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Student Number: ________

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(16)

Oxley College

TRIAL EXAMINATION 2000

3 UNIT MATHEMATICS

Time Allowed - 2 hours
(Plus 5 minutes reading time)

INSTRUCTIONS

* Attempt **all** questions.
* All questions are of **equal** value.
* Show all necessary working in every question.
* Marks may be deducted for poorly arranged or careless work.
* *Board-approved* calculators may be used.
* Clearly label each question and part on your answer sheet.
* **Start each question on a new page.**

Question 1. (12 marks) (Start on a new page)

[4] (a) (i) Write down the expansion of $\tan(A+B)$

(ii) Find the value of $\tan 105^0$ in simplest surd form.

[3] (b) Solve the inequality: $\dfrac{2x+1}{2x-1} \geq 2$

[2] (c) Evaluate: $\displaystyle\int_0^{\pi/4} \cos x \sin^2 x \, dx$.

[3] (d) Solve: $x^6 - 9x^3 + 8 = 0$

Question 2. (12 marks) (Start on a new page)

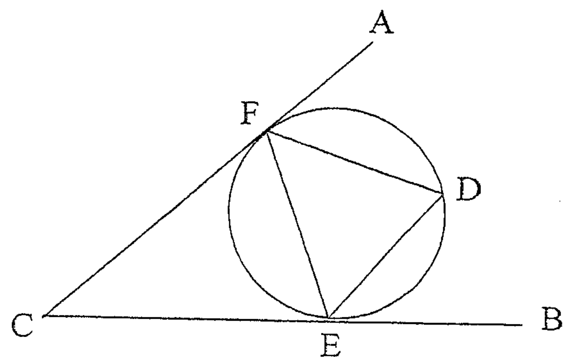
[4] (a) (i) Find $\frac{d}{dx}(x \log x - x)$

(ii) Hence evaluate $\int_2^e \log x \, dx$. Leave the answer in exact form.

[3] (b) In the diagram, AC and BC are tangents to the circle, touching at F and E respectively. $\angle ACB$ equals 50°.

(i) Show that $\angle CEF$ is 65°.

(ii) Hence, find $\angle EDF$, giving reasons for your answer.



5] (c) (i) Show that $\cos 6x = 2\cos^2 3x - 1$.

(ii) The arc of the curve $y = \cos 3x$ between the lines $x = 0$ and $x = \frac{\pi}{6}$ is rotated about the x-axis.

Using (i), or otherwise, find the <u>exact</u> volume of the solid formed.

Question 3. (12 marks) (Start on a new page)

[5] (a) Consider the equation $x^2 - 4 + \log_e x = 0$

(i) Show, by means of calculations, that the root of the equation lies between 1 and 2.

(ii) Use two applications of the 'halving the interval' method to find a smaller interval containing the root.

(iii) By drawing a graph of $y = \log_e x$, and any other appropriate graph on the same set of axes, verify that the equation has only one root.

[7] (b) Given the function: $y = \dfrac{x^2 - 2x - 3}{x - 1}$

(i) Find the coordinates of the points of intersection with the axes.

(ii) Find the equation of any asymptotes.

(iii) Show that the curve has no stationary points.

(iv) Sketch the curve.

Question 4. (12 marks) **(Start on a new page)**

[4] (a) The roots, α, β and γ of the equation $8x^3 - 36x^2 + 22x + 21 = 0$ are in an arithmetic progression.

(i) Show that $\alpha + \gamma = 2\beta$

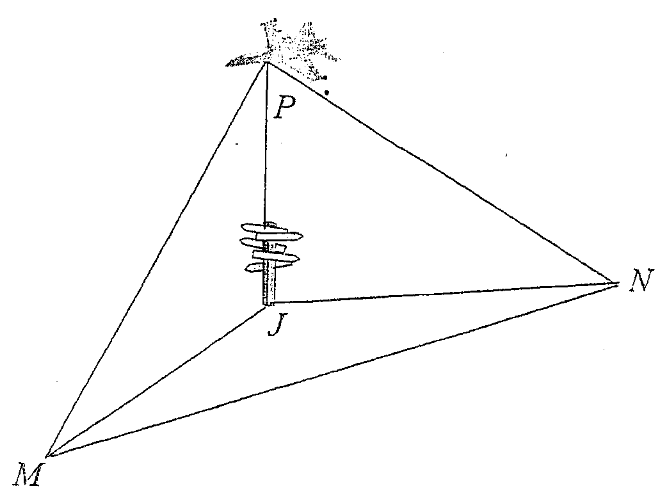
(ii) Write down the value of $\alpha + \beta + \gamma$

(iii) Find α, β and γ.

[4] (b) (i) Show that $\sum_{r=1}^{n} (5r-4) = 1 + 6 + 11 + \ldots + (5n-4)$

(ii) Hence, prove by Mathematical Induction that $\sum_{r=1}^{n} (5r-4) = \frac{1}{2}n(5n-3)$

4] (c) From a plane 500 metres above a road junction J, the angle of depression to a point M, due south of the junction is 42°. To another point N, bearing 080° from the junction, the angle of depression is 32°.



(i) Find the lengths of JM and JN

(ii) How far apart are M and N ?

Question 5. (12 marks) (Start on a new page)

[7] (a) $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are two points on the parabola $x^2 = 4ay$. The variable chord PQ is such that it is always parallel to the line $y = x$.

(i) Find the gradient of PQ and hence show that $p + q = 2$

(ii) Show that the equation of the **normal** at P is: $x + py = 2ap + ap^3$, given that the gradient of the *tangent* at P equals p.

(iii) Write down the equation of the normal at Q, and hence find the coordinates of the point of intersection R, of these normals.

(iv) Prove that the locus of R is the straight line $x - 2y + 12a = 0$

[5] (b) Newton's Law of Cooling states that when an object at temperature $T\,^{\circ}$C is placed in an environment at temperature $T_0\,^{\circ}$C, the rate of the temperature loss is given by the equation:

$$\frac{dT}{dt} = k(T - T_0)$$ *where* t is the time in seconds and k is a constant

(i) Show that $T = T_0 + Ae^{kt}$ is a solution to the equation.

(ii) A packet of peas, initially at 24°C is placed in a snap-freeze refrigerator in which the internal temperature is maintained at -40°C. After 5 seconds, the temperature of the packet is 19°C.

(α) Show that the value of the constant A is 64.

(β) Show that the value of the constant $k \approx -0.0163$

(γ) How long will it take for the packet's temperature to reduce to 0°C?

Question 6. (12 marks) (Start on a new page)

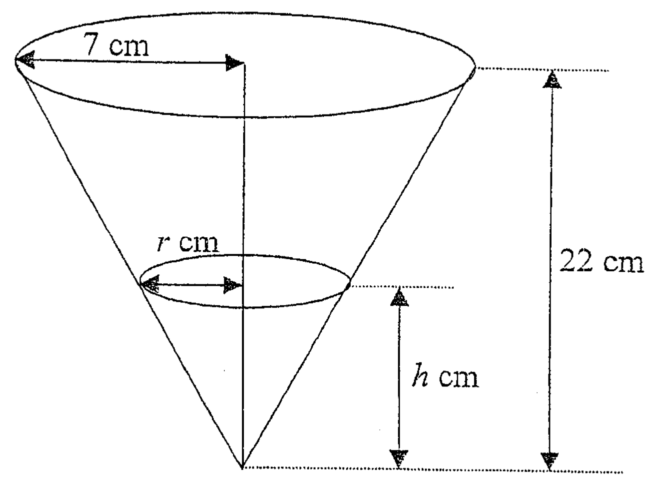
[2] (a) Find the general solution for: $\tan 2x = 1$

[5] (b) A golf ball is lying on a horizontal fairway when a golfer hits it. It just passes over a 2.25 metre high tree 1.5 seconds later. The tree is 60 metres away from the point from which the ball was hit. Taking $g = 10\,m\,s^{-1}$, calculate:

(i) the initial velocity and the angular projection.

(ii) how far away, from where the golfer hits it, does the ball land.

[5] (c) Soft serve ice cream is served into a jumbo-sized right circular cone as shown in the diagram. The cone has height 22 cm and radius 7 cm.



Ice cream is leaking through a hole of negligible size in the bottom of the cone at a constant rate of 7 cm^3 per minute.

(i) Use similar triangles to find a relationship between r and h.

(ii) Show that when the depth of the ice cream in the cone is h cm, the volume of ice cream is $\frac{7}{66}h^3$, using the approximate value $\pi = \frac{22}{7}$

(ii) At what rate is the depth of ice cream in the cone decreasing when $h = 11$

Question 7. **(12 marks)** **(Start on a new page)**

[4] (a) A particle is moving in a straight line. At time t seconds its velocity, v metres per second, and its displacement, x metres, are such that:

$$v^2 = 48 - 3x^2$$

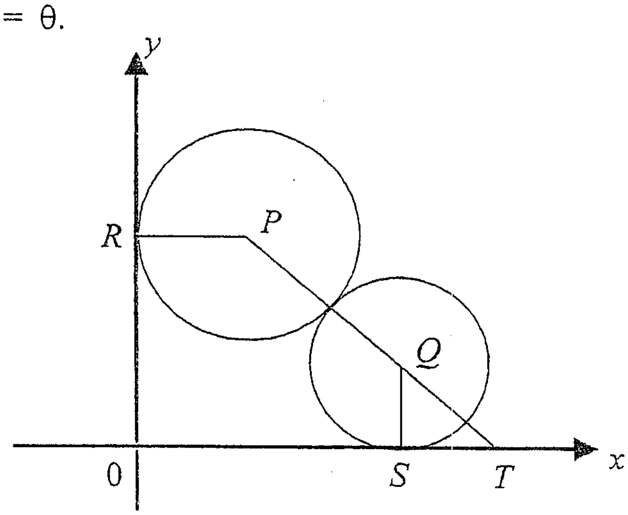
(i) Show that the motion is simple harmonic.

(ii) Find the amplitude of the motion.

(iii) Determine the particle's maximum speed.

(iv) Determine the particle's maximum acceleration.

[8] (b) The diagram shows two touching circles, with centres P and Q. The circle with centre P has a radius of 4 units and touches the y-axis at R. The circle with centre Q has a radius of 3 units and touches the x-axis at S. PQ produced meets the x-axis at T and $\angle QTS = \theta$.



(i) Show that $OR = 3 + 7\sin\theta$ and $OS = 4 + 7\cos\theta$

(ii) Show that $RS^2 = 42\sin\theta + 56\cos\theta + 74$

(iii) Hence express RS^2 in the form $74 + r\cos(\theta - \alpha)$, clearly stating the values of r and α.

(iv) Find the maximum length of RS and the value of θ for which this occurs.

END OF EXAMINATION

1. (a) (i) $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$ — Aw-①

(ii) $\tan 105° = \tan(45° + 60°)$ — A1-1

$= \dfrac{\tan 45 + \tan 60}{1 - \tan 45 \tan 60}$

$= \dfrac{1 + \sqrt{3}}{1 - \sqrt{3}}$ — A1-1

$= \dfrac{1+\sqrt{3}}{1-\sqrt{3}} \times \dfrac{1+\sqrt{3}}{1+\sqrt{3}}$

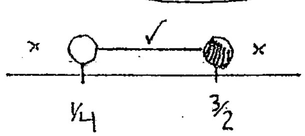
$= -2 - \sqrt{3}$ — Aw-③

(b) $\dfrac{2x+1}{2x-1} \geq 2$

Critical Value when $2x - 1 = 0$
ie. $x = \frac{1}{2}$ — A1-1

Now, $2x + 1 \geq 2(2x-1)$
$2x + 1 \geq 4x - 2$
$3 \geq 2x$
$\therefore x \leq \frac{3}{2}$ — A1-1

TEST:



$\therefore \frac{1}{2} < x \leq 1\frac{1}{2}$ — Aw-③

Q1 – continued...

(c) $\displaystyle\int_0^{\pi/4} \cos x \cdot \sin^2 x \, dx = \left[\tfrac{1}{3}\sin^3 x\right]_0^{\pi/4}$ — A1-

$= \frac{1}{3}(\sin^3 \frac{\pi}{4} - \sin^3 0)$

$= \frac{1}{3}\left(\frac{1}{\sqrt{2}}\right)^3 - 0$

$= \dfrac{1}{6\sqrt{2}} = \dfrac{\sqrt{2}}{12} \approx 0.1178.$ — Aw-

(d) $x^6 - 9x^3 + 8 = 0$

let $a = x^3$

$\therefore a^2 - 9a + 8 = 0$ — A1-
$(a-8)(a-1) = 0$
$\therefore a = 8, 1.$ — A1-

Now, $x^3 = 8 \longrightarrow x = 2$ — Aw-
$x^3 = 1 \longrightarrow x = 1$

Q2. (a)(i) $\frac{d}{dx}(x\log x - x) = (x\cdot\frac{1}{x} + \log x\cdot 1) - 1$ — A1-1

$= 1 + \log x - 1$

$= \log x$ — Aw-(2)

(ii) $\int_2^e \log x\,dx = \left[x\log x - x\right]_2^e$ — A1-1

$= (e\log e - e) - (2\log 2 - 2)$

$= (e - e) - 2(\log 2 - 1)$

$= 2(1 - \log 2)$ — Aw-(2)

(b)(i) CF = CE (tangents from an external point are equal). — A1-1.

∴ ΔECF is isosceles.

∴ ∠EFC = ∠CEF = 65° (base ∠'s isosc. Δ) — Aw-(2)

(ii) ∠EDF = ∠CEF = 65° (alt. segment theorem) — Aw-(1)

(c)(i) $\cos 2\alpha = 2\cos^2\alpha - 1$

let $\alpha = 3x$

$\therefore \cos 6x = 2\cos^2 3x - 1$ — Aw-(1)

(ii) $V = \pi\int y^2\,dx$

$= \pi\int_0^{\pi/6} \cos^2 3x\,dx$ — A1-1

$= \pi\int_0^{\pi/6} \frac{1}{2}(\cos 6x + 1)\,dx$ — A1-1

$= \frac{\pi}{2}\left[x - \frac{1}{6}\sin 6x\right]_0^{\pi/6}$ — A1-1

$= \frac{\pi}{2}\left[\left(\frac{\pi}{6} - \frac{1}{6}\sin\pi\right) - \left(0 - \frac{1}{6}\sin 0\right)\right]$ ↓

$= \frac{\pi}{2}\left(\frac{\pi}{6}\right)$

$= \boxed{\frac{\pi^2}{12}\text{ units}^3}$ — Au

Q3. (a)(i) Let $f(x) = x^2 - 4 + \log_e x$

$\therefore f(1) = 1^2 - 4 + \log_e(1) = -3$

$f(2) = 2^2 - 4 + \log_e(2) = 0.693$

Since $f(x)$ changes sign, the root lies between 1 and 2. — A

(ii) Half the interval of 1 and 2 is $1\frac{1}{2}$.

$f(1\frac{1}{2}) = 1.5^2 - 4 + \log_e(1.5) = -1.34$

Since $f(1\frac{1}{2}) < 0$, the root lies between 1.5 and 2 — A

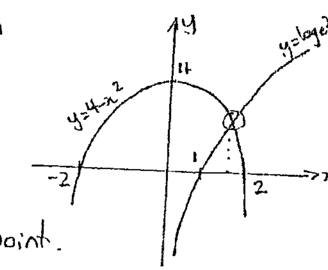
Half the interval of 1.5 and 2 is 1.75

$f(1.75) = 1.75^2 - 4 + \log_e(1.75) = -0.378$

Since $f(1.75) < 0$, the root lies between 1.75 and 2 — Au

(iii) The root is the intersection of $y = \log_e x$ and $y = 4 - x^2$.

ie. $\log_e x = 4 - x^2$



There is only one such point. — Aw-

Q3. continued...

(b) $y = \dfrac{x^2-2x-3}{x-1}$

(i) When $x=0$: $y = \frac{-3}{-1}$ → ($y=3$)

When $y=0$: $x^2-2x-3=0$

$(x-3)(x+1)=0$ → ($x = 3, -1$) Aw-(1)

(ii) Vertical Asymptotes: when $x-1=0$

→ ($x=1$)

Oblique Asymptotes: when ($y = x-1$) Aw-(2)

DIVIDE: $\dfrac{x^2-2x+1}{x-1} - \dfrac{4}{x-1}$

$(x-1) - \dfrac{4}{x-1}$.

(iii) $\dfrac{dy}{dx} = \dfrac{(x-1)(2x-2)-(x^2-2x-3)}{(x-1)^2}$

$= \dfrac{2x^2-2x-2x+2-x^2+2x+3}{(x-1)^2}$

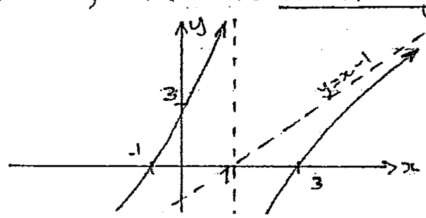
$= \dfrac{x^2-2x+5}{(x-1)^2}$ A1 - 1

TP when $dy/dx = 0$

ie. $x^2-2x+5=0$

Since $\Delta < 0$, there are no stationary points. Aw-(2)

(iv)



Aw-(2)

Q4. (a) (i) AP: $\beta - \alpha = \gamma - \beta$

$\therefore 2\beta = \alpha + \gamma$ Aw

(ii) $\alpha + \beta + \gamma = -b/a = 36/8 = 4.5$ Aw

(iii) $(\alpha+\gamma)+\beta = 4.5$

$2\beta + \beta = 4.5$

$3\beta = 4.5$ → ($\beta = 1.5$) A1 -

Now, $\alpha + \gamma = \quad = 3$ ——(1)

Also, $\alpha\beta\gamma = -d/a = -21/8$

$\therefore \alpha\gamma = -7/4$ ——(2)

Eq.(1): $\gamma = 3 - \alpha$

Sub into (2): $\alpha(3-\alpha) = -7/4$

$4\alpha^2 - 12\alpha - 7 = 0$

$(2\alpha+1)(2\alpha-7) = 0$

$\therefore \alpha = -1/2, 7/2$.

When $\alpha = -1/2, \gamma = 7/2$
$\alpha = 7/2, \gamma = -1/2$ }

$\therefore$ The roots are $-1/2, 3/2, 7/2$ Aw.

(b) (i) $\sum_{r=1}^{n} (5r-4) = 1+6+11+\cdots+(5n-4)$

(ii) Prove: $1+6+11+\cdots+(5n-4) = \frac{1}{2}n(5n-3)$

* Prove true for $n=1$:

LHS = 5-4 = 1 ... A1 -

Q4. continued---

* Assume true for $n=k$:

ie. $1+6+11+\cdots+5k-4 = \frac{1}{2}k(5k-3)$

* Prove true for $n=k+1$:

RTP: $[1+6+11+\cdots+5k-4] + 5(k+1)-4 = \frac{1}{2}(k+1)(5(k+1)-3)$

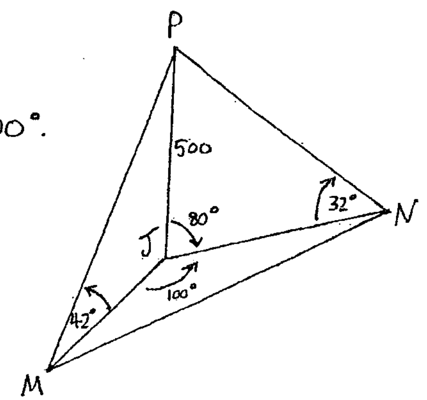
ie. $1+6+11+\cdots+(5k-4)+(5k+1) = \frac{1}{2}(k+1)(5k+2)$ — A1-1

$$\begin{aligned} LHS &= \tfrac{1}{2}k(5k-3)+(5k+1) \\ &= \tfrac{1}{2}[5k^2-3k+10k+2] \\ &= \tfrac{1}{2}[5k^2+7k+2] \\ &= \tfrac{1}{2}(k+1)(5k+2) \\ &= RHS. \end{aligned}$$

Since proved true for $n=k+1$ and show true for $n=1$; then the result must be true for $n=2,3,$ etc. by the process of Mathematical Induction. — Aw-(4)

(c)

$\angle MJN = 100°$.



Q4 continued...

(i) In ΔJMP: $\tan 42° = \frac{500}{JM}$ $\therefore JM = \frac{500}{\tan 42°}$ — A

In ΔJNP: $\tan 32° = \frac{500}{JN}$ $\therefore JN = \frac{500}{\tan 32°}$ — A

(ii) In ΔJMN: By the Cosine Rule:

$$MN^2 = \left(\frac{500}{\tan 42}\right)^2 + \left(\frac{500}{\tan 32}\right)^2 - 2\left(\frac{500}{\tan 42}\right)\left(\frac{500}{\tan 32}\right)\cos 100°$$

$\therefore MN = 1050.2$ metres — Aw

Q5. (a) (i) $M_{PQ} = \frac{ap^2-aq^2}{2ap-2aq} = \frac{a(p-q)(p+q)}{2a(p-q)} = \frac{p+q}{2}$

PQ parallel to $y=x$ ie. $m=1$. } $\therefore \frac{p+q}{2} = 1$

$p+q=2$ — Aw

(ii) $m = -1/p$ at $P(2ap, ap^2)$

$y-y_1 = m(x-x_1)$

$\therefore y-ap^2 = -1/p(x-2ap)$

$yp - ap^3 = -x + 2ap$

$\therefore x+py = 2ap^2+ap^3$ ——— ① — Aw-

(iii) Similarly, eqn. of normal at Q is:

$x+qy = 2aq^2+aq^3$ ——— ②

Eq. ①-②: $4(p-q) = 2a(p-q) + a(p^3-q^3)$

$4 = 2a + a(p^2+q^2+pq)$ — A1-

Q5. Continued...

Sub into Eq ①: $x + p(2a + ap^2 + aq^2 + apq) = 2ap + ap^3$

$\therefore x = -2ap - ap^3 - apq^2 - ap^2q + 2ap + ap^3$

$x = -apq(p+q)$

But $p+q = 2 \Rightarrow \therefore x = -2apq$

Aw-②

$\therefore$ Coordinates of R: $x = -2apq$ —— ③

$y = 2a + a(p^2+q^2+pq)$ — ④

(iv) Eq ④: $y = 2a + a[(p+q)^2 - pq]$

$y = 2a + a(4 - pq)$

$y = 6a - apq$.

Eq ③: $pq = \frac{-x}{2a}$

$\therefore y = 6a + x/2$

$2y = 12a + x$

$\therefore x - 2y + 12a = 0$ is the eqn of the locus of R.

Aw ③

(b)(i) $T = T_0 + Ae^{kt}$

$\frac{dT}{dt} = 0 + Ae^{kt} \times k$

$= (T - T_0) \times k$

$\therefore dT/dt = k(T - T_0)$

Aw-①

Q5. Continued...

(ii) $T_0 = -40°$, $T = 19°$

(α) When $t=0$: $24 = -40 + Ae^0$

$\therefore A = 64$

(β) When $t=5$: $19 = -40 + 64e^{5k}$

$59 = 64e^{5k}$

$e^{5k} = \frac{59}{64}$

$5k = \ln(59/64)$

$\therefore k = 1/5 \ln(59/64)$

ie. $k = -0.0163$ (3sf)

A

(γ) For $T=0$: $0 = -40 + 64e^{-0.0163t}$

$e^{-0.0163t} = 40/64$

$-0.0163t = \ln(40/64)$

$\therefore t = \frac{\ln(40/64)}{-0.0163}$

ie. $t = 28.9$ seconds

Aw

Q6. (a) $\tan 2x = 1$

$2x = n\pi + \frac{\pi}{4}$

$\therefore x = \frac{n\pi}{2} + \frac{\pi}{8}$ — Aw-(2)

(b) $\ddot{x} = 0$; $\ddot{y} = -10$

$\dot{x} = V\cos\theta$; $\dot{y} = -10t + V\sin\theta$

$x = Vt\cos\theta$; $y = -5t^2 + Vt\sin\theta$

(assuming originally at origin (0,0))

(i) At $t = 1.5$, $x = 60$:

$60 = V(1.5)\cos\theta$

$V\cos\theta = 40$ —— ①

At $t = 1.5$, $y = 2.25$:

$2.25 = -5(1.5)^2 + V(1.5)\sin\theta$

$V\sin\theta = 9$ —— ② — A1-1

Eq. ②÷①: $\tan\theta = 9/40 \rightarrow \theta = 12.68°$ — A1-1

Sub into Eq.①: $V\cos(12.68°) = 40 \rightarrow V = 41\,ms^{-1}$ — Aw-(3)

(ii) When $y = 0$: $-5t^2 + Vt\sin\theta = 0$

$t(-5t + V\sin\theta) = 0$

$\therefore t = 0$ or $t = \frac{V\sin\theta}{5}$

ie. $t = 0, 9/5$. — A1-1

When $t = 9/5$: $x = Vt\cos\theta$

$x = 41 \times 9/5 \times \cos 12.68°$

$\therefore x = 72$ metres. — Aw-(2)

Q6. continued...

(c) (i) $\frac{r}{7} = \frac{h}{22}$

$22r = 7h \rightarrow \therefore r = \frac{7h}{22}$ — Aw-

(ii) $V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3} \cdot \frac{22}{7} \cdot \left(\frac{7h}{22}\right)^2 \cdot h$

$= \frac{22}{21} \cdot \frac{49h^2}{484} \cdot h$

$\therefore V = \frac{1078h^3}{10164} = \frac{7}{66}h^3$ — Aw-

(iii) Now, $\frac{dV}{dh} = \frac{7}{22}h^2$ and $\frac{dV}{dt} = 7$ cm/min. — A1-

$\therefore \frac{dh}{dt} = \frac{dV}{dt} \cdot \frac{dh}{dV}$

$= 7 \cdot \frac{22}{7h^2}$

$\therefore \frac{dh}{dt} = \frac{22}{h^2}$ — A1-1

When $h = 11$, $\frac{dh}{dt} = \frac{22}{121} = \frac{2}{11}$ cm/min — Aw-(

Q7. (a) (i) $\frac{1}{2}v^2 = 24 - \frac{3}{2}x^2$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -3x$

$\therefore \ddot{x} = -3x$ — Aw-(1)

Since the acceleration is proportional to its distance from the origin, it is moving in SHM.

(ii) $v^2 = 3(16 - x^2)$

ie. $v^2 = n^2(a^2 - x^2)$ $\rightarrow \therefore a = 4$ metres — Aw-(1)

(iii) Max. Speed when $x = 0$:

ie. $v^2 = 3(16 - 0)$

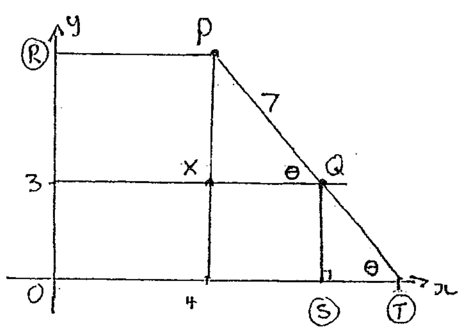
$\therefore v = \sqrt{48}\ ms^{-1}$ is the max. speed. — Aw-(1)

(iv) Max. Acceleration when $x = \pm 4$

ie. $\ddot{x} = -3 \times -4$

$\therefore \ddot{x} = 12\ ms^{-2}$ is the max. acceleration — Aw-(1)

(b)



(i) $OR = 3 + PX = 3 + 7\sin\theta$

$OS = 4 + QX = 4 + 7\cos\theta$ — Aw-(1)

Q7. *Continued...*

(ii) $RS^2 = OR^2 + OS^2$ (Pythagoras' Theorem)

$= (3 + 7\sin\theta)^2 + (4 + 7\cos\theta)^2$

$= 9 + 42\sin\theta + 49\sin^2\theta + 16 + 56\cos\theta + 49\sin^2\theta.$

$= 74 + 42\sin\theta + 56\cos\theta$ — Aw

(iii) Let $42\sin\theta + 56\cos\theta = r\cos(\theta - \alpha)$

where $r = \sqrt{42^2 + 56^2} = 70$

$\alpha = \tan^{-1}\left(\frac{42}{56}\right) = 36°52'$ — A1

$\therefore 42\sin\theta + 56\cos\theta = 70\cos(\theta - 36°52')$

$\therefore RS^2 = 74 + 70\cos(\theta - 36°52')$ — Aw

(iv) Maximum value of RS occurs when:

$\cos(\theta - 36°52') = 1$

ie. when $\theta = 36°52'$. — A1

$\therefore RS^2 = 74 + 70 = 144$

ie. $RS = 12$ units — Aw