{k}\left(h^{-\frac{1}{2}}\right)$

Then integrate both sides.

$t = -\frac{4a\pi}{k}\left(h^{\frac{1}{2}} \div \frac{1}{2}\right) + C$

$t = -\frac{8a\pi}{k}\sqrt{h} + C$

To find the value of C use the **initial** and **final** conditions.

When $t = 0 \quad h = H$

$0 = -\frac{8a\pi}{k}\sqrt{H} + C$

$C = \frac{8a\pi}{k}\sqrt{H}$

When $t = T \quad h = 0$

$T = -\frac{8a\pi}{k}\sqrt{0} + C$

$T = C$

$T = \frac{8a\pi}{k}\sqrt{H}$

$\frac{T}{\sqrt{H}} = \frac{8a\pi}{k}$

Therefor the equation for t in terms of T, H and h is:

$t = -\left(\frac{T}{\sqrt{H}}\right)\sqrt{h} + T$

Rearrange this equation to make h the subject

$$t=-\left(\frac{T}{\sqrt{H}}\right)\sqrt{h}+T$$

$$t=-T\left(\frac{\sqrt{h}}{\sqrt{H}}-1\right)$$

$$-\frac{t}{T}=\frac{\sqrt{h}}{\sqrt{H}}-1$$

$$1-\frac{t}{T}=-\frac{\sqrt{h}}{\sqrt{H}}$$

$$\left(1-\frac{t}{T}\right)^2=\frac{h}{H}$$

$$H\left(1-\frac{t}{T}\right)^2=h$$

$$h=H\left(1-\frac{t}{T}\right)^2$$

Most students could not get past the first step, which was to take the reciprocal of $\frac{dh}{dt}$ and integrate.

A number of students were integrating with respect to *t* but still had an *h* in the expression

If students realised that they were required to both the ***initial*** and ***final*** condition in then generally they were able to find the required expression for *h* in terms of *H*, *t* and *T*